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ON CONTINUOUS INTERVAL FUNCTIONS

LADISLAV MIŠÍK

Dedicated to Academician Štefan Schwarz on the occasion of his 70th birthday

In 1966, in the references of my paper ([7]) I announced that my paper entitled "Über stetige Intervalfunktionen" was in the press. But I had written only the first draft of the paper and as I did not find it interesting enough, I did not finish it for publication. Unfortunaly, I forgot to correct the announcement in the references of [7]. My first aim for writting the mentioned paper was the study of Darboux functions on the spaces of intervals. Some statements of the original draft are introduced in [7].

On January 29, 1981, M. Laczkovich sent me preprints [2] and [3] of his papers, which he had submitted to Acta Math. Acad. Sci. Hung. Simultaneously, he asked me for a reprint of my mentioned unpublished paper. I sent him the copy of the mentioned first draft with some comments concerning the proof of theorem 5 in the draft. In August, I received a letter from H. W. Pu in which he asked me for the full reference of my paper "Über stetige Intervalfunktionen". I sent him the English translation of the draft of the paper. In that translation, I made some corrections and suitable modifications in the proof of theorem 5, as I mentioned in my letter to M. Laczkovich. Simultaneously, I wrote a letter to H. W. Pu in which I mentioned that M. Laczkovich in [2] proves that any additive interval function defined on the space of all closed subintervals of a given closed interval I is uniformly continuous if it has a finite strong derivative on I ([2], theorem 3). I also added some comments concerning the relations between the theorem of M. Laczkovich and my theorem 5.

The facts introduced above were an impulse to my decision to adapt for publication the English translation which I had sent to H. W. Pu.

1. Let there be $n \geqq 1$ and E_n the euclidean n-dimensional space. Let $a_i < b_i$ for $i = 1, \ldots, n$. By the closed interval $I = \langle a_1, b_1; \ldots; a_n, b_n \rangle$ in E_n we understand the set $\{(x_1, \ldots, x_n) \in E_n\colon a_i \leqq x_i \leqq b_i \text{ for } i = 1, \ldots, n\}$. The boundary (the interior) of the interval I will be denoted by $\mathrm{Fr}(I)$ ($\mathrm{Int}(I)$). The intervals $I_1, \ldots, I_n$ will be called non-overlapping iff $\mathrm{Int}(I_i) \cap \mathrm{Int}(I_j) = \emptyset$ for all $i \neq j$, $i, j = 1, \ldots, n$, where $\emptyset$ is the empty set. The Lebesgue measure of a Lebesgue measurable set A in E_n will be denoted by $m(A)$. A set expressible as the sum of a finite number of closed

intervals in E_n will be termed a figure in E_n. If R is a figure in E_n or G is an open set in E_n, then $X(R)$ or $X(G)$ will denote the space of all closed intervals contained in R or in G, respectively, and $X_0(R)=X(R)\cup\{\emptyset\}$, $X_0(G)=X(G)\cup\{\emptyset\}$.

Lemma 1. *Let I and J be elements of $X(E_n)$. Let $\{s_k\}_{k=1}^{\infty}$ be a sequence of naturals defined as follows: $s_1=2$, $s_{k+1}=3s_k+2$ for $k=1,2,3,\ldots$. Then the difference $I-J$ is either the empty set or there exists a finite system $\{I_1,\ldots,I_t\}$ of non-overlapping closed intervals such that*

$$\bigcup_{i=1}^{t}\operatorname{Int}(I_i)\subset I-J\subset\bigcup_{i=1}^{t}I_i \quad \text{and} \quad t\leqq s_n.$$

Proof. If $I\subset J$, then $I-J=\emptyset$.

If $I\cap J\subset \operatorname{Fr}(I)$, then $\operatorname{Int}(I)\subset I-J\subset I$. Since $I-J=I-(I\cap J)$, we can assume that $J\subset I$ and $I-J\neq\emptyset$.

Let be $J\subset I$ and $I-J\neq\emptyset$. We prove the lemma for $n=1$. In this case for $I=\langle a,b\rangle\supset\langle c,d\rangle=J$ there holds: $(a,c)\cup(d,b)\subset I-J\subset\langle a,c\rangle\cup\langle d,b\rangle$ if $a<c<d<b$; $(a,c)\subset I-J\subset\langle a,c\rangle$ if $a<c<d=b$ and $(d,b)\subset I-J\subset\langle d,b\rangle$ if $a=c<d<b$. Lemma 1 holds for $n=1$.

Let Lemma 1 be true for n. Let I and $J\in X(E_{n+1})$, $J\subset I$ and $I-J\neq\emptyset$. Then there exist $\tilde Y$, $\tilde Y_2\in X(E_n)$ and $a\leqq c<d\leqq b$ such that $I=\tilde Y_1\times\langle a,b\rangle$ and $J=\tilde Y_2\times\langle c,d\rangle$. If $\tilde Y_1-\tilde Y_2\neq\emptyset$, then there exists a finite system $\{Y_1,\ldots,Y_s\}$, of non-overlapping closed intervals in E_n such that $\bigcup_{i=1}^{s}\operatorname{Int}(Y_i)\subset\tilde Y_1-\tilde Y_2\subset\bigcup_{i=1}^{s}Y_i$ and $s\leqq s_n$. If $\langle a,b\rangle-\langle c,d\rangle\neq\emptyset$, then there exists a system $\mathcal{T}$ of maximally two non-overlapping closed intervals in E_1 such $\cup\{\operatorname{Int}(T)\colon T\in\mathcal{T}\}\subset\langle a,b\rangle-\langle c,d\rangle\subset\cup\{T\colon T\in\mathcal{T}\}$. The following cases are possible: a) $\tilde Y_1-\tilde Y_2\neq\emptyset$, $\langle a,b\rangle-\langle c,d\rangle\neq\emptyset$, b) $\tilde Y_1-\tilde Y_2\neq\emptyset$, $\langle a,b\rangle=\langle c,d\rangle$ and c) $\tilde Y_1=\tilde Y_2$, $\langle a,b\rangle-\langle c,d\rangle\neq\emptyset$.

In case a) the system $\{Y_i\times T\colon i=1,\ldots,s,\ T\in\mathcal{T}\}\cup\{Y_i\times\langle c,d\rangle\colon i=1,\ldots,s\}\cup\{\tilde Y_2\times T\colon T\in\mathcal{T}\}$, in case b) the system $\{Y_i\times\langle c,d\rangle: i=1,\ldots,s\}$ and in case c) the system $\{\tilde Y_2\times T\colon T\in\mathcal{T}\}$ is the system $\{I_i\colon i=1,\ldots,t\}$ mentioned in lemma 1.

Let be $\varrho(A,B)=m(A\triangle B)$, where $A\triangle B$ is the symmetric difference of A and B for each A, $B\in X_0(E_n)$. Then $(X_0(E_n),\varrho)$ and $(X(E_n),\varrho)$ are metric spaces.

Lemma 2. *Let $I=\langle a_1,b_1;\ \ldots;\ a_n,b_n\rangle$, $0<\varepsilon<m(I)$, $K=\max\{b_i-a_i\colon i=1,\ldots,n\}$, $\delta\geqq\dfrac{K\varepsilon}{m(I)-\varepsilon}$ and $Y=\langle a_1-\delta,\ b_1+\delta;\ldots;\ a_n-\delta,\ b_n+\delta\rangle$. Then $\cup\overline{O_I(\varepsilon)}\subset Y$, where $\overline{O_I(\varepsilon)}$ is the closure in $(X(E_n),\varrho)$ of the ε-neighbourhood of I.*

Proof. Let $J=\langle c_1,d_1;\ldots;c_n,d_n\rangle\in\overline{O_I(\varepsilon)}$. Let there exist a $j\in\{1,\ldots,n\}$ for which either $c_j<a_j-\delta$ or $b_j+\delta<d_j$. Let $e_i=\max(a_i,c_i)$ and $f_i=\min(b_i,d_i)$ for $i=1,\ldots,n$. Then

$$\varrho(I, J) = m(I - (I \cap J)) + m(J - (I \cap J)) =$$

$$= \prod_{i=1}^{n} (b_i - a_i) - \prod_{i=1}^{n} (f_i - e_i) + \prod_{i=1}^{n} (d_i - c_i) - \prod_{i=1}^{n} (f_i - e_i) \geqq$$

$$\geqq \prod_{i=1}^{n} (f_i - e_i) \frac{b_i - f_i + e_i - a_i + d_i - f_i + e_i - c_i}{f_i - e_i} >$$

$$> \prod_{i=1}^{n} (f_i - e_i) \frac{\delta}{f_i - e_i} \geqq \prod_{i=1}^{n} (f_i - e_i) \frac{\delta}{K} \geqq \frac{m(I \cap J)\varepsilon}{m(I) - \varepsilon} \geqq \varepsilon .$$

However, this is a contradiction and therefore $J \subset Y$ if $J \in \overline{O_I(\varepsilon)}$.

