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ERROR BOUNDS FOR THE SECANT METHOD

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ABSTRACT. Error bounds for the secant method eventually better than those presented in literature, under the same assomptions.

I. Introduction

In this paper we study the iterative procedure

$$x_{n+1} = x_n - \delta f(x_{n-1}, x_n)^{-1} f(x_n) \tag{1}$$

to approximate solutions x^* of the equation

$$f(x) = 0, \tag{2}$$

where f is a nonlinear operator between two Banach spaces E and $\hat{E}$, x_{-1} and x_0 are two points in the domain of f, and δf is a consistent approximation of f'.

The iterative procedure (1) is called the *secant method* but it is also known under the name of *regula falsi* or *the method of chords*. This procedure has been known since the time of early Italian algebraists [5] and it was extended to the solution of nonlinear equations in Banach spaces by Sergeev [12] and Schmidt [11]. Potra [6], [7], [8], Pták [9], [10], Dennis [3], Gragg and Tapia [4] and others have done similar work on the secant as well as the Newton method.

Here we provide a priori and a posteriori error estimates which are proven to be eventually better than those presented in literature [7], [8], [9], under the same assumptions.

Finally, a simple example is provided, where our results are compared favourably with the corresponding results obtained in [7] and [8].

II. Error estimates for the secant method.

In the study of the secant method we shall use the method of nondiscrete mathematical induction. This method was developed by V. Pták by refining the closed graph theorem [9], [10].

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Let T denote either the set of all positive numbers, or an interval of the form $(0, b] = \{x \in \mathbb{R}; 0 < x \leq b\}$. Let w be a mapping of the cartesian product T^2 into T and let us consider the "iterates" $w^{(n)}$ of w given for each $t = (t_1, t_2) \in T^2$ by

$$w^{(0)}(t) = t_2, \quad w^{(n+1)}(t) = w^{(n)}(t_2, w(t)), \quad n = 0, 1, 2, \ldots . \tag{3}$$

Definition 1. *A mapping* $w\colon T^2 \to T$ *with the above iteration law is called a rate of convergence of type* (2, 1) *on* T *if the series*

$$\sigma(t) = \sum_{k=0}^{\infty} w^{(k)}(t) \tag{4}$$

is convergent for all $t \in T^2$.

We will need:

Definition 2. *Let* E *and* $\hat{E}$ *be two Banach spaces and let* V *be a convex and open subset of* E. *Let* $f\colon V \to \hat{E}$ *be a nonlinear operator which is Fréchet-differentiable on* V. *A mapping* $\delta f\colon V \times V \to L(E, \hat{E})$ *will be called a consistent approximation of* f' *if there exists a constant* $H > 0$ *such that*

$$\|\delta f(x, y) - f'(z)\| \leq H(\|x - z\| + \|y - z\|) \quad \textit{for all} \quad x, y, z \in V. \tag{5}$$

The above condition implies the Lipschitz continuity of f'. In this case we can by using a standard argument (see [7], p. 432) easily deduce the following:

$$\|f(u) - f(v) - f'(v)(u - v)\| \leq H\|u - v\|^2; \quad u, v \in V \tag{6}$$

and

$$\|f(u) - f(v) - \delta f(x, y)(u - v)\| \leq H(\|u - v\| + \|x - v\| + \|y - v\|)\,\|u - v\|. \tag{7}$$

Let $C(h_0, q_0, r_0)$ be the class of all the triplets (f, x_0, x_{-1}) satisfying the following properties for some μ:

(p_1) *f is a nonlinear operator having the domain of definition* $V \subset E$ *and taking values in* $\hat{E}$.

(p_2) x_0 *and* x_{-1} *are two points of* V *such that*

$$\|x_0 - x_{-1}\| \leq q_0, \quad \|x_0 - x_{-1}\| < \mu. \tag{8}$$

(p_3) *f is Fréchet-differentiable in the open ball* $U = U(x_0, \mu) = \{x \in V \mid \|x_0 - x\| < \mu\}$ *and continuous on its closure* $\bar{U}$.

