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ON THE OSCILLATION OF A VOLTERRA INTEGRAL EQUATION

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1. Introduction

A vast literature exists on the oscillation theory of functional differential equations. The references [1] and [4] present a fairly exhaustive listing for the interested reader. Our purpose here is to add to the pioneering work of Onose [2] who recently obtained some oscillation criteria for the oscillation of the integral equation:

$$X(t) = f(t) - \int_0^t a(t,s)g\big(s, X(s)\big)\,\mathrm{d}s, \quad t \geqslant 0. \tag{1}$$

Oscillation results for integral equations of the Volterra type are scant and only a few references exist on this subject. Related studies can also be found in Parhi and Misra [3].

In this work, we have obtained somewhat stronger results than those of Onose [2] who obtained sufficient conditions for bounded solutions of equation 1 to be oscillatory. We have not only found sufficient conditions for all solutions of equation 1 to oscillate but also given growth estimates on solutions of equation 1.

2. Assumptions and definitions

(i) $f\colon [0,\infty) \to \mathbb{R}$, $g\colon [0,\infty] \times \mathbb{R} \to \mathbb{R}$ and continuous, where $\mathbb{R}$ is the real line;

(ii) $a\colon [0,\infty) \times \mathbb{R} \to \mathbb{R}^+$, continuous, $0 \leqslant t \leqslant \infty$ and $0 \leqslant s \leqslant t$, $a(t,s) = 0$, $s > t$.

We only consider those solutions of (1) which are continuously extendable on $[0,\infty)$ and are nontrivial. The term "solution" henceforth applies to such solutions of (1).

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A solution of equation 1 is said to be *oscillatory* if it has arbitrarily large zeros on the positive half real line $\mathbb{R}^+$; otherwise it is called *nonoscillatory*. A solution $y(t)$ of (1) is said to be *slowly oscillating* if the set

$$S = \{|t_\alpha - t_\beta| \colon y(t_\alpha) = y(t_\beta) = 0, \quad |y(t)| > 0 \text{ for } t \in (t_\alpha, t_\beta)\}$$

is bounded on $\mathbb{R}^+$. In the last section of this work, we study slowly oscillating solutions of (1). Qualitative behavior of the nonoscillatory solutions of equation 1 is also examined. An oscillatory solution $X(t)$ is said to be *properly unbounded* if $\limsup\limits_{t\to\infty} X(t) = \infty$ and $\liminf_{t\to\infty} X(t) = -\infty$. A nonoscillatory solution is *properly unbounded* if it is unbounded.

3. Main results

Theorem 1. *Suppose that*

$$Xg(t, X) > 0 \qquad \text{for } X \neq 0,\ t \geqslant 0; \tag{2}$$

$$\frac{g(t, X)}{X} \leqslant M; \qquad \text{for some } M > 0,\ t \geqslant 0 \text{ and } X \neq 0. \tag{3}$$

Further suppose there exists positive and continuous functions $p(t)$, $h(t)$ on $[0, \infty)$ such that $h(s) = 0$ for $s > t$,

$$a(t, s) \leqslant p(t)h(s), \tag{4}$$

$$\int^{\infty} h(t)\,\mathrm{d}t < \infty, \tag{5}$$

and

$$p(t) \text{ and } f(t)/t \quad \text{are bounded for } t \geqslant 0. \tag{6}$$

Let $X(t)$ be any solution of (1). Then

$$X(t) = O(t), \qquad \text{i.e. } \overline{\lim_{t\to\infty}} \frac{X(t)}{t} < +\infty.$$

Proof. Form equation (1),

$$\frac{X(t)}{t} \leqslant \frac{|f(t)|}{t} + \int_0^t p(t)h(s) \cdot \frac{g(s, X(s))}{|X(s)|} \cdot \frac{|X(s)|}{s}\,\mathrm{d}s$$

$$\leqslant K + L \int_0^t h(s) \cdot \frac{|X(s)|}{s}\,\mathrm{d}s$$

for some positive constants K and L. The conclusion follows by Gronwall's inequality. □

Theorem 2. *Suppose conditions* (2) *through* (4) *and* (6) *of Theorem* 1 *hold. Further suppose that condition* (5) *is modified to*

$$\int^{\infty} th(t)\,\mathrm{d}t < \infty, \tag{7}$$

and

$$\limsup_{t\to\infty} f(t) = \infty, \quad \liminf_{t\to\infty} f(t) = -\infty. \tag{8}$$

Then all solutions of equation (1) *are oscillatory.*

Proof. Since condition (5) of Theorem 1 is implied by (7) for $t \geqslant 1$, we can safely assume that the conclusion of Theorem 1 holds. Without any loss of generality, suppose $T > 1$ is large enough so that $X(t) > 0$ for $t \geqslant T$.