Theorem 1. *The space $(X_0(E_n), \varrho)$ is a metric connected space and $(X(E_n), \varrho)$ is a metric locally compact connected space.*

Proof. Let I be in $X(E_n)$ and $0 < \varepsilon < m(I)$. According to lemma 2, there exists a $Y \in X(E_n)$ such that $\cup \overline{O_I(\varepsilon)} \subset Y$. Let $\{I_k\}_{k=1}^{\infty}$ be a sequence of elements in $\overline{O_I(\varepsilon)}$ and let $I_k = \langle a_{1,k}, b_{1,k}; \ldots; a_{n,k}, b_{n,k} \rangle$ for $k = 1, 2, 3, \ldots$. Since $\bigcup_{k=1}^{\infty} I_k \subset Y$, the sequences $\{a_{i,k}\}_{k=1}^{\infty}$ and $\{b_{i,k}\}_{k=1}^{\infty}$ are bounded for $i = 1, \ldots, n$. Therefore there exists a sequence $\{k_\nu\}_{\nu=1}^{\infty}$ such that all sequences $\{a_{i,k_\nu}\}_{\nu=1}^{\infty}$ and $\{b_{i,k_\nu}\}_{\nu=1}^{\infty}$ for $i = 1, \ldots, n$ are convergent. Let $a_i = \lim_{\nu \to \infty} a_{i,k_\nu}$ and $b_i = \lim_{\nu \to \infty} b_{i,k_\nu}$ for $i = 1, \ldots, n$. Since $0 < \varepsilon < m(I)$, $\varrho(I, I_k) \leqq \varepsilon$ for $k = 1, 2, 3, \ldots$ and ϱ is continuous, we have $a_i < b_i$ for $i = 1, \ldots, n$. Let $J = \langle a_1, b_1; \ldots; a_n, b_n \rangle$. Then $\lim_{\nu \to \infty} \varrho(J, I_{k_\nu}) = 0$ and $\varrho(I, J) \leqq \varepsilon$. Thus $J \in \overline{O_I(\varepsilon)}$ and we have proved that $\overline{O_I(\varepsilon)}$ is a compact set in $X(E_n)$. Therefore $(X(E_n), \varrho)$ is a locally compact space.

Assume that $X(E_n) = O_1 \cup O_2$, where O_1 and O_2 are two nonempty open disjoint sets. Then there exist $I, J \in X(E_n)$ such that $I = \langle a_1, b_1; \ldots; a_n, b_n \rangle \in O_1$ and $J = \langle c_1, d_1; \ldots; c_n, d_n \rangle \in O_2$. Let $J_t = \langle ta_1 + (1-t)c_1, tb_1 + (1-t)d_1; \ldots; ta_n + (1-t)c_n, tb_n + (1-t)d_n \rangle$ for $t \in \langle 0, 1 \rangle$. The map φ: $\langle 0, 1 \rangle \to X(E_n)$ defined as follows: $\varphi(t) = J_t$ for each $t \in \langle 0, 1 \rangle$ is continuous. Thus $\varphi^{-1}(O_1)$ and $\varphi^{-1}(O_2)$ are two non-empty disjoint open sets in $\langle 0, 1 \rangle$ for which $\langle 0, 1 \rangle = \varphi^{-1}(O_1) \cup \varphi^{-1}(O_2)$. However, this is a contradiction, since $\langle 0, 1 \rangle$ is a connected set.

Since $X(E_n)$ is a connected set in $X_0(E_n)$, $\{\emptyset\}$ is not open in $X_0(E_n)$ and $X_0(E_n) - \{\emptyset\}$ is open in $X_\emptyset(E_n)$, the space $X_0(E_n)$ is connected. Analogously we can prove the following theorem:

Theorem 2. *Let R be a figure in E_n. Then $(X_\emptyset(R), \varrho)$ is a compact metric space. If $I \in X(E_n)$, then $(X_0(I), \varrho)$ is a compact connected metric space.*

If R is a figure in E_n, the space $(X_0(R), \varrho)$ is a one-point compactification of $(X(R), \varrho)$.

Theorem 3. *The space $(X_0(E_n), \varrho)$ is a complete metric space in which the base $\mathcal{B}=\{O_I(\varepsilon): I\in X(E_n),\ 0<\varepsilon\leqq m(I)\}\cup\{O_\emptyset(\varepsilon): \varepsilon>0\}$ has the following two properties:*

(1) Let $I\in X_0(E_n)$ and U be an open set for which $I\in U$. Then there exists a $B\in\mathcal{B}$ such that $B\subset U$ and $I\in\bar{B}-B$.

(2) Let $B\in\mathcal{B}$ and let be $B=A_1\cup A_2$, where $A_1\neq\emptyset$, $A_2\neq\emptyset$, $A_1\cap A_2=\emptyset$ and $\bar{C}\cap B\subset A_1$ or $\bar{C}\cap B\subset A_2$ if $C\in\mathcal{B}$ and $C\subset A_1$ or $C\subset A_2$, respectively. Then the sets $A_1'\cap A_2$ and $A_1\cap A_2'$ are non-empty. By A_1' or A_2' we denote the set of all points of accumulation of A_1 or A_2, respectively.

Proof. Let $\{I_k\}_{k=1}^{\infty}$ be a Cauchy sequence in $(X_0(E_n), \varrho)$. If there exists a sequence $\{k_s\}_{s=1}^{\infty}$ such that $\lim_{s\to\infty} m(I_{k_s})=0$, then $\lim_{k\to\infty}\varrho(I_k, \emptyset)=0$ and the sequence $\{I_k\}_{k=1}^{\infty}$ converges to $\emptyset$ in $(X_0(E_n), \varrho)$.

If there does not exist a sequence $\{k_s\}_{s=1}^{\infty}$ such that $\lim_{s\to\infty} m(I_{k_s})=0$, then there exists a positive number K and a natural number N such that $K<m(I_N)$ and $\varrho(I_p, I_q)<\frac{K}{2}$ for all $p, q\geqq N$. Then $I_p\in O_{I_N}\left(\frac{K}{2}\right)$ for all $p\geqq N$. According to lemma 2, there exists a $J\in X(E_n)$ such that $\overline{\cup O_{I_N}\left(\frac{K}{2}\right)}\subset J$. It is easy to prove that $\{I_k\}_{k=1}^{\infty}$ converges to some $I\in X(J)$.

Now let $I\in X_0(E_n)$, U an open set in $X_0(E_n)$ and $I\in U$. Then there exists a $\delta>0$ such that $O_I(\delta)\subset U$. If $I=\emptyset$, we choose a $J\in O_I(\delta)\cap X(E_n)$ such that $m(J)<\frac{\delta}{2}$. Then $B=O_J(m(J))\in\mathcal{B}$, $B\subset U$ and $I=\emptyset\in\bar{B}-B$. Let be $I\in X(E_n)$. Then we choose a $J\in X(E_n)$ such that $I\subset \mathrm{Int}(J)$ and $\varrho(I, J)<\frac{\delta}{2}$. Then for $B=O_J(\varrho(I, J))$ we have: $B\in\mathcal{B}$, $B\subset U$ and $I\in\bar{B}-B$.

Let $B\in\mathcal{B}$ and $B=A_1\cup A_2$, where A_1 and A_2 are two nonempty disjoint sets with the following property: $\bar{C}\cap B\subset A_1$ or $\bar{C}\cap B\subset A_2$ if $C\in\mathcal{B}$ and $C\subset A_1$ or $C\subset A_2$, respectively.