(p_4) *There exists a consistent approximation* δf *of* f' *such that* $D_0 \equiv \delta f(x_{-1}, x_0)$ *is invertible and*

$$\|D_0^{-1}(\delta f(x, y) - f'(z))\| \leq h_0(\|x - z\| + \|y - z\|) \quad \textit{for all} \quad x, y, z \in U. \tag{9}$$

(p_5) *The following inequalities are satisfied*:

$$\|D_0^{-1}f(x_0)\| \leq r_0, \tag{10}$$

$$h_0 q_0 + 2\sqrt{h_0 r_0} \leq 1, \tag{11}$$

$$\mu \geq \frac{1}{2h_0}(1 - h_0 q_0 - \sqrt{(1 - h_0 q_0)^2 - 4h_0 r_0}) \equiv \mu_0. \tag{12}$$

Using the iterative procedure (1) Potra showed in ([7], Thm. 3) and ([8], Thm. 1) that if $(f, x_0, x_{-1}) \in C(h_0, q_0, r_0)$, then the equation $f(x) = 0$ has a locally unique solution x^*, and certain error estimates are valid.

In particular he showed:

Theorem 1. *If* $(f, x_0, x_{-1}) \in C(h_0, q_0, r_0)$, *then via the iterative procedure* (1) *one obtains a sequence* $\{x_n\}$, $n \geq 0$ *of points from the open ball* $U(x_0, \mu_0)$ *which converges to a unique root* x^* *of the equation* $f(x) = 0$ *in* $\bar{U}(x_0, \mu_0)$ *and the following estimates are satisfied*:

$$\|x_n - x^*\| \leq \sigma_0(w_0^{(n)}(t_0)),\ t_0 = (q_0, r_0), \quad n = 0, 1, 2, \ldots \tag{13}$$

$$\|x_n - x^*\| \leq c(n) \equiv [a_0^2 + \|x_n - x_{n-1}\|(\|x_{n-1} - x_{n-2}\| + \|x_n - x_{n-1}\|)]^{1/2} - a_0,$$
$$n = 1, 2, \ldots \tag{14}$$

$$\|x_{n+1} - x_n\| \leq w_0^{(n)}(t_0), \quad n = 0, 1, 2, \ldots \tag{15}$$

$$\|x_n - x^*\| \leq c_0(n) \equiv s_0 - \|x_n - x_0\| -$$
$$[(s_0 - \|x_n - x_0\|)^2 - (\|x_n - x_{n-1}\| + \|x_{n-1} - x_{n-2}\|)\,\|x_n - x_{n-1}\|]^{1/2}, \tag{16}$$
$$n = 1, 2, \ldots$$

where

$$a_0 = \frac{1}{2h_0}[(1 - h_0 q_0)^2 - 4h_0 r_0]^{1/2}, \tag{17}$$

$$s_0 = \frac{1 - q_0 h_0}{2h_0}, \tag{18}$$

$$w_0(t) = w_0(q, r) = \frac{r(q + r)}{r + 2\sqrt{r(q + r) + a_0^2}} \tag{19}$$

and

$$\sigma_0(t) = \sigma_0(q, r) = r - a_0 + \sqrt{r(q + r) + a_0^2}. \tag{20}$$

We can prove the following theorem:

Theorem 2. *Under the hypothesis of Theorem 1 the following inequalities hold for* $n = 1, 2, 3, \ldots$,

$$\|x_n - x^*\| \le c_1(n) \equiv$$
$$\equiv [a_{n-1}^2 + \|x_n - x_{n-1}\|(\|x_{n-1} - x_{n-2}\| + \|x_n - x_{n-1}\|)]^{1/2} - a_{n-1} \tag{21}$$

provided that for $n = 1, 2, \dots$ *the following estimate is true*:

$$\frac{w_0^{(n-1)}(t_0)}{w_0^{(n-1)}(t_0) + 2\sqrt{w_0^{(n-1)}(t_0)(q_0 + w_0^{(n-1)}(t_0)) + a_0^2}} +$$
$$+ 2\sqrt{\frac{w_0^{(n)}(t_0)}{w_0^{(n-1)}(t_0) + 2\sqrt{w_0^{(n-1)}(t_0)(q_0 + w_0^{(n-1)}(t_0)) + a_0^2}}} \le 1, \tag{H}$$