From equation 1,

$$X(t) = f(t) - \int_0^T a(t,s)g\big(s, X(s)\big)\,\mathrm{d}s - \int_T^t a(t,s)g\big(s, X(s)\big)\,\mathrm{d}s \tag{9}$$

$$\leqslant f(t) - \int_0^T p(t)h(s)g\big(s, X(s)\big)\,\mathrm{d}s \tag{10}$$

$$+ \int_T^t p(t)\cdot sh(s)\frac{g(s,X(s))}{X(s)}\cdot\frac{X(s)}{s}\,\mathrm{d}s.$$

Now $p(t)$ is bounded, and by Theorem 1, $\big(X(t)\big)/t$ is bounded. In view of condition (7), the last two integrals on the right hand side of (10) are finite. Since $X(t) > 0$, and (8) holds, we reach a contradiction which completes the proof. □

Remark 1. Our Theorem 2 does not generalize Theorem 1 of Onose [2], but presents an extended set of conditions which apply to all solutions of equation (1).

Example 1. Consider the equation

$$X(t) = (t+1)(\sin t) - \frac{1}{2}(\mathrm{e}^{-t}\cos t + \mathrm{e}^{-t}\sin t - 1) \tag{11}$$

$$- \int_0^t \frac{1}{s+1}\cdot\mathrm{e}^{-s}\big(X(s)\big)\,\mathrm{d}s, \quad t \geqslant 0.$$

Equation (11) satisfies all conditions of Theorem 2. Hence, all solutions of (11) are oscillatory. In fact

$$X(t) = (t+1)\sin t$$

is one such solution.

Remark 2. Our next theorem improves the condition 7 of Theorem 2.

Theorem 3. *Suppose all conditions of Theorem* 1 *hold. Further suppose that (8) holds and*

$$\limsup_{t\to\infty} p(t) \int^{t} sh(s)\,\mathrm{d}s < \infty. \tag{12}$$

Then all solutions of equation (1) *are oscillatory.*

Proof. Without any loss of generality, let $X(t) > 0$ for $t \geqslant T$ be a solution of (1). In a manner of Theorem 1, we see that

$$X(t) = O(t). \tag{13}$$

From equation (1),

$$\begin{aligned} X(t) &\leqslant f(t) - \int_0^T a(t,s)g\bigl(s, X(s)\bigr)\,\mathrm{d}s + \int_T^t p(t)\cdot sh(s)\,\mathrm{d}s \cdot \frac{g(s,X(s))}{X(s)}\cdot\frac{X(s)}{s}\,\mathrm{d}s \\ &\leqslant f(t) - \int_0^T a(t,s)g\bigl(s, X(s)\bigr)\,\mathrm{d}s + Mp(t)\int_T^t sh(s)\frac{X(s)}{s}\,\mathrm{d}s. \end{aligned} \tag{14}$$

From (12), (13), and boundedness of $p(t)$, we see that the last two integrals in inequality (14) are bounded. Since

$$\limsup_{t\to\infty} f(t) = \infty$$

and

$$\liminf_{t\to\infty} f(t) = -\infty$$

we reach a contradiction. The proof is complete. □

Remark 3. Our next theorem does not require that $f(t)$ be unbounded.

Theorem 4. *Suppose conditions* (2) *through* (5) *of Theorem* 1 *hold;* $p(t)$ *and* $f(t)$ *are bounded; and*

$$\liminf_{t\to\infty}\int^{t} f(t)\,\mathrm{d}t=-\infty,\qquad \limsup_{t\to\infty}\int^{t} f(t)\,\mathrm{d}t=\infty. \tag{15}$$

Further suppose

$$\int^{\infty} p(s)\int^{s} h(r)\,\mathrm{d}r\,\mathrm{d}s<\infty \tag{16}$$

and

$$\int^{\infty} p(t)\,\mathrm{d}t<\infty. \tag{17}$$

Then all solutions of equation (1) *are* bounded and oscillatory simultaneously.