First let $B=O_I(\varepsilon)$, where $I\in X(E_n)$ and $0<\varepsilon\leqq m(I)$. We shall prove the following proposition: There exist two closed intervals $J_1=\langle u_{1,1}, v_{1,1};\ \ldots;\ u_{n,1}, v_{n,1}\rangle$ and $J_2=\langle u_{1,2}, v_{1,2};\ \ldots;\ u_{n,2}, v_{n,2}\rangle$ in B and a natural $i\in\{1, \ldots, n\}$ such that $J_1\in A_1$, $J_2\in A_2$, either $u_{i,1}\neq u_{i,2}$, $u_{j,1}=u_{j,2}$ for $j=1, \ldots, i-1, i+1, \ldots, n$ and $v_{k,1}=v_{k,2}$ for $k=1, \ldots, n$ or $u_{k,1}=u_{k,2}$ for $k=1, \ldots, n$ and $v_{i,1}\neq v_{i,2}$ and $v_{j,1}=v_{j,2}$ for $j=1, \ldots, i-1, i+1, \ldots, n$ and $\bar{J}_t=\langle u_{1,1}, v_{1,1};\ \ldots;\ u_{i-1,1}, v_{i-1,1};\ tu_{i,1}+(1-t)u_{i,2}, v_{i,1};\ u_{i+1,1}, v_{i+1,1};\ \ldots;\ u_{n,1}, v_{n,1}\rangle\in B$ for all $t\in\langle 0, 1\rangle$ in the first case or $\tilde{Y}_t = \langle u_{1,1}, v_{1,1}; \ldots; u_{i-1,1}, v_{i-1,1}; u_{i,1}, tv_{i,1} + (1-t)\, v_{i,2}; u_{i+1,1}, v_{i+1,1}; \ldots; u_{n,1}, v_{n,1} \rangle \in B$ for all $t \in \langle 0, 1 \rangle$ in the second case.

We can assume that $I \in A_1$. In the case of $I \in A_2$ we proceed similarly. Let $I = \langle a_1, b_1; \ldots; a_n, b_n \rangle \in A_1$ and $J = \langle c_1, d_1; \ldots; c_n, d_n \rangle \in A_2$. Since $\varrho(I, J) = m(I \cup J) - m(I \cap J) < \varepsilon \leqq m(I)$, there must be $\mathrm{Int}(I \cap J) \neq \emptyset$. Thus $Y = I \cap J \in X(E_n)$. Since $\varrho(Y, I) \leqq \varrho(I, J) < \varepsilon$, Y is in B. Therefore either $Y \in A_1$ or $Y \in A_2$.

Let $Y = \langle e_1, f_1; \ldots; e_n, f_n \rangle \in A_1$. Then $e_i = \max(a_i, c_i)$ and $f_i = \min(b_i, d_i)$ for $i = 1, \ldots, n$. If $e_1 = \max(a_1, c_1) = a_1 > c_1$, then the intervals $I_t = \langle tc_1 + (1-t)e_1, f_1; e_2, f_2; \ldots; e_n, f_n \rangle$ belong to B for all $t \in \langle 0, 1 \rangle$, because $Y \subset I \cap I_t \subset I \cup I_t \subset I \cup J$ and $\varrho(I, I_t) \leqq \varrho(I, J) < \varepsilon$ for all $t \in \langle 0, 1 \rangle$. If $I_1 \in A_2$, the proposition is proved.

If either $I_1 \in A_1$ or $e_1 = c_1$ and if $f_1 = \min(b_1, d_1) = b_1 < d_1$, we consider the following system of intervals $\tilde{I}_t = \langle c_1, td_1 + (1-t)f_1; e_2, f_2; \ldots; e_n, f_n \rangle$ for $t \in \langle 0, 1 \rangle$. Then $\tilde{I}_t \in B$ for all $t \in \langle 0, 1 \rangle$, because $Y \subset I \cap \tilde{I}_t \subset I \cup \tilde{I}_t \subset I \cup J$ and $\varrho(I, \tilde{I}_t) \leqq \varrho(I, J) < \varepsilon$. If $\tilde{I}_1 \in A_2$, the proposition is proved.

If $e_1 = c_1$ or $I_1 \in A_1$, $f_1 = d_1$ or $\tilde{I}_1 \in A_1$ and $e_2 = \max(e_2, c_2) = a_2 > c_2$, then we take the system of intervals $Y_t = \langle c_1, d_1; tc_2 + (1-t)e_2, f_2; e_3, f_3; \ldots; e_n, f_n \rangle$ for $t \in \langle 0, 1 \rangle$ in consideration and we proceed as we proceed in the case of the systems $\{I_t: t \in \langle 0, 1 \rangle\}$ and $\{\tilde{I}_t: t \in \langle 0, 1 \rangle\}$. Since $Y \in A_1$ and $J \in A_2$, we get by induction the existence of some $i \in \{1, \ldots, n\}$ such that either $T_1 = \langle c_1, d_1; \ldots; c_{i-1}, d_{i-1}; e_i, f_i; \ldots; e_n, f_n \rangle \in A_1$ and $\tilde{T}_1 = \langle c_1, d_1; \ldots; c_{i-1}, d_{i-1}; c_i, f_i; e_{i+1}, f_{i+1}; \ldots; e_n, f_n \rangle \in A_2$ or $T_2 = \langle c_1, d_1; \ldots; c_{i-1}, d_{i-1}; c_i, f_i; e_{i+1}, f_{i+1}; \ldots; e_n, f_n \rangle \in A_1$ and $\tilde{T}_2 = \langle c_1, d_1; \ldots; c_{i-1}, d_{i-1}; c_i, d_i; e_{i+1}, f_{i+1}; \ldots; e_n, f_n \rangle \in A_2$.

If $Y \in A_2$, it is easy to see that we must proceed similarly to prove the proposition.

Now let $\{\tilde{J}_t: t \in \langle 0, 1 \rangle\}$ or $\{\tilde{Y}_t: t \in \langle 0, 1 \rangle\}$ be the system mentioned in the proposition in the first case, or in the second case, respectively. We shall deal only with the system $\{\tilde{J}_t: t \in \langle 0, 1 \rangle\}$ which corresponds to the first case. The second case can be treated similarly.

The system $\{\tilde{J}_t: t \in \langle 0, 1 \rangle\}$ is a compact subset of $O_I(\varepsilon)$ and therefore $\delta = \inf\{\varrho(\tilde{J}_t, Y): \; Y \in \overline{O_I(\varepsilon)} - O_I(\varepsilon), \; t \in \langle 0, 1 \rangle\} > 0$. Let $\delta_t = \inf\{\varrho(\tilde{J}_t, Y): Y \in \overline{O_I(\varepsilon)} - O_I(\varepsilon)\}$ for $t \in \langle 0, 1 \rangle$. Then $0 < \delta \leqq \delta_t$ for all $t \in \langle 0, 1 \rangle$. If $\varepsilon = m(I)$, then $\emptyset \in \overline{O_I(\varepsilon)} - O_I(\varepsilon)$ and therefore $\delta_t \leqq \varrho(\tilde{J}_t, \emptyset) = m(\tilde{J}_t)$. If $0 < \varepsilon < m(I)$, then $m(I - (I \cap J)) \leqq \varrho(I, J) < \varepsilon < m(I)$. Therefore there exists a closed interval T such that $T \subset I \cap J$ and $m(I - T) = \varepsilon$. But then $T \in \overline{O_I(\varepsilon)} - O_I(\varepsilon)$ and $\delta_t \leqq \varrho(\tilde{J}_t, T) = m(\tilde{J}_t - T) < m(\tilde{J}_t)$.

For each $t \in \langle 0, 1 \rangle$ there exists an $\varepsilon_t \geqq 0$ such that either $O_{\tilde{J}_t}(\varepsilon_t) \subset A_1$ and $(O_{\tilde{J}_t}(\varepsilon') \cap A_2) \cup (O_{\tilde{J}_t})\varepsilon') - O_I(\varepsilon)) \neq \emptyset$ or $O_{\tilde{J}_t}(\varepsilon_t) \subset A_2$ and $(O_{\tilde{J}_t}(\varepsilon') \cap A_1 \cup (O_{\tilde{J}_t}(\varepsilon') - O_I(\varepsilon))$

$\neq \emptyset$ for all $\varepsilon' > \varepsilon_t$ if $\tilde{J}_t \in A_1$ or $\tilde{J}_t \in A_2$, respectively. If $\varepsilon_t = 0$, we put $O_{J_t}(0) = \emptyset$. If $0 < \varepsilon_t < \delta_t$, then there exists an interval $J \in \overline{O_{\tilde{J}_t}(\varepsilon_t)}$ such that either $J \in A_1 \cap A_2'$ or $J \in A_1' \cap A_2$ when either $\tilde{J}_t \in A_1$ or $\tilde{J}_t \in A_2$, respectively. This is a consequence of the compactness of $\overline{O_{\tilde{J}_t}(\varepsilon_t)}$. If $\varepsilon_t = 0$ and if $\tilde{J}_t \in A_1$ or $\tilde{J}_t \in A_2$, then either $\tilde{J}_t \in A_1 \cap A_2'$ or $\tilde{J}_t \in A_1' \cap A_2$, respectively.