where

$$h_n = \sup_{x, y, z \in U} \frac{\|D_n^{-1}(\delta f(x, y) - f'(z))\|}{\|x - z\| + \|y - z\|}, \tag{22}$$

$$D_n = \delta f(x_{n-1}, x_n), \tag{23}$$

$$q_{n+1} = r_n = \|x_n - x_{n+1}\|, \tag{24}$$

and

$$a_n(h_n, q_n, r_n) = a_n = \frac{1}{2h_n}[(1 - h_n q_n)^2 - 4h_n r_n]^{1/2}, \quad n = 0, 1, 2, \dots. \tag{25}$$

Proof. First let us observe that with the constant a_0 given by (17) we have $\sigma_0(r_0) = \mu_0$. Hence the closed ball with centre x_0 and radius μ_0 is included in U. Consider the triplet $(f, x_{-1}, x_0) \in C(h_0, q_0, r_0)$. We will prove that $(f, x_{n-1}, x_n) \in C(h_n, q_n, r_n)$. It suffices to show that the inequality

$$h_n q_n + 2\sqrt{h_n r_n} \le 1. \tag{26}$$

Using (9), (15), and (20), we have:

$$\begin{aligned}
\|D_0^{-1}(D_0 - D_n)\| &\le \|D_0^{-1}(D_0 - f'(x_{n-1}))\| + \|D_0^{-1}(f'(x_{n-1}) - D_n)\| \\
&\le h_0[\|x_0 - x_{n-1}\| + \|x_{-1} - x_{n-1}\| + \|x_n - x_{n-1}\|] \\
&\le h_0[2\|x_0 - x_{n-1}\| + \|x_{-1} - x_0\| + \|x_{n-1} - x_n\|] \\
&\le h_0[2(\mu_0 - \sigma_0(w_0^{(n-1)}(t_0)) + q_0 + w_0^{(n-1)}(t_0)] \\
&\le 1 - h_0[w_0^{(n-1)}(t_0) + 2\sqrt{w_0^{(n-1)}(t_0)(q_0 + w_0^{(n-1)}(t_0)) + a_0^2} < 1.
\end{aligned} \tag{27}$$

According to Banach's lemma this implies that

$$\|(D_0^{-1}D_n)^{-1}\| \le \{1 - h_0[\|x_0 - x_{n-1}\| + \|x_{-1} - x_{n-1}\| + \|x_{n-1} - x_n\|]\}^{-1}. \tag{28}$$

From the identity

$$D_n^{-1}(\delta f(x, y) - f'(z)) = (D_0^{-1}D_n)^{-1} D_0^{-1}(\delta f(x, y) - f'(z)),$$

(9) and (28) we obtain

$$\|D_n^{-1}(\delta f(x, y) - f'(z))\| \leq h_0\|(D_0^{-1}D_n)^{-1}\| (\|x - z\| + \|y - z\|), \quad (29)$$

that is

$$h_n \leq z_n \equiv h_0\{1 - h_0[\|x_0 - x_{n-1}\| + \|x_{-1} - x_{n-1}\| + \|x_{n-1} - x_n\|]\}^{-1}. \quad (30)$$

By (27) and (28) we can easily obtain that

$$h_n \leq [w_0^{(n-1)}(t_0) + 2\sqrt{w_0^{(n-1)}(t_0)(q_0 + w_0^{(n-1)}(t_0)) + a_0^2}]^{-1}. \quad (31)$$

To show (26), using (24), and (31), it suffices to show (H), which is true by hypothesis.

By applying Theorem 1 to the triplet (f, x_{n-1}, x_n) we deduce (21). That completes the proof of the theorem.

