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SEMILATTICES OF FINITE ARITHMETICAL ALGEBRAS

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An algebra A is *arithmetical* if the congruence lattice $\text{Con } A$ is distributive and every two congruences $\Theta, \Phi \in \text{Con } A$ permute, i.e. $\Theta \circ \Phi = \Phi \circ \Theta$. It is easy to see that A is *arithmetical* if and only if the identity

$$(*) \quad \Theta \cap (\Phi \circ \Psi) \subseteq (\Theta \cap \Psi) \circ (\Theta \cap \Phi)$$

holds for every $\Theta, \Phi, \Psi \in \text{Con } A$, see e.g. [3], [4] (the symbol \circ denotes relational product). As was shown in [3], an algebra A is arithmetical if and only if it satisfies the *Chinese remainder theorem*. The famous characterization was given by A. F. Pixley [3] for finite algebras:

Proposition 1. *A finite algebra A is arithmetical if and only if there exists a ternary function $t: A^3 \rightarrow A$ compatible with $\text{Con } A$ satisfying*

$$(**) \quad t(x, x, z) = z, \quad t(x, y, x) = x, \quad t(x, z, z) = x.$$

This function t is called a *Pixley function*. Recall that an n -ary function $f: A^n \rightarrow A$ is compatible with $\text{Con } A$ if for each $\Theta \in \text{Con } A$, $\langle x_i, y_i \rangle \in \Theta$ for $i = 1, \dots, n$ imply

$$\langle f(x_1, \dots, x_n), f(y_1, \dots, y_n) \rangle \in \Theta.$$

A similar characterization was given by A. F. Pixley also for varieties of arithmetical algebras. Using it, we gave in [1] a characterization of algebras in such a variety as algebras whose every two-element subset forms a (semi)lattice with respect to polynomial operations. This cannot be immediately used for a single algebra since the proof depends on the construction of the free algebra $F_{\mathcal{V}}(x, y)$. However, we can modify it to

Theorem 1. *Let A be a finite algebra. The following conditions are equivalent:*

- (1) A is arithmetical;
- (2) for every element b of A there exists a binary function \wedge compatible with $\text{Con } A$ such that for each $a \in A$ the algebra $(\{a, b\}, \wedge)$ is a semilattice with the greatest element b .

Proof. (1) \Rightarrow (2): Put $x \wedge y = t(x, b, y)$ for a Pixley function $t(x, y, z)$. It is routine to show that for any $a \in A$, $(\{a, b\}, \wedge)$ is a semilattice with the greatest element b .

(2) \Rightarrow (1): Suppose $\Theta, \Phi, \Psi \in \text{Con } A$ and

$$\langle a, c \rangle \in \Theta \cap (\Phi \circ \Psi)$$

for some a, c of A , i.e. there exists an element $b \in A$ such that

$$\langle a, c \rangle \in \Theta, \quad \langle a, b \rangle \in \Phi, \quad \langle b, c \rangle \in \Psi.$$

Let $\wedge: A^2 \rightarrow A$ be a binary function compatible with $\text{Con } A$ such that $(\{a, b\}, \wedge)$, $(\{c, b\}, \wedge)$ are \wedge -semilattices with the (common) greatest element b . Then

$$\begin{aligned} \langle a, a \wedge c \rangle &= \langle a \wedge a, a \wedge c \rangle \in \Theta(a, c) \subseteq \Theta, \\ \langle a, a \wedge c \rangle &= \langle a \wedge b, a \wedge c \rangle \in \Theta(b, c) \subseteq \Psi, \\ \text{i.e. } \langle a, a \wedge c \rangle &\in \Theta \cap \Psi, \end{aligned}$$

and further

$$\begin{aligned} \langle a \wedge c, c \rangle &= \langle a \wedge c, c \wedge c \rangle \in \Theta(a, c) \subseteq \Theta, \\ \langle a \wedge c, c \rangle &= \langle a \wedge c, b \wedge c \rangle \in \Theta(a, b) \subseteq \Phi, \\ \text{i.e. } \langle a \wedge c, c \rangle &\in \Theta \cap \Phi \end{aligned}$$

whence $\langle a, c \rangle \in (\Theta \cap \Psi) \circ (\Theta \cap \Phi)$. By the above quoted result of [3], [4] concerning (*) we obtain (1). \square