Let $\alpha = \inf \{t \in \langle 0, 1 \rangle : \varepsilon_u = \delta_u$ for all $u \in (t, 1\rangle\}$ and $\beta = \sup \{t \in \langle 0, 1 \rangle : \varepsilon_u = \delta_u$ for all $u \in \langle 0, t)\}$. From the properties of A_1 and A_2, from the definition of ε_t and from $\tilde{J}_1 \in A_1$ and $\tilde{J}_0 \in A_2$ we conclude that $0 \leqq \beta < \beta + \delta \leqq \alpha \leqq 1$. But then there exist two numbers r and s such that $\beta \leqq r < s \leqq \alpha$, $\tilde{J}_r \in A_2$, $\tilde{J}_s \in A_1$, $\varepsilon_r < \delta_r$ and $\varepsilon_s < \delta_s$. However, from the consideration in the preceding paragraph it follows that $A_1' \cap A_2 \neq \emptyset$ and $A_1 \cap A_2' \neq \emptyset$.

Now let $B = O_\emptyset(\varepsilon)$, where $\varepsilon > 0$. There can neither $A_1 = \{\emptyset\}$ nor $A_2 = \{\emptyset\}$. Then there exist two intervals I and J such that $I = \langle a_1, b_1; \ldots; a_n, b_n \rangle \in A_1$ and $J = \langle c_1, d_1; \ldots; c_n, d_n \rangle \in A_2$. Two cases are possible: either a) $\{\langle ta_1, tb_1; \ldots; ta_n, tb_n \rangle : t \in (0, 1\rangle\} \subset A_1$ or b) $\{\langle ta_1, tb_1; \ldots; ta_n, tb_n \rangle : t \in (0, 1\rangle\} - A_1 \neq \emptyset$. We shall prove that in both cases there exist an interval $Y \in O_\emptyset(\varepsilon)$ and $\delta > 0$ such that $O_Y(\delta) \subset O_\emptyset(\varepsilon)$ and $O_Y(\delta) \cap A_1 \neq \emptyset$ and $O_Y(\delta) \cap A_2 \neq \emptyset$.

Let there be $\{\langle ta_1, tb_1; \ldots; ta_n, tb_n \rangle : t \in (0, 1\rangle\} \subset A_1$. Then there exists a $t > 0$ such that $t(b_i - a_i) \leqq d_i - c_i$ for $i = 1, \ldots, n$. Let $Y_s = \langle sta_1 + (1-s)c_1, stb_1 + (1-s)d_1; \ldots; sta_n + (1-s)c_n, stb_n + (1-s)d_n \rangle$ for $s \in \langle 0, 1 \rangle$ and $\alpha = \inf \{s \in \langle 0, 1 \rangle : Y_s \in A_1\}$. If $\alpha = 1$, then $Y_\alpha \in A_1 \cap A_2'$, if $\alpha = 0$, then $Y_\alpha \in A_1' \cap A_2$ and if $0 < \alpha < 1$, then either $Y_\alpha \in A_1 \cap A_2'$ or $Y_\alpha \in A_1' \cap A_2$. Since $0 < \alpha t(b_i - a_i) + (1-\alpha)(d_i - c_i) \leqq d_i - c_i$ for $i = 1, \ldots, n$, there holds: $\varrho(\emptyset, Y_\alpha) = m(Y_\alpha) = \prod_{i=1}^{n} (\alpha t(b_i - a_i) + (1-\alpha)(d_i - c_i)) \leqq \prod_{i=1}^{n} (d_i - c_i) = m(J) < \varepsilon$. Therefore $Y_\alpha \in O_\emptyset(\varepsilon)$. There exists a $\delta > 0$ such that $O_{Y_\alpha}(\delta) \subset O_\emptyset(\varepsilon)$ and $O_{Y_\alpha}(\delta) \cap A_1 \neq \emptyset$, $O_{Y_\alpha}(\delta) \cap A_2 \neq \emptyset$.

Let there $\{\langle ta_1, tb_1; \ldots; ta_n, tb_n \rangle : t \in (0, 1\rangle\} - A_1 \neq \emptyset$. Let $\beta = \sup \{t \in (0, 1\rangle : \langle ta_1, tb_1; \ldots; ta_n, tb_n \rangle \in A_2\}$. Then $0 < \beta \leqq 1$. If $\beta = 1$, then $Y = \langle a_1, b_1; \ldots; a_n, b_n \rangle \in A_1 \cap A_2'$ and if $0 < \beta < 1$, then either $Y = \langle \beta a_1, \beta b_1; \ldots; \beta a_n, \beta b_n \rangle \in A_1 \cap A_2'$ or $Y = \langle \beta a_1, \beta b_1; \ldots; \beta a_n, \beta b_n \rangle \in A_1' \cap A_2$. Since $\varrho(\emptyset, Y) = \beta m(I) < \varepsilon$, Y is in $O_\emptyset(\varepsilon)$. There also exists a positive number δ such that $O_Y(\delta) \subset O_\emptyset(\varepsilon)$ and $O_Y(\delta) \cap A_1 \neq \emptyset$ and $O_Y(\delta) \cap A_2 \neq \emptyset$.

Let $Y \in O_\emptyset(\varepsilon)$ and $\delta > 0$ such that $O_Y(\delta) \subset O_\emptyset(\varepsilon)$, $O_Y(\delta) \cap A_1 \neq \emptyset$ and $O_Y(\delta) \cap A_2 \neq \emptyset$. Let $\tilde{A}_1 = O_Y(\delta) \cap A_1$ and $\tilde{A}_2 = O_Y(\delta) \cap A_2$. Then $O_Y(\delta) = \tilde{A}_1 \cup \tilde{A}_2$, $\tilde{A}_1 \cap \tilde{A}_2 = \emptyset$ and $\tilde{A}_1 \neq \emptyset$, $\tilde{A}_2 \neq \emptyset$. Let $C \in \mathscr{B}$ and either $C \subset \tilde{A}_1$ or $C \subset \tilde{A}_2$. Therefore either $\bar{C} \cap O_\emptyset(\varepsilon) \subset A_1$ or $\bar{C} \cap O_\emptyset(\varepsilon) \subset A_2$ and thus $\bar{C} \cap O_Y(\delta) \subset \bar{C} \cap O_\emptyset(\varepsilon) \cap O_Y(\delta) \subset A_1 \cap O_Y(\delta) = \tilde{A}_1$, or $\bar{C} \cap O_Y(\delta) \subset \bar{C} \cap O_\emptyset(\varepsilon) \cap O_Y(\delta) \subset A_2 \cap O_Y(\delta) = \tilde{A}_2$ if either $C \subset \tilde{A}_1$ or $C \subset \tilde{A}_2$, respectively. But then there holds: $\tilde{A}_1 \cap \tilde{A}_2' \neq \emptyset$ and $\tilde{A}_1' \cap \tilde{A}_2 \neq \emptyset$ and therefore $A_1 \cap A_2' \neq \emptyset$ and $A_1' \cap A_2 \neq \emptyset$.

Thus the property (2) for the base $\mathscr{B}$ is proved.

We remark that the properties (1) and (2) are important in the study of the $\mathscr{B}$-Darboux Baire 1 functions (see [4] and [5]).

We note that also the base $\mathscr{B}$ in $X(E_n)$ of all sets $U(I, \varepsilon)=\{J\in X(E_n):(a_1+\varepsilon, b_1-\varepsilon; \ldots; a_n+\varepsilon, b_n-\varepsilon)\subset J\subset(a_1-\varepsilon, b_1+\varepsilon; \ldots; a_n-\varepsilon, b_n+\varepsilon)\}$ for $I=\langle a_1, b_1; \ldots; a_n, b_n\rangle\in X(E_n)$ and $0<\varepsilon<\frac{1}{2}\min(b_1-a_1, \ldots, b_n-a_n)$ satisfies the properties (1) and (2).

2. There are several definitions of the continuity of interval functions.

The first is as follows: *An interval function f is continuous on a figure R in E_n (or on an open set G in E_n) iff for each $x\in R$ ($x\in G$) and $\varepsilon>0$ there exists a $\delta>0$ such that $|f(I)|<\varepsilon$ holds whenever $I\in X(R)$ ($I\in X(G)$), $x\in I$ and $m(I)<\delta$* ([10]). If an interval function is continuous on a figure R (or on G) in this sense, we shall say that f is pointwise continuous on R (on G).