Note that by using (3) we can easily deduce that

$$w_0^{(1)}(t) = w_0(t)$$

and

$$w_0^{(n)}(t) = w_0[w_0^{(n-2)}(t), w_0^{(n-1)}(t)], \quad n = 2, 3, \ldots . \quad (32)$$

Let us assume that the sequence $\{w_0^{(n)}(t_0)\}$ is bounded above by q_0. Then, using (32) and (19) it can easily be seen that (H) is satisfied for all $n = 1, 2, \ldots$.

We can now improve the results of Theorem 1 through Theorem 2 as follows:

Proposition 1. *Under the hypotheses of Theorem 2 the following are true:*

(a) *For all* $n = 1, 2, \ldots$, *the triplet* $(f, x_{n-1}, x_n) \in C(z_n, q_n, r_n)$,

$$\|x_n - x^*\| \leq c_2(n) \equiv$$

$$\equiv [d_{n-1}^2 + \|x_n - x_{n-1}\| (\|x_{n-1} - x_{n-2}\| + \|x_n - x_{n-1}\|)]^{1/2} - d_{n-1} \quad (33)$$

and if

$$d_n \geq a_0, \quad n = 0, 1, 2, \ldots, \quad (34)$$

then

$$c_2(n) \leq c(n), \quad (35)$$

where we have denoted

$$d_n(z_n, q_n, r_n) = d_n = \frac{1}{2z_n}[(1 - z_nq_n)^2 - 4z_nr_n]^{1/2}, \quad n = 0, 1, 2, \ldots . \quad (36)$$

(b) *If*

$$a_n \geq a_0, \quad n = 1, 2, \dots, \tag{37}$$

then

$$c_1(n) \leq c(n) \quad \textit{for all} \quad n = 1, 2, \dots . \tag{38}$$

(c) *Moreover, for all* $n = 1, 2, \dots$, *the triplet* $(f, x_{n-1}, x_n) \in C(h_0, q_n, r_n)$,

$$\|x_n - x^*\| \leq c_3(n) \equiv$$

$$\equiv [e_{n-1}^2 + \|x_n - x_{n-1}\| (\|x_{n-1} - x_{n-2}\| + \|x_n - x_{n-1}\|)]^{1\,2} - e_{n-1} \tag{39}$$

and if

$$e_n \geq a_0, \quad n = 1, 2, \dots, \tag{40}$$

then

$$c_3(n) \leq c(n) \quad \textit{for all} \quad n = 1, 2, \dots, \tag{41}$$

where we have denoted

$$e_n(q_n, r_n) = e_n = \frac{1}{2h_0} [(1 - h_0 q_n)^2 - 4h_0 r_n]^{1/2}, \quad n = 0, 1, 2, \dots . \tag{42}$$

Proof. (a) By (30)—(32) it follows that the triplet $(f, x_{n-1}, x_n) \in C(z_n, q_n, r_n)$. By applying Theorem 1 to the triplet (f, x_{n-1}, x_n) we obtain (33). Using (14), (33) and (34), the inequality (35) follows immediately.

(b) Using (14), (21) and (31), the result (38) follows.

(c) The triplet $(f, x_{n-1}, x_n) \in C(h_0, q_n, r_n)$, since the proof of (26) can be repeated with $h_n = h_0$ and h_0 dominated by the right-hand side of (31). Applying Theorem 1 to the triplet (f, x_{n-1}, x_n) we obtain (39). Using (14), (39) and (40) the result (41) follows.

That completes the proof of the proposition.

Facts. The functions a_n, d_n, and e_n are decreasing with respect to each of their variables separately. Therefore, they are decreasing in the sense that if P is a function of the three variables h, q and r, $h_1 \leq h_2$, $q_1 \leq q_2$ and $r_1 \leq r_2$ implies

$$P(h_2, q_2, r_2) \leq P(h_1, q_1, r_1). \tag{43}$$

Indeed, we get

$$P(h_1, q_1, r_1) \geq P(h_2, q_1, r_1) \geq P(h_2, q_2, r_1) \geq P(h_2, q_2, r_2). \tag{44}$$

Note that:

(a) Inequality (34) holds if

$$q_n \leq q_0, \tag{45}$$

$$r_n \leq r_0, \tag{46}$$

and

$$z_n \leq z_0, \quad \text{for all} \quad n = 0, 1, 2, \ldots. \tag{47}$$

(b) Inequality (37) holds if

$$q_n \leq q_0,$$

$$r_n \leq r_0,$$

and

$$h_n \leq h_0, \quad \text{for all} \quad n = 0, 1, 2, \ldots. \tag{48}$$

(c) Inequality (41) holds if

$$r_n \leq r_0$$

and

$$q_n \leq q_0, \quad \text{for all} \quad n = 0, 1, 2, \ldots.$$

Moreover, since the sequence $\{x_n\}$, $n = -1, 0, 1, 2, \ldots$ converges, there exists an integer $N \geq 1$ such that (45) and (46) hold for all $n \geq N$.

With the exception of the scalar case, the cost of computing h_n may be very high. However, the d_n's and z_n's can be computed.

By (30) we can easily check that (47) is true if

$$\|x_0 - x_{n-1}\| + \|x_{-1} - x_{n-1}\| + \|x_{n-1} - x_n\| \leq 2q_0 \quad \text{for all} \quad n = 0, 1, 2, \ldots. \tag{49}$$

It turns out that under certain assumptions the conditions (45) and (46) are satisfied.

In particular, we can show the following:

Proposition 2. *Assume*:

(a) *the hypotheses of Theorem* 1 *are true.*

(b) *The following estimates are true*:

$$r_0 \leq q_0, \tag{50}$$

and

$$2h_0(r_0 + q_0) \leq 1. \tag{51}$$

Then

$$w_0^{(n)}(t_0) \leq w_0^{(n-1)}(t_0) \leq r_0, \quad \textit{for all} \quad n = 1, 2, \ldots, \tag{52}$$

$$r_n \leq r_0$$

and

$$q_n \leq q_0.$$

Proof. It suffices to show (52). The rest will follow from (14), (24) and (50). We first show that (52) is valid for $n = 1$.

That is

$$w_0^{(1)}(t_0) \leq r_0, \tag{53}$$

which is true by (19) and (61).

Assume now that

$$v_1 = w_0^{(n-1)}(t_0) \leq w_0^{(n-2)}(t_0) = v_2 \leq w_0^{(n-3)}(t_0) = v_3.$$

We must show that

$$w_0(v_2, v_1) = w_0^{(n)}(t_0) \leq w_0^{(n-1)}(t_0) = w_0(v_3, v_2),$$

which is true since the function w_0 is increasing in both variables.

That completes the proof of the proposition.

Let us denote by A and B the left-hand sides of (11) and (51), respectively. It can easily be seen that

$$A \leq 1 \not\Leftrightarrow B \leq 1,$$

but both can hold at the same time.

Let us take for example:

$$h_0 = 1,\ q_0 = \ .5 \text{ and } r_0 = \frac{1}{49};\ \text{then } A = \ .785744285 \text{ and } B = 1.040816327$$

or

$$h_0 = 1,\ q_0 = \ .29 \text{ and } r_0 = \ .2,\ \text{then } A = 1.184427191 \text{ and } B = \ .98$$

or

$$h_0 = 1,\ q_0 = r_0 = \ .1,\ \text{then } A = \ .732455532 \text{ and } B = \ .4.$$

Furthermore, we can produce the following a posteriori error estimates on the distances $\|x_n - x^*\|$.

Theorem 3. *Assume*:

(a) *the hypotheses of Theorem 2 are true*

and

(b) *the linear operator* $\delta f(x, y)$ *is such that*

$$\delta f(x,y)(x-y) = f(x) - f(y), \quad \text{for all} \quad x, y \in V. \tag{54}$$