Proof. Let $X(t)$ be any solution of equation (1). Then boundedness of $X(t)$ follows by Gronwall's inequality since $f(t)$ is now bounded. Now suppose to the contrary that $X(t)$ is nonoscillatory. Without any loss of generality suppose there exists a large $T>0$ such that $X(t)>0$ for $t\geqslant T$. From equation (1)

$$\begin{aligned}
\int_T^t X(r)\,\mathrm{d}r &= \int_T^t f(r)\,\mathrm{d}r-\int_T^t\int_0^s a(s,r)g(r,X(r))\,\mathrm{d}r\,\mathrm{d}s\\
&= \int_T^t f(r)\,\mathrm{d}r-\int_T^t\int_0^T a(s,r)g(r,X(r))\,\mathrm{d}r\,\mathrm{d}s\\
&\quad -\int_T^t\int_T^s a(s,r)g(r,X(r))\,\mathrm{d}r\,\mathrm{d}s\\
&\leqslant \int_T^t f(r)\,\mathrm{d}r+\int_T^t p(s)\int_0^T h(r)\cdot\frac{g(r,X(r))}{X(r)}\cdot X(r)\,\mathrm{d}r\,\mathrm{d}s\\
&\quad +\int_T^t p(s)\int_T^s h(r)\cdot\frac{g(r,X(r))}{X(r)}\cdot X(r)\,\mathrm{d}r\,\mathrm{d}s
\end{aligned} \tag{18}$$

since $X(t)$ and $[g(t,X(t))]/X(t)$ are bounded, and conditions (16) and (17) hold, the last two integers on the right hand side of (18) are finite. Since

$$\int_T^t X(t)>0$$

for $t \geqslant T$, a contradiction is immediately seen in view of (15). The proof is complete. □

Example 2. Consider the equation

$$X(t) = \sin\big((\ln(t+1)\big) - \frac{1}{(t+1)^2}\int_0^t \frac{1}{(s+1)^2}X(s)\,\mathrm{d}s, \quad t \geqslant 0. \tag{19}$$

If we choose

$$a(t,s) = \frac{1}{(t+1)^2(s+1)^2}, \quad t \geqslant s \geqslant 0,$$

then all conditions of this theorem are satisfied. All solutions of this equation are oscillatory and bounded.

Example 3. Consider the equation

$$\begin{aligned} X(t) = 2\sin\big(\ln(t+1)\big) &- \frac{\sin(\ln(t+1))}{(t+1)^3} \\ &- \frac{\cos(\ln(t+1))}{(t+1)^3} + \frac{1}{(t+1)^2} \\ &- \int_0^t \frac{X(s)}{(t+1)^2(s+1)^2}\,\mathrm{d}s, \quad t \geqslant 0. \end{aligned} \tag{20}$$

Here if we choose

$$\begin{aligned} a(t,s) &= \frac{1}{(t+1)^2(s+1)^2}, \quad t \geqslant s \\ &= 0, \quad s > t \end{aligned}$$

Then all conditions of Theorem 4 are satisfied. Therefore, all solutions of equation (20) are bounded and oscillatory. In fact, $X(t) = 2\sin\big(\ln(t+1)\big)$, $t \geqslant 0$ is one such solution.

Remark 4. The solution $X(t) = 2\sin\big(\ln(t+1)\big)$ of the preceding example is slowly oscillating since its zeros occur at $t_n = \mathrm{e}^{n\pi} - 1$. It is easily seen that $t_{n+1} - t_n \to \infty$ as $n \to \infty$. Our next theorem gives conditions which ensure that solutions of equation (1) with non-vanishing first derivatives are indeed slowly oscillating.

Theorem 5. *Suppose conditions of Theorem 4 hold. Let $X(t)$ be any solution of (1) which satisfies*

$$\limsup_{t\to\infty} |X(t)| > 0. \tag{21}$$

Then $X(t)$ is bounded and oscillatory, and either

$$\limsup_{t\to\infty} |X'(t)| > 0 \tag{22}$$

or else $X(t)$ is slowly oscillating.

Proof. We only need to show that $X(t)$ is slowly oscillating if (22) does not hold. Since by Theorem 4, $X(t)$ is oscillatory and (21) holds, there exists a sequence $\{t_n\}_{n=0}^{\infty}$ such that

$$t_n \to \infty \text{ as } n \to \infty, \quad t_n \geqslant T, \quad n \geqslant 0; \tag{23}$$

$$X(t_n) > d, \quad n \geqslant 1 \text{ for some } d > 0; \tag{24}$$

for each $n \geqslant 1$, let $[\alpha_n, \beta_n]$ be the largest interval around t_n such that for $n \geqslant 1$, $X(\alpha_n) = X(\beta_n) = 0$, $X(t) > 0$, $t \in (\alpha_n, \beta_n)$. Then by the mean value theorem we have

$$X'(s_n) = \frac{X(t_n) - X(\alpha_n)}{t_n - \alpha_n}$$

$$|X'(s_n)| \geqslant \frac{|X(t_n)| - |X(\alpha_n)|}{t_n - \alpha_n} = \frac{X(t_n) - X(\alpha_n)}{t_n - \alpha_n} \geqslant \frac{d}{\beta_n - \alpha_n}$$

where $s_n \in (\alpha_n, t_n)$ and $\alpha_n < t_n < \beta_n$. In view of (22) if $X'(t) \to 0$ as $t \to \infty$, then $\limsup_{t\to\infty}(\beta_n - \alpha_n) = \infty$ which completes the proof. □