The second is the following: *An interval function f is continuous on a figure R (on an open set G) iff for each $\varepsilon>0$ there exists $\delta>0$ such that $|f(I)|<\varepsilon$ whenever $I\in X(R)$ ($I\in X(G)$) and $m(I)<\delta$* ([9]). If f is continuous on R (on G) in this sense, we shall say that f is uniformly continuous on R (on G). However, this continuity is usually, also by M. Laczkovich, called simply continuity.

In [8] C. J. Neugebauer gives the following definition of the continuity of an interval function: *An interval function f is continuous on a figure R (on an open set G) iff for each $\varepsilon>0$ and for each $I\in X(R)$ ($I\in X(G)$) there exists a $\delta>0$ such that $|f(I)-f(J)|<\varepsilon$ holds whenever $J\in X(R)$ ($J\in X(G)$) and $\varrho(I, J)<\delta$.* Thus f is continuous on R (on G) in this sense iff f is continuous on $(X(R), \varrho)$ (on $(X(G), \varrho)$). This continuity is introduced also in my paper [6] in theorem 3. In such a case of the continuity we shall say that f is metrically continuous on R (on G).

An interval function f is called additive on a figure R (on an open set G) iff $f(I\cup J)=f(I)+f(J)$ for each nonoverlapping interval I and J of $X(R)$ ($I, J\in X(G)$) for which $I\cup J$ is an interval. Let f be an additive interval function on R (on G). Then the function f: $X_0(R)\to(-\infty, \infty)$ (f: $X_0(G)\to(-\infty, \infty)$ defined as follows: $f(I)=f(I)$ for each $I\in X(R)$ ($I\in X(G)$) and $f(\emptyset)=0$, is an additive extension of f to $X_0(R)$ (to $X_0(G)$).

Theorem 4. *Let f be an additive interval function on a figure R (on an open set G). Then the following properties are equivalent:*

a) *f is uniformly continuous on R (on G)*

b) *The additive extension f is continuous at $\emptyset$ in the metric space $(X_0(R), \varrho)$ $(X_0(G), \varrho)$,*

c) *f is metrically continuous on R (on G) and there holds* b).

Proof. The equivalence of a) and b) is evident. The theorem will be proved if we prove that b) implies c).

Let f be continuous at $\emptyset$ in $(X_0(R), \varrho)$ ($(X_0(G), \varrho)$). Let s_n be the number introduced in Lemma 1. Let $\varepsilon>0$ Let $\delta>0$ such that $|f(I)|<\frac{\varepsilon}{2s_n}$ whenever $I\in X(R)$ ($I\in X(G)$) and $m(I)<\delta$. Let $I\in X(R)$ ($I\in X(G)$). According to Lemma 1 for each $J\in X(R)$ ($J\in X(G)$) for which $\varrho(I, J)<\delta$ there exists a finite system $\{J_1, \ldots, J_s\}$ of non-overlapping closed intervals in $X(R)$ (in $X(G)$) such that $\bigcup_{i=1}^{p}\mathrm{Int}(J_i)\subset I-J\subset\bigcup_{i=1}^{p}J_i$, $\bigcup_{i=p+1}^{s}\mathrm{Int}(J_i)\subset J-I\subset\bigcup_{i=p+1}^{s}J_i$, $0\leqq p\leqq s_n$, $p\leqq s\leqq 2s_n$, where $\bigcup_{i=1}^{p}\mathrm{Int}(J_i)$ and $\bigcup_{i=1}^{p}J_i$ for $p=0$ and $\bigcup_{i=p+1}^{s}\mathrm{Int}(J_i)$ and $\bigcup_{i=p+1}^{s}J_i$ for $s=p$ we shall put equal to $\emptyset$. If $I\cap J$ is an interval, then $f(I)=f(I\cap J)+\sum_{i=1}^{p}f(J_i)$ and $f(J)=f(I\cap J)+\sum_{i=p+1}^{s}f(J_i)$ and if $I\cap J$ is not an interval, then $f(I)=\sum_{i=1}^{p}f(J_i)$ and $f(J)=\sum_{i=p+1}^{s}f(J_i)$. Therefore $|f(I)-f(J)|\leqq\sum_{i=1}^{s}|f(J_i)|<\varepsilon$. We put here $\sum_{i=1}^{p}f(J_i)=0$ if $p=0$ and $\sum_{i=p+1}^{s}f(J_i)=0$ if $s=p$.

If I is a closed interval, then according to theorem 2, the metric space $(X_0(I), \varrho)$ has some properties as interval. Any continuous function on $(X_0(I), \varrho)$ is uniformly continuous on $X_0(I)$ and has a maximum and a minimum on $X_0(I)$. All continuous functions on $(X_0(I), \varrho)$ form a separable Banach space with the norm $\|f\|=\max\{|f(J)|: J\in X_0(I)\}$, where f is continuous function on $X_0(I)$ ([1], p. 397). Any continuous function on $(X_0(I), \varrho)$ has the Darboux property according to the base $\mathscr{B}$ which is introduced in theorem 3, thus: for each $B\in\mathscr{B}$ each Y, $J\in\bar{B}$ each c such that $f(J)<c<f(I)$ there exists $T\in B$ such that $f(T)=c$.

It is well known that any real function of a real variable is continuous on an interval J if it has a finite derivative at any point of J. We shall prove that any additive interval function which has at any point of an open set (of a figure) a finite strong derivative is metrically continuous. We recall the definition of the strong derivative. Let f be an interval function on an open set G (on a figure R). Then a number a is a strong derivative of f in $X\in G$ ($X\in R$) iff for each $\varepsilon>0$ there exists $\delta>0$ such that $\left|\frac{f(I)}{m(I)}-a\right|<\varepsilon$ for each $I\in X(G)$ ($I\in X(R)$), $X\in I$ and $d(I)<\delta$, where $d(I)$ is the diameter of I.

Lemma 3. *Let $J=\langle a_1, b_1; \ldots; a_n, b_n\rangle$ be an interval, $X=(x_1, \ldots, x_n)\in J$, let f be an additive interval function on J and K a positive number. If $|f(I)|\leqq Km(I)$ whenever I is in $X(J)$, $X\in I$ then there holds: $|f(I)|\leqq 2^nKm(J)$ for each $I\in X(J)$.*

Proof. Let $I=\langle c_1, d_1, \ldots; c_n, d_n\rangle\in X(J)$ and $N_I=\{i\in\{1, \ldots, n\}: c_i\leqq x_i\leqq d_i\}$. We shall prove that $|f(I)|\leqq 2^jKm(J)$ if the cardinality of N_I is $n-j$.

If the cardinality of N_I is n, then $X\in I$ and therefore $|f(I)|\leqq Km(I)\leqq Km(J)$.

We shall assume that $|f(Y)| \leqq 2^j K m(J)$ for each $Y \in X(J)$ such that the cardinality of N_Y is $n-j$. Let the cardinality of N_I be $n-(j+1)$. Then there exists an $i \in \{1, \ldots, n\} - N_I$. There must hold either $x_i < c_i$ or $d_i < x_i$. If $x_i < c_i$, we put $Y_1 = \langle c_1, d_1; \ldots; c_{i-1}, d_{i-1}; a_i, c_i; c_{i+1}, d_{i+1}; \ldots; c_n, d_n \rangle$ and $Y_2 = \langle c_1, d_1; \ldots; c_{i-1}, d_{i-1}; a_i, d_i; c_{i+1}, d_{i+1}; \ldots; c_n, d_n \rangle$. Then $Y_2 = Y_1 \cup I$ and Y_1 and I are nonoverlapping closed intervals. Therefore $f(Y_2) = f(Y_1) + f(I)$ and $|f(I)| = |f(Y_2) - f(Y_1)| \leqq |f(Y_2)| + |f(Y_1)| \leqq 2^{j+1} K m(J)$ because $N_{Y_1} = N_{Y_2} = N_I \cup \{i\}$. We treat the case $d_i < x_i$ analogously.

Theorem 5. *Let f be an additive interval function on an open set G (on a figure R) which has a finite strong derivative at any point of G (of R). Then f is metrically continuous on G (on R).*

Proof. Let $I = \langle a_1, b_1; \ldots; a_n, b_n \rangle \in X(G)$ and $\varepsilon > 0$. For every $X \in \mathrm{Fr}(I)$ there exists a positive number $\delta(X)$ such that $(\mathrm{D}f(X) - \frac{1}{2})\ m(J) < f(J) < (\mathrm{D}f(X) + \frac{1}{2})\ m(J)$ if $J \in X(G)$, $X \in J$ and the diameter $d(J)$ of J is less than $\delta(X)$, where $\mathrm{D}f(X)$ is the strong derivative of f at X.