Then the following inequalities are true:

$$c_2(n) \leq \|x_{n+1} - x_n\| \tag{55}$$

and

$$\|x_n - x^*\| \leq c_4(n) = \frac{1 - 2h_0\|x_0 - x_n\| - \sqrt{(1 - 2h_0\|x_0 - x_n\|)^2 - 4h_0\|D_0^{-1}f(x_n)\|}}{2h_0},$$

$$n = 1, 2, \dots . \tag{56}$$

Proof. Using (30) and (33) it can easily follow that

$$\lim_{n\to\infty} d_n \geq \frac{1 - h_0[\|x_0 - x^*\| + \|x_{-1} - x^*\|]}{2h_0} > 0,$$

which implies (55) for sufficiently large n. By reordering the sequence $\{x_n\}$, $n = -1, 0, 1, 2$ we can assume that (55) is true for $n = 1, 2, \dots$. Let us consider the linear operator D, given by

$$D = \delta f(x^*, x_n). \tag{57}$$

We will show that D is invertible for all $n \geq N$. Indeed, we have by (9), (15), and (27) and (55) for $n \geq N$,

$$\begin{aligned}
\|D_0^{-1}(f'(x_0) - D)\| &\leq h_0[\|x_0 - x^*\| + \|x_0 - x_n\|] \\
&\leq h_0[2\|x_0 - x_n\| + \|x_n - x^*\|] \\
&\leq h_0[2\|x_0 - x_n\| + c_2(n)] \\
&\leq h_0[2\|x_0 - x_n\| + \|x_{n+1} - x_n\|] \\
&\leq h_0[2(\mu_0 - \sigma_0(w_0^{(n)}(t_0)) + w_0^{(n)}(t_0)] \\
&\leq 1 - h_0[w_0^{(n)}(t_0) + q_0 + 2\sqrt{w_0^{(n)}(t_0)(q_0 + w_0^{(n)}(t_0)) + a_0^2}] < 1.
\end{aligned}$$

According to Banach's lemma it follows that the linear operator D is invertible for $n \geq N$ and that

$$\|(D_0^{-1}D)^{-1}\| \leq [1 - h_0(2\|x_0 - x_n\| + \|x_n - x^*\|)]^{-1}. \tag{58}$$

Using the identity

$$D(x_n - x^*) = f(x_n),$$

(57) and (58), we obtain

$$\begin{aligned}
\|x_n - x^*\| &\leq \|(D_0^{-1}D)^{-1}\| \cdot \|D_0^{-1}f(x_n)\| \\
&\leq [1 - h_0(2\|x_0 - x_n\| + \|x_n - x^*\|)]^{-1} \|D_0^{-1}f(x_n)\|.
\end{aligned} \tag{59}$$

The inequality (56) follows now from (59).

That completes the proof of the theorem.

We now compare the estimates c_4 and c_0.

Proposition 3. *Under the hypotheses of Theorem 2 the following inequality is true:*

$$c_4(n) \leq c_0(n), \quad n = 1, 2, \dots, \tag{60}$$

where c_4 and c_0 are defined by (56) *and* (16), *respectively.*

Proof. Using the identity

$$f(x_n) = f(x_n) - f(x_{n-1}) + \delta f(x_{n-1}, x_{n-2})(x_n - x_{n-1})$$

and (7) we obtain

$$\|D_0^{-1} f(x_n)\| \leq h_0(\|x_n - x_{n-1}\| + \|x_{n-2} - x_{n-1}\|)\,\|x_n - x_{n-1}\|. \tag{61}$$

Moreover, it can easily be seen that

$$1 - 2h_0\|x_0 - x_n\| \geq 2h_0(s_0 - \|x_n - x_0\|). \tag{62}$$

The estimates $c_4(n)$ and $c_0(n)$ can be written, respectively,

$$c_4(n) = \|D_0^{-1} f(x_n)\|\{2[1 - 2h_0\|x_0 - x_n\| + ((1 - 2h_0\|x_0 - x_n\|)^2 - \\ - 4h_0\|D_0^{-1} f(x_n)\|)^{1/2}]\}^{-1} \tag{63}$$

and

$$c_0(n) = h_0(\|x_n - x_{n-1}\| + \|x_{n-1} - x_{n-2}\|)\|x_n - x_{n-1}\|\{h_0[(s_0 - \|x_n - x_0\|) + \\ + ((s_0 - \|x_n - x_0\|)^2 - (\|x_n - x_{n-1}\| + \|x_{n-1} - x_{n-2}\|)\,\|x_n - x_{n-1}\|)^{1/2}\}^{-1}. \tag{64}$$

Using (61) and (62) it can easily be seen that the numerator of (63) is smaller than or equal to the numerator of (64). Whereas the denominator of (63) is greater than or equal to the denominator of (64). The estimate (60) now follows.