Remark 5. Our next theorem is somewhat stronger and does not require that $\limsup_{t\to\infty} |X(t)| > 0$ where $X(t)$ is a solution of equation (1). The solution $X(t) = 2\sin\big(\ln(t+1)\big)$ of equation 20 in Example 3 is slowly oscillating; but does not satisfy the conclusion of Theorem 5 since $\limsup_{t\to\infty} |X'(t)| = 0$. However, it satisfies the conditions and conclusion of Theorem 6.

Theorem 6. *In addition to conditions of Theorem 4, suppose $p(t)$, and $f(t)$ are continuously differentiable on $(0,\infty)$ and*

$$p'(t) \to 0, \quad f'(t) \to 0, \quad h(t)p(t) \to 0 \text{ as } t \to \infty. \tag{25}$$

Further suppose

$$\left|\frac{\partial}{\partial t} a(t,s)\right| \leqslant |p'(t)h(s)|, \quad s \leqslant t, \ t \geqslant 0. \tag{26}$$

Let $X(t)$ be any solution of equation (1). *Then the following conclusions hold:*

$$X(t) \text{ is bounded,} \tag{27}$$

$$X'(t) \to 0 \text{ as } t \to \infty. \tag{28}$$

Either

$$\limsup |X(t)| > 0 \tag{29}$$

or

$$X(t) \text{ is slowly oscillating.} \tag{30}$$

Proof. From Equation (1),

$$X'(t) = f(t) - a(t,t)g(t, X(t)) - \int_0^t \frac{\partial}{\partial t} a(t,s)g(s, X(s))\,\mathrm{d}s \tag{31}$$

$$|X'(t)| \leqslant |f'(t)| + Mp(t)h(t)|X(t)| + p'(t)\int_0^t h(s)g(s, X(s))\,\mathrm{d}s.$$

Since conditions of Theorem 4 hold, $X(t)$ is bounded. From conditions (25) and (26), we see that (31) implies

$$X'(t) \to 0 \text{ as } t \to \infty. \tag{32}$$

Since conditions of Theorem 5 are also satisfied and (22) is no longer true, $X(t)$ is slowly oscillating. This completes the proof. □

Example 3 satisfies all the conditions of this theorem.

Remark 6. We have the following partial converse of Theorem 2.

Theorem 7. *Suppose conditions* (2) *through* (4) *and* (6) *of Theorem* 1 *hold, and condition* (7) *of Theorem* 2 *is satisfied. Further suppose that* $g(t, X)/X \geqslant K > 0$. *Let $X(t)$ be a properly unbounded oscillatory solution of equation* (1). *Then*

$$\limsup_{t\to\infty} f(t) = \infty \quad \textit{and} \quad \liminf_{t\to\infty} f(t) = -\infty.$$

Proof. From equation (1),

$$X(t)=f(t)-\int_0^T a(t,s)g(s,X(s))\,\mathrm{d}s-\int_T^t a(t,s)g(s,X(s))\,\mathrm{d}s. \tag{33}$$

Since conditions of Theorem 1 are implied, we find that $X(t)/t$ is bounded. Now

$$\left|\int_T^t a(t,s)g(s,X(s))\,\mathrm{d}s\right|\leqslant\int_T^t p(t)sh(s)\frac{g(s,X(s))}{X(s)}\cdot\frac{X(s)}{s}\,\mathrm{d}s. \tag{34}$$

In view of condition (7) of Theorem 2 and the fact that $g(t,X)/X\geqslant K$, we find that the left side of (34) is bounded. Thus the last two integrals in (33) remain finite. The conclusion follows from the fact that $X(t)$ is the properly unbounded oscillatory solution of equation (1). □

Corollary 1. *Suppose* (2) *through* (4) *and* (6) *of Theorem* 1 *and* (7) *of Theorem* 2 *hold. Then a necessary and sufficient condition for all properly unbounded solutions to be oscillatory is that*

$$\limsup_{t\to\infty} f(t)=\infty \quad \textit{and} \quad \liminf_{t\to\infty} f(t)=-\infty.$$

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