Let $\mathscr{F}$ be the system of all triple (N_1, N_2, N_3) such that $\{N_1, N_2, N_3\}$ is a disjoint decomposition of $\{1, \ldots, n\}$ such that $N_1 \cup N_2 \neq \emptyset$. Let $(N_1, N_2, N_3) \in \mathscr{F}$ and let $F_{(N_1, N_2, N_3)}$ be the set $\{(x_1, \ldots, x_n)$: for each $i \in N_1$ there is $x_i = a_i$ for each $i \in N_2$ there is $x_i = b_i$ and for each $i \in N_3$ there holds $a_i \leqq x_i \leqq b_i\}$. The cardinality of the system $\mathscr{F}$ is $p_n = \sum_{k=0}^{n-1} \binom{n}{k} 2^{n-k}$.

Let $J(X)$ be an n-dimensional cube in $X(G)$ with the centre X and with the diameter less than $\delta(X)$. Since $\mathrm{Fr}(I) = \cup\{F_{(N_1, N_2, N_3)}: (N_1, N_2, N_3) \in \mathscr{F}\}$ and since each set $F_{(N_1, N_2, N_3)}$ for $(N_1, N_2, N_3) \in \mathscr{F}$ is compact, there exists a finite system $\{X_1, \ldots, X_s\}$ of points of $\mathrm{Fr}(I)$ such that $F_{(N_1, N_2, N_3)} \subset \cup\{\mathrm{Int}(J(X_i)): X_i \in F_{(N_1, N_2, N_3)}\}$ for each $(N_1, N_2, N_3) \in \mathscr{F}$. Let $K = \max\{\max(|\mathrm{D}f(X_i) - \frac{1}{2}|, |\mathrm{D}f(X_i) + \frac{1}{2}|)$: $i = 1, \ldots, s\}$ and l be the minimum of the lengths of the cubes $J(X_i)$ for $i = 1, \ldots, s$.

Let $\mathscr{S}_0$ be the system $\{J: J$ is a closed interval for which there exists a finite set $K_J \subset \{1, \ldots, s\}$ such that $J = \cap\{J(X_i): i \in K_J\}\}$ and let $\mathscr{S}$ be the system of all minimal elements of $\mathscr{S}_0$; thus $\mathscr{S} = \{J \in \mathscr{S}_0$: for each $T \in \mathscr{S}_0$ there holds: if $T \subset J$, then $T = J\}$. Then the system $\mathscr{S}$ is a finite system of non-overlapping closed intervals, for $J \in \mathscr{S}$ there exists a point X_k such that $J \subset J(X_k)$ and $\cup\{J(X_i)$: $X_i \in F_{(N_1, N_2, N_3)}\} = \cup\{J \in \mathscr{S}: J \subset J(X_i), X_i \in F_{(N_1, N_2, N_3)}\}$ for each $(N_1, N_2, N_3) \in \mathscr{F}$. Let j be the cardinality of the system $\mathscr{S}$.

It is easy to prove that there exists an $\eta > 0$ such that: $\eta < \frac{l}{3}$, $\eta < \frac{1}{2} \min\{b_i - a_i$: $i = 1, \ldots, n\}$, $I_\eta - \mathrm{Int}(J_\eta) \subset \cup\{J(X_i): i = 1, \ldots, s\}$ and $j(p_n)^2\ 2^n K m(I_\eta - J_\eta) < \varepsilon$, where $J_\eta = \langle a_1 + \eta, b_1 - \eta; \ldots; a_n + \eta, b_n - \eta \rangle \subset \langle a_1 - \eta, b_1 + \eta; \ldots; a_n - \eta, b_n + \eta \rangle = I_\eta$. Now for each $(N_1, N_2, N_3) \in \mathscr{F}$ we define two intervals: $Y_{(N_1, N_2, N_3)} = \langle t_1, u_1; \ldots; t_n, u_n \rangle$ and $Y^*_{(N_1, N_2, N_3)} = \langle v_1, w_1; \ldots; v_n, w_n \rangle$, where $t_i = a_i$, $u_i = a_i + \eta$, $v_i = a_i - \eta$, $w_i = a_i$ if $i \in N_1$, $t_i = b_i - \eta$, $u_i = b_i$, $v_i = b_i$, $w_i = b_i + \eta$ if $i \in N_2$ and

$t_i = a_i + \eta$, $u_i = b_i - \eta$, $v_i = a_i$, $w_i = b_i$ if $i \in N_3$. Let $\mathcal{Y} = \{Y_{(N_1, N_2, N_3)}: (N_1, N_2, N_3) \in \mathcal{F}\}$ and $\mathcal{Y}^* = \{Y^*_{(N_1, N_2, N_3)}: (N_1, N_2, N_3) \in \mathcal{F}\}$.

We shall prove that $\mathcal{Y} \cup \mathcal{Y}^*$ is a finite system of non-overlapping closed intervals and $\overline{I - J_\eta} = \cup\mathcal{Y}$, $\overline{I_\eta - I} = \cup\mathcal{Y}^*$.

Let $P = (p_1, \ldots, p_n) \in \overline{I - J_\eta}$ and let $N_1 = \{i \in \{1, \ldots, n\}: p_i \leqq a_i + \eta\}$, $N_2 = \{i \in \{1, \ldots, n\}: b_i - \eta \leqq p_i\}$ and $N_3 = \{1, \ldots, n\} - (N_1 \cup N_2)$. Since $P \in \overline{I - J_\eta}$, there is either $P \in I - J_\eta$ or $P \in \mathrm{Fr}(J_\eta)$. Therefore $N_1 \cup N_2 \neq \emptyset$ and $(N_1, N_2, N_3) \in \mathcal{F}$. But then $P \in Y_{(N_1, N_2, N_3)}$ and therefore $\overline{I - J_\eta} \subset \cup\mathcal{Y}$. Let $P \in \cup\mathcal{Y}$. Then there exists a $(N_1, N_2, N_3) \in \mathcal{F}$ such that $P \in Y_{(N_1, N_2, N_3)}$. From the definition of $Y_{(N_1, N_2, N_3)}$ it follows that $a_i \leqq p_i \leqq b_i$ for each $i \in \{1, \ldots, n\}$ and therefore $P \in I$. For each $i \in N_1 \cup N_2$ there cannot hold $a_i + \eta < p_i < b_i - \eta$ and since $N_1 \cup N_2 \neq \emptyset$ there holds: $P \notin \mathrm{Int}(J_\eta)$. Thus $P \in \overline{I - J_\eta}$. This gives that $\cup\mathcal{Y} \subset \overline{I - J_\eta}$ and we have proved that $\overline{I - J_\eta} = \cup\mathcal{Y}$. The proof of $\overline{I_\eta - I} = \cup\mathcal{Y}^*$ is similar.

Let T, $U \in \mathcal{Y} \cup \mathcal{Y}^*$ and $\mathrm{Int}(T) \cap \mathrm{Int}(U) \neq \emptyset$. Since $\mathrm{Int}(T) \cap \mathrm{Int}(U) \neq \emptyset$ and $\mathrm{Int}(\overline{I - J_\eta}) \cap \mathrm{Int}(\overline{I_\eta - I}) = \emptyset$, there can be either T, $U \in \mathcal{Y}$ or T, $U \in \mathcal{Y}^*$. Let T, $U \in \mathcal{Y}$. Then there exists a (N_1, N_2, N_3) and (K_1, K_2, K_3) of $\mathcal{F}$ such that $T = Y_{(N_1, N_2, N_3)} = \langle t_1, u_1; \ldots; t_n, u_n\rangle$ and $U = Y_{(K_1, K_2, K_3)} = \langle v_1, w_1; \ldots; v_n, w_n\rangle$. Since $\mathrm{Int}(T) \cap \mathrm{Int}(U) \neq \emptyset$, there exists a point $P = (p_1, \ldots, p_n) \in \mathrm{Int}(T) \cap \mathrm{Int}(U)$. Therefore there holds: $t_i \leqq p_i \leqq u_i$ and $v_i \leqq p_i \leqq w_i$ for each $i \in \{1, \ldots, n\}$. Since $a_i < a_i + \eta < b_i - \eta < b_i$ for each $i \in \{1, \ldots, n\}$, there must be $t_i = v_i$ and $u_i = w_i$ for each $i \in \{1, \ldots, n\}$. Thus $(N_1, N_2, N_3) = (K_1, K_2, K_3)$ and $T = U$.

For T, $U \in \mathcal{Y}^*$ the proof is similar.