Remark 1. Theorem 1 can be paraphrased: A finite algebra A is arithmetical iff A is the union of two-element \wedge -semilattices with a common greatest element, where the semilattice operation is compatible with $\text{Con } A$. This can be considered a description of the structure of a finite arithmetical algebra.

Remark 2. If A is a finite arithmetical algebra and $a, b, c \in A$ and if \wedge is defined as above, then $(\{a, b\}, \wedge)$, $(\{c, b\}, \wedge)$ are semilattices with the greatest element b but the four-element subset $\{a \wedge c, a, c, b\}$ is not a lattice in general since the operation \wedge need be neither commutative nor associative for the elements a, c .

Another nice characterization was established by K. Kaarli [2]:

Proposition 2. *An algebra A is arithmetical if and only if the following compatible function property holds: For any positive integer k and finite subsets $X, Y \subseteq A^k$ with $X \subseteq Y$, any function $f: X \rightarrow A$ compatible with $\text{Con } A$ has an extension from Y to A which is also compatible with $\text{Con } A$.*

A ternary function $d(x, y, z)$ satisfying $d(x, x, z) = z$ and $d(x, y, z) = x$ for $x \neq y$ is called a *discriminator*.

If A is an algebra and a, b are different elements of A , then we can uniquely define a discriminator d on $\{a, b\}$, and it is evidently a partial function compatible with $\text{Con } A$. We are going to show that in a finite case the extension property of discriminator on two non-disjoint two-element subsets characterizes the arithmeticity as well:

Theorem 2. *For a finite algebra A , the following conditions are equivalent:*

- (1) A is arithmetical;
- (2) for every three elements x, y, z of A and the discriminator d_1 on $\{x, y\}$ and d_2 on $\{y, z\}$ there exists a common extension $t: A^3 \rightarrow A$ of d_1, d_2 which is compatible with $\text{Con } A$.

Proof. (1) \Rightarrow (2): It is easy to show that a Pixley function compatible with $\text{Con } A$ is an extension of both d_1, d_2 .

(2) \Rightarrow (1): Similarly to the proof of Theorem 1, if $\langle x, z \rangle \in \Theta \cap (\Phi \circ \Psi)$ for some $\Theta, \Phi, \Psi \in \text{Con } A$, i.e. $\langle x, z \rangle \in \Theta, \langle x, y \rangle \in \Phi, \langle y, z \rangle \in \Psi$ for some $y \in A$, we conclude

$$\begin{aligned} \langle x, t(x, y, z) \rangle &= \langle d_1(x, y, x), t(x, y, z) \rangle = \langle t(x, y, x), t(x, y, z) \rangle \in \Theta, \\ \langle x, t(x, y, z) \rangle &= \langle d_1(x, y, y), t(x, y, z) \rangle = \langle t(x, y, y), t(x, y, z) \rangle \in \Psi, \\ \langle t(x, y, z), z \rangle &= \langle t(x, y, z), d_2(z, y, z) \rangle = \langle t(x, y, z), t(z, y, z) \rangle \in \Theta, \\ \langle t(x, y, z), z \rangle &= \langle t(x, y, z), d_2(y, y, z) \rangle = \langle t(x, y, z), t(y, y, z) \rangle \in \Phi, \end{aligned}$$

thus $\langle x, z \rangle \in (\Theta \cap \Psi) \circ (\Theta \cap \Phi)$, where $t(x, y, z)$ is the extension of d_1, d_2 which is compatible with $\text{Con } A$. By the above quoted result of [3], [4] concerning (*) we obtain (1). \square

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