That completes the proof of the proposition.

Moreover, we can show:

Proposition 4. *Assume*

(a) *the set $C_1(z_k, q_k, r_k)$ denoting the class of all triplets $(f, x_{k-1}, x_k) \in \in C(z_k, q_k, r_k)$ satisfies the estimates* (50) *and* (51) *for some fixed k, $k = 0, 1, 2, \ldots$.*

(b) *For $k = 0, 1, 2, \ldots$ the sequences $\{z_k\}$ and $\{q_k\}$ are decreasing.*

Then for every fixed k, $k = 0, 1, 2, \ldots$ the following inequalities are true:

$$\|x_n - x^*\| \leq \sigma_k(w_k^{(n)}(t_k)), \quad \textit{with} \quad t_k = (q_k, r_k) \tag{65}$$

and

$$\sigma_k(w_k^{(n)}(t_k)) \leq \sigma_0(w_0^{(n)}(t_0)) \quad \textit{for all} \quad n = k,\ k+1, \ldots, \tag{66}$$

where we have denoted

$$w_k(t) = w_k(q, r) = \frac{r(q + r)}{r + 2\sqrt{r(q + r) + d_k^2}} \tag{67}$$

and

$$\sigma_k(t) = r - d_k + \sqrt{r(q + r) + d_k^2}. \tag{68}$$

Proof. The result (65) follows immediately by applying Theorem 1 to the triplet $(f, x_{k-1}, x_k) \in C(z_k, q_k, r_k) \subset C_1(z_k, q_k, r_k)$, for each fixed k, $k = 0, 1, 2, \ldots$.

By (3) we have for $k = 0, 1, 2, \ldots$

$$\sigma_0(w_0^{(n)}(t_0)) = \sigma_0(w_0(p_1, p_2)) \text{ with } p_1 = w_0^{(n-2)}(t_0),\ p_2 = w_0^{(n-1)}(t_0) \tag{69}$$

and

$$\sigma_k(w_k^{(n)}(t_k)) = \sigma_k(w_k(p_3, p_4)) \text{ with } p_3 = (w_k^{(n-2)}(t_k),\ p_4 = w_k^{(n-1)}(t_k)$$

$$n = k+1, k+2, \ldots. \tag{70}$$

The functions w_0, w_k, σ_0, σ_k are increasing in both variables. Therefore (66) is true if

$$w_0^{(m)}(t_0) \geq w_k^{(m)}(t_k), \quad m = k, k+1, \ldots.$$

The above inequality is true as equality for $k = 0$.

Assume

$$w_0^{(m)}(t_0) \geq w_k^{(m)}, \text{ for fixed } k = 0, 1, 2, \ldots \text{ and all } m = k, k+1, \ldots. \tag{71}$$

Then we must show

$$\begin{aligned} w_0(w_0^{(m-1)}(t_0), w_0^{(m)}(t_0)) &= w_0^{(m+1)}(t_0) \geq w_k^{(m+1)}(t_{k+1}) = \\ &= w_{k+1}(w_{k+1}^{(m-1)}(t_{k+1}), w_{k+1}^{(m)}(t_{k+1})) \end{aligned} \tag{72}$$

or, since

$$w_k^{(m+1)}(t_k) \geq w_k^{(m+1)}(t_{k+1}) \geq w_{k+1}^{(m+1)}(t_{k+1}),$$

we must have

$$w_0^{(m+1)}(t_0) \geq w_k(w_k^{(m-1)}(t_k), w_k^{(m)}(t_k)),$$

which is true by (71) and the fact that the functions w_0, w_m are increasing in both variables.

That completes the proof of the proposition.