Let $\mathcal{J} = \{J: J$ is a closed interval for which there exist a $T \in \mathcal{S}$ and a $(N_1, N_2, N_3) \in \mathcal{F}$ such that $J = T \cap Y_{(N_1, N_2, N_3)}\}$ and let $\mathcal{I} = \{Y: Y$ is a closed interval for which there exist a $T \in \mathcal{S}$ and a $(N_1, N_2, N_3) \in \mathcal{F}$ such that $Y = Y^*_{(N_1, N_2, N_3)} \cap T\}$. Then the systems $\mathcal{J}$ and $\mathcal{I}$ are finite systems of nonoverlapping closed intervals of the cardinality not greater than $j \cdot p_n$ and $\cup\mathcal{J} = \overline{I - J_\eta}$ and $\cup\mathcal{I} = \overline{I_\eta - I}$.

Let $J \in \mathcal{J}$. Then there exist a $T \in \mathcal{S}$ and a $(N_1, N_2, N_3) \in \mathcal{F}$ such that $J = T \cap Y_{(N_1, N_2, N_3)}$. Let $\hat{Y} = \langle t_1, u_1; \ldots; t_n, u_n\rangle$, where $t_i = a_i$, $u_i = a_i + \eta$ for each $i \in N_1$, $t_i = b_i - \eta$, $u_i = b_i$ for each $i \in N_2$ and $t_i = a_i$ and $u_i = b_i$ for each $i \in N_3$. Then $J \subset Y_{(N_1, N_2, N_3)} \subset \cup\{J(X_i) \cap \hat{Y}: X_i \in F_{(N_1, N_2, N_3)}\}$ and therefore there exists an $i \in \{1, \ldots, s\}$ such that $J \subset J(X_i) \cap \hat{Y}$, $J(X_i) \cap \hat{Y}$ is an interval contained in $\overline{I - J_\eta}$ and $X_i \in J(X_i) \cap \hat{Y}$.

Let $Y \in \mathcal{I}$. Then there exist a $T \in \mathcal{S}$ and a $(N_1, N_2, N_3) \in \mathcal{F}$ such that $Y = T \cap Y^*_{(N_1, N_2, N_3)}$. Then $Y \subset Y^*_{(N_1, N_2, N_3)} \subset \cup\{J(X_i) \cap Y^*_{(N_1, N_2, N_3)}: X_i \in F_{(N_1, N_2, N_3)}\}$.

Therefore there exists an X_j, $j \in \{1, \ldots, s\}$ such that $Y \subset J(X_j) \cap Y^*_{(N_1, N_2, N_3)} \subset \overline{I_\eta - I}$, $J(X_j) \cap Y^*_{(N_1, N_2, N_3)}$ is an interval and $X_j \in J(X_j) \cap Y^*_{(N_1, N_2, N_3)}$.

It is easy to see that there exists a $\delta > 0$ such that $J_\eta \subset J \subset I_\eta$ for each $J \in X(G)$ satisfying $\varrho(I, J) < \delta$.

Let $J = \langle c_1, d_1; \ldots; c_n, d_n \rangle$ an interval of $X(G)$ such that $\varrho(I, J) < \delta$. Then $J_\eta \subset J \subset I_\eta$. Let there for each $(N_1, N_2, N_3) \in \mathscr{F}$ be $\hat{J}_{(N_1, N_2, N_3)} = \langle e_1, f_1; \ldots; e_n, f_n \rangle$, where $e_i = a_i$, $f_i = \max(a_i, c_i)$ for each $i \in N_1$, $e_i = \min(b_i, d_i)$, $f_i = b_i$ for each $i \in N_2$ and $e_i = \max(a_i, c_i)$, $f_i = \min(b_i, d_i)$ for each $i \in N_3$. Then the system $\hat{\mathscr{J}} = \{\hat{J}_{(N_1, N_2, N_3)}: \hat{J}_{(N_1, N_2, N_3)}$ is a closed interval in E_n, $(N_1, N_2, N_3) \in \mathscr{F}\}$ is a finite system of non-overlapping closed intervals for which $\overline{I - (J \cap I)} = \cup\{\hat{J}_{(N_1, N_2, N_3)}: \hat{J}_{(N_1, N_2, N_3)} \in \hat{\mathscr{J}}\}$. We give here only the proof of the last relation.

Let $P = (p_1, \ldots, p_n) \in \overline{I - (J \cap I)}$. Then $a_i \leqq p_i \leqq b_i$ for each $i \in \{1, \ldots, n\}$ and there exists a $j \in \{1, \ldots, n\}$ such that either $p_j \leqq \max(a_j, c_j)$ or $\min(b_j, d_j) \leqq p_j$ holds. Let $N_1 = \{i \in \{1, \ldots, n\}: a_i < c_i$ and $p_i \leqq c_i\}$, $N_2 = \{i \in \{1, \ldots, n\}: d_i < b_i$ and $d_i \leqq p_i\}$ and $N_3 = \{1, \ldots, n\} - (N_1 \cup N_2)$. Since either $P \in I - (J \cap I)$ or P is a limit point of $I - (J \cap I)$, there must be $N_1 \cup N_2 \neq \emptyset$. Thus $(N_1, N_2, N_3) \in \mathscr{F}$ and $P \in \hat{J}_{(N_1, N_2, N_3)}$, because $a_i \leqq p_i \leqq c_i$ for each $i \in N_1$, $d_i \leqq p_i \leqq b_i$ for each $i \in N_2$ and $c_i \leqq p_i \leqq d_i$ for each $i \in N_3$. Let there be $(N_1, N_2, N_3) \in \mathscr{F}$, let $\hat{J}_{(N_1, N_2, N_3)}$ be a closed interval and $P = (p_1, \ldots, p_n) \in \hat{J}_{(N_1, N_2, N_3)}$. Then $a_i \leqq p_i \leqq c_i < a_i + \eta < b_i - \eta < b_i$ for each $i \in N_1$, $a_i < a_i + \eta < b_i - \eta < d_i \leqq p_i \leqq b_i$ for each $i \in N_2$ and $a_i \leqq \max(a_i, c_i) \leqq p_i \leqq \min(b_i, d_i) \leqq b_i$ for each $i \in N_3$. Therefore $P \in I$. Since $N_1 \cup N_2 \neq \emptyset$, there exists a $j \in \{1, \ldots, n\}$ such that the inequality $\max(a_j, c_j) < p_j < \min(b_j, d_j)$ does not hold. Therefore $P \notin \operatorname{Int}(J \cap I)$. Thus $P \in I - \operatorname{Int}(J \cap I) = \overline{I - (J \cap I)}$.

Let there be $\hat{J}^*_{(N_1, N_2, N_3)} = \langle g_1, h_1; \ldots; g_n, h_n \rangle$ for each $(N_1, N_2, N_3) \in \mathscr{F}$, where $g_i = c_i$, $h_i = \max(a_i, c_i)$ for each $i \in N_1$, $g_i = \min(b_i, d_i)$, $h_i = d_i$ for each $i \in N_2$ and $g_i = \max(a_i, c_i)$, $h_i = \min(b_i, d_i)$ for each $i \in N_3$. Then the system $\hat{\mathscr{J}} = \{\hat{J}^*_{(N_1, N_2, N_3)}: \hat{J}^*_{(N_1, N_2, N_3)}$ is a closed interval in E_n and $(N_1, N_2, N_3) \in \mathscr{F}\}$ is a finite system of non-overlapping closed intervals for which $\overline{J - (J \cap I)} = \cup\{\hat{J}^*_{(N_1, N_2, N_3)}: \hat{J}^*_{(N_1, N_2, N_3)} \in \hat{\mathscr{J}}^*\}$.

Let $\mathscr{U}$ be the system $\{T$: T is a closed interval for which there exist an $J \in \mathscr{J}$ and $\hat{J}_{(N_1, N_2, N_3)} \in \hat{\mathscr{J}}$ such that $T = J \cap \hat{J}_{(N_1, N_2, N_3)}\}$ and let $\mathscr{V}$ be the system $\{T$: T is a closed interval for which there exist an $Y \in \mathscr{J}$ and $\hat{J}^*_{(N_1, N_2, N_3)} \in \hat{\mathscr{J}}^*$ such that $T = Y \cap \hat{J}^*_{(N_1, N_2, N_3)}\}$. It is evident that $\mathscr{U}$ and $\mathscr{V}$ are finite systems of non-overlapping closed intervals of the cardinality not greater than $j(p_n)^2$ and for each $T \in \mathscr{U}$ there exists a closed interval $J(X_i) \cap \hat{Y}$ contained in $\overline{I - J_\eta}$ such that $T \subset J(X_i) \cap \hat{Y}$ and $X_i \in J(X_i) \cap \hat{Y}$ and for $T \in \mathscr{V}$ there exists a closed interval $J(X_j) \cap \hat{Y}^*_{(N_1, N_2, N_3)}$ and $X_j \in J(X_j) \cap \hat{Y}^*_{(N_1, N_2, N_3)}$. According to lemma 3 we have $|f(T)| \leqq 2^n K m(J(X_i)$

$\cap \hat{Y}) \leqq 2^n K m(\overline{I - J_\eta})$ for each $T \in \mathcal{U}$ and $|f(T)| \leqq 2^n K m(J(X_j) \cap \hat{Y}^*_{(N_1, N_2, N_3)}) \leqq$ $2^n K m(\overline{I_\eta - I})$ for $T \in \mathcal{V}$. Then we have: $f(J) = f(J \cap I) + \Sigma\{f(T)\colon\ T \in \mathcal{V}\}$ and $f(I) = f(J \cap I) + \Sigma\{f(T)\colon\ T \in \mathcal{U}\}$. Therefore $|f(J) - f(I)| = |\Sigma\{f(T)\colon\ T \in \mathcal{V}\} - \Sigma\{f(T)\colon T \in \mathcal{U}\}| \leqq \Sigma\{|f(T)|\colon T \in \mathcal{U} \cup \mathcal{V}\} \leqq j(p_n)^2\, 2^n K\, m(I_\eta - J_\eta) < \varepsilon$.

The proof of the mentioned theorem 3 of Laczkovich [2] is based on theorem 1 of [2] and on theorem 7 of [3].

We give here Laczkovich's theorems 1 of [2] and 7 of [3]. We do not give here the definition of the C_k property because this property is not important for Laczkovich's proof of theorem 3.

If φ is an additive interval function defined on $X(I)$, where $I = \langle a_1, b_1; \ldots; a_n, b_n \rangle$ is a closed interval, M. Laczkovich defines a function f_φ on I as follows: $f_\varphi(x) = 0$ if $x = (x_1, \ldots, x_n)$ and $x_i = a_i$ for some $i \in \{1, \ldots, n\}$ and $f_\varphi(x) = \varphi(\langle a_1, x_1; \ldots; a_n, x_n \rangle)$ if $x = (x_1, \ldots, x_n)$ and $a_i < x_i \leqq b_i$ for each $i \in \{1, \ldots, n\}$.

Laczkovich's theorem 7 of [3] says that f_φ is differentiable on I if φ is an additive interval function defined on $X(I)$, which has a finite strong derivative at each point of I. Theorem 1 of [2] says that for any additive interval function φ defined on $X(I)$ the following three assertions are equivalent:

(i) φ is uniformly continuous on I,

(ii) φ has the C_k property in I for every $k = 0, 1, \ldots, n-1$,

(iii) f_φ is continuous on I.

It is easy to prove that theorem 5 is a consequence of Laczkovich's theorem 3 of [2].

We give here another proof of Laczkovich's theorem 3 of [2] in a way similar to the one used in the proof of our theorem 5.

Theorem 6. (M. Laczkovich) *Let f be an additive interval function on $X(I)$, where I is a closed interval. If f has at each point of I a finite strong derivative, then f is uniformly continuous on I.*

Proof. Let f be an additive interval function on I which has a finite strong derivative at each point of I. If we suppose that f is not uniformly continuous on I, then there exist a $\varepsilon > 0$ and a sequence $\{J_k\}_{k=1}^{\infty}$ of $X(I)$ such that $|f(J_k)| \geqq \varepsilon$ for $k = 1, 2, 3, \ldots$ and $\lim_{k \to \infty} m(J_k) = 0$. Let there be $J_k = \langle a_{1,k}, b_{1,k}; \ldots; a_{n,k}, b_{n,k} \rangle$ for $k = 1, 2, 3, \ldots$. Then there exists an $i \in \{1, \ldots, n\}$ such that $\liminf_{k \to \infty} (b_{i,k} - a_{i,k}) = 0$. Since the sequences $\{a_{j,k}\}_{k=1}^{\infty}$ and $\{b_{j,k}\}_{k=1}^{\infty}$ for $j = 1, \ldots, n$ are bounded, there exist a sequence $\{k_s\}_{s=1}^{\infty}$ and numbers a_j and b_j for $j = 1, \ldots, n$ such that $a_j = \lim_{s \to \infty} a_{j,k_s}$, $b_j = \lim_{s \to \infty} b_{j,k_s}$ and $a_i = b_i$.

Let $H=\{(x_1, \ldots, x_n)\in I\colon x_i=a_i\}$. Since f has a finite strong derivative at each point of I, there exist a finite system $\{X_r=(x_1^r, \ldots, x_n^r)\colon r=1, \ldots, s\}$ of points of H and δ such that $0<\delta<1$, for each $r\in\{1, \ldots, s\}$ and $Y\in X(I)$, we have $(\mathrm{D}f(X_r)-\frac{1}{2})m(Y)<f(Y)<(\mathrm{D}f(X_r)+\frac{1}{2})m(Y)$ if $X_r\in Y$ and $d(Y)<\delta$, and $H\subset\cup\{Y_r\colon r=1, \ldots, s\}$, where $Y_r=I\cap\left\langle x_1^r-\frac{\delta}{2n},\ x_1^r+\frac{\delta}{2n};\ \ldots;\ x_n^r-\frac{\delta}{2n},\ x_n^r+\frac{\delta}{2n}\right\rangle$ for $r=1, \ldots, s$.

Let $\mathscr{S}_0$ be the system $\{J\colon J$ is a closed interval for which there exists a finite set $K_J\subset\{1, \ldots, s\}$ such that $J=\cap\{Y_r\colon r\in K_J\}\}$ and let $\mathscr{S}$ be the system of all minimal elements of $\mathscr{S}_0$. Then $\mathscr{S}$ is a finite system of non-overlapping closed intervals such that $\cup\{Y_r\colon r=1, \ldots, s\}=\cup\{J\colon J\in\mathscr{S}\}$ and for each $J\in\mathscr{S}$ there exists an $r\in\{1, \ldots, s\}$ such that $J\subset Y_r$. Let j be the cardinality of $\mathscr{S}$. Let $\eta>0$ such that $Y=\{(x_1, \ldots, x_n)\in I\colon a_i-\eta\leqq x_i\leqq a_i+\eta\}\subset\cup\{Y_r\colon e=1, \ldots, s\}$ and $j\ 2^n\ K\ m(Y)<\varepsilon$, where $K=\max\{\max(\|\mathrm{D}f(X_r)-\frac{1}{2}|,\ \mathrm{D}f(X_r)+\frac{1}{2}|\colon r=1, \ldots, s\}$.

Since $\{J_{k_s}\}_{s=1}^{\infty}$ is a sequence of intervals of $X(I)$ such that $\lim_{s\to\infty} a_{i,k_s}=\lim_{s\to\infty} b_{i,k_s}=a_i$, there exists a natural m such that $J_{k_m}\subset Y$. The interval J_{k_m} is a union of the system $\mathscr{J}^*=\{J_{k_m}\cap J\cap Y\colon J_{k_m}\cap J\cap Y$ is a closed interval, $J\in\mathscr{S}\}$ of non-overlapping closed intervals. According to Lemma 3, we have $|f(J_{k_m}\cap J\cap Y|\leqq 2^n\ K\ m(Y_r\cap Y)\leqq 2^n\ K\ m(Y)$ for each $J_{k_m}\cap J\cap Y\in\mathscr{J}^*$ if $J\subset Y_r$ for $r\in\{1, \ldots, s\}$. Therefore we get $\varepsilon\leqq|f(J_{k_m})=|\Sigma\{f(J_{k_m}\cap J\cap Y)\colon J_{k_m}\cap J\cap Y\in\mathscr{J}^*\}|\leqq\Sigma\{|f(J_{k_m}\cap\cap Y)|\colon J_{k_m}\cap J\cap Y\in\mathscr{J}^*\}\leqq j\ 2^n\ K\ m(Y)<\varepsilon$. But, this is a contradiction and the theorem is proved.

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О НЕПРЕРЫВНЫХ ФУНКЦИЯХ ИНТЕРВАЛА

Ladislav Mišík

Резюме

В статье изучается пространство замкнутых интервалов в n-размерном эвклидовом пространстве и доказывается, что аддитивная функция интервала метрически непрерывна, если имеет конечную сильную производную. Приводится также другое доказательство теоремы М. Лацковича.