A lower bound on $\|x_n - x^*\|$ can be given by the following:

Proposition 5. *Under the hypotheses of Theorem 2 the following inequality holds for* $n = 1, 2, \ldots$

$$\|x_{n-1} - x^*\| \geq q,$$

where q is the positive root of the equation

$$\begin{aligned} &h_0\|(D_0^{-1}D_{n-1})^{-1}\|\ |x_{n-1} - x^*\|^2 + \\ &+ (1 + \|(D_0^{-1}D_{n-1})^{-1}\|\ \|x_{n-1} - x_{n-2}\| h_0)\|x_{n-1} - x^*\| - \|x_{n-1} - x_n\| = 0. \end{aligned}$$

Proof. Using the identity

$$x_n - x_{n-1} = x^* - x_n + (D_0^{-1}D_{n-1})^{-1}D_0^{-1}[(f(x^*) - f(x_{n-1})) - D_{n-1}(x^* - x_{n-1})],$$

(7) and the triangle inequality, the result follows immediately.

That completes the proof of the proposition.

Note that q depends on $\|(D_0^{-1}D_{n-1})^{-1}\|$, which in practice can be replaced by the right-hand side of (28). Denote by $\bar{q}$ the resulting quantity. Then we will certainly have

$$\|x_{n-1} - x^*\| \geq q \geq \bar{q} \quad \text{for all} \quad n = 1, 2, \ldots.$$

III. Applications

Let us now compare the estimates (33) with (14) and (56) with (16), on a very simple example. We consider the quadratic equation

$$f(x) = x^2 - 16. \tag{73}$$

Take $x_{-1} = 3$, and $x_0 = 3.2$ and $\delta f(x, y)(x - y) = f(x) - f(y)$. Then $h_0 = \frac{10}{62}$, $q_0 = \ .2$ and $r_0 = \ .92903225$.

The condition (11) is now satisfied, since

$$h_0 q_0 + 2\sqrt{h_0 r_0} = \ .806451609 < 1.$$

It is easy to see that $(f, x_{-1}, x_0) \in C(h_0, q_0, r_0)$.

The secant method for (73) becomes

$$x_{n+1} = \frac{x_{n-1}x_n + 16}{x_{n-1} + x_n}, \quad n = 0, 1, 2, \ldots.$$

Note that $x^* = 4$.

We can now compute

$x_1 = 4.129032258$,
$x_2 = 3.985915493$,
$x_3 = 3.999776048$,
$x_4 = 4.000000395$,
$x_5 = 4$,
$a_0 = 1.8$,
$d_0 = a_0$,
$d_1 = 2.461392145$,
$d_2 = 3.014220162$,
$d_3 = 3.092844865$,

and

$d_4 = 3.099887432.$

Using the above values, (33), (14), (56) and (16), we can tabulate the following results $\left(\text{within a precision of } \frac{1}{2} 10^{-8}\right)$.

n	error	error estimates (33)	error estimates (14)	error estimates (56)	error estimates (16)
1	.12903226	.27096774	.27096774	.25675941	.27096774
2	.014084507	.030974961	.042129881	.02443182	.03492694
3	$2.23952 \cdot 10^{-4}$	$3.608997 \cdot 10^{-4}$	$6.042855 \cdot 10^{-4}$	$3.8946524 \cdot 10^{-4}$	$4.945036 \cdot 10^{-4}$
4	$3.95 \cdot 10^{-7}$	$5.108 \cdot 10^{-7}$	$8.777498 \cdot 10^{-7}$	$6.8693502 \cdot 10^{-7}$	$7.182 \cdot 10^{-7}$
5	$1.1 \cdot 10^{-11}$	$1.4 \cdot 10^{-11}$	$2.46 \cdot 10^{-8}$	$1.9 \cdot 10^{-11}$	$2.0 \cdot 10^{-11}$

The above table indicates that our estimates (33) and (56) are better than the corresponding ones given by (14) and (16), respectively. Note, however, that the additional information (computation) $\|D_0^{-1} f(x_n)\|$ is used by (56).

Similar favourable comparisons can be made between the lower bound obtained here and the corresponding one in [[8], formula 12].

The above strongly recommends the usefulness of our estimates in numerical applications.

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