Ján Rusnák
Existence theorems for a certain nonlinear boundary value problem of the third order


Persistent URL: [http://dml.cz/dmlcz/128622](http://dml.cz/dmlcz/128622)

**Terms of use:**

© Mathematical Institute of the Slovak Academy of Sciences, 1987

Institute of Mathematics of the Academy of Sciences of the Czech Republic provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these Terms of use.

This paper has been digitized, optimized for electronic delivery and stamped with digital signature within the project [DML-CZ: The Czech Digital Mathematics Library](http://project.dml.cz)
EXISTENCE THEOREMS FOR A CERTAIN
NON-LINEAR BOUNDARY VALUE PROBLEM OF
THE THIRD ORDER

JÁN RUSNÁK

Introduction

In the paper we shall investigate a boundary value problem (abbr. BVP) of the form

(1) \[ x''' = f(t, x, x', x'') , \quad (t, x, x', x'') \in [a_1, a_3] \times \mathbb{R}^3 , \]
\[ a_2 x'(a_1) - a_3 x''(a_1) = A_1 \]
(2) \[ x(a_2) = A_2 \]
\[ \gamma_2 x'(a_3) + \gamma_3 x''(a_3) = A_3 , \]
\[ a_2, a_3, \gamma_2, \gamma_3 \geq 0 , \quad a_2 + a_3 > 0 , \quad \gamma_2 + \gamma_3 > 0 , \quad a_2 + \gamma_2 > 0 , \]
\[ a_1 < a_2 < a_3 . \]

Denote \( I = [a_1, a_3] , \quad I_1 = [a_1, a_2] , \quad I_2 = [a_2, a_3] \).

This BVP is a special case of BVP with general linear boundary conditions at boundary points investigated in [3, Theorem 1] where under assumptions that \( f \) is continuous and bounded on \( I \times \mathbb{R}^3 \) there was deduced that BVP (1) and (2) has at least one solution.

We shall prove existence theorems for a solution of (1) and (2) which lies between its lower and upper solutions and its derivative also lies between the derivative of lower and upper solutions, all this under hypotheses that both \( f \) is bounded and \( f \) need not be bounded. Similarly as in [3] there are utilized methods of proofs from [1]. In [1] K. Schmitt considers a non-linear two point BVP of the second order with general linear conditions at boundary points.

A function \( \alpha \in C_3(I) \) (\( C_3(I) \) is the space of all functions with continuous derivatives of order \( \leq 3 \), on \( I \)) will be called a lower solution of BVP (1) and (2) if there holds

(3) \[ \alpha''' \geq f(t, \alpha, \alpha', \alpha'') , \]
\[ a_2 a'(a) - a_3 a''(a) \leq A_1, \quad a(a_2) = A_2, \]
\[ \gamma_2 a'(a_3) + \gamma_3 a''(a_3) \leq A_3. \]

Similarly \( \beta \in C_3(\mathcal{I}) \) will be an upper solution of BVP (1) and (2) if
\[ \beta'' \leq f(t, \beta, \beta', \beta''), \]
\[ a_2 \beta'(a) - a_3 \beta''(a) \geq A_1, \quad \beta(a_2) = A_2, \]
\[ \gamma_2 \beta'(a_3) + \gamma_3 \beta''(a_3) \geq A_3. \]

Moreover let there hold for \( \alpha \) and \( \beta \) (from above)
\[ \alpha'(t) \leq \beta'(t), \quad \forall t \in \mathcal{I}. \]

Denote
\[ \omega_1 = \{(t, x, x') : t \in \mathcal{I}, \beta(t) \leq x \leq \alpha(t), \alpha'(t) \leq x' \leq \beta'(t)\}, \]
\[ \omega_2 = \{(t, x, x') : t \in \mathcal{I}, \alpha(t) \leq x \leq \beta(t), \alpha'(t) \leq x' \leq \beta'(t)\}, \]
\[ \omega = \omega_1 \cup \omega_2, \]
\[ \delta(y_1, y_2, y_3) = \begin{cases} y_1, y_2 < y_1 \leq y_3 \\ y_2, y_1 \leq y_2 \leq y_3 \\ y_3, y_1 \leq y_3 < y_2 \end{cases}, \quad y_1, y_2, y_3 \in \mathbb{R}. \]

Let the function \( f \) be continuous and bounded on \( \mathcal{I} \times \mathbb{R}^3 \) and let there exist functions \( \alpha, \beta \in C_3(\mathcal{I}) \) that are lower and upper solutions of BVP (1) and (2), respectively. Let us modify the function \( f \) to \( F \) as follows:
for \( t \in I_1 \):
\[ F(t, x, x', x'') = f(t, \delta(\beta(t), x, \alpha(t)), \delta(\alpha'(t), x', \beta'(t)), x'') + \]
\[ + \frac{x' - \delta(\alpha'(t), x', \beta'(t))}{1 + x'^2}, \]
for \( t \in I_2 \) \( F \) has the same form as for \( t \in I_1 \), we just must formally interchange the symbols \( \alpha(t) \) and \( \beta(t) \).

Hypotheses of \( f, \alpha \) and \( \beta \) imply that \( F \) is continuous and bounded on \( \mathcal{I} \times \mathbb{R}^3 \).

**Existence theorems**

**Theorem 1.** Let the function \( f \) be continuous and bounded on \( I \times \mathbb{R}^3 \) and let there exist functions \( \alpha, \beta \in C_3(\mathcal{I}) \) which are lower and upper solutions of BVP (1) and (2), respectively.

Let \( f \) be non-decreasing in \( x \) for \( t \in I_1 \) and \( x : \beta(t) \leq x \leq \alpha(t) \) and be non-increasing in \( x \) for \( t \in I_2 \) and \( x : \alpha(t) \leq x \leq \beta(t) \).
Then there exists at least one solution $x$ of BVP (1) and (2) such that

\[(t, x(t), x'(t)) \in \omega, \ \forall \ t \in I.\]

Proof. Consider a modified equation to (1):

\[x''' = F(t, x, x', x'').\]

From properties of $F$ we have that there exists at least one solution $x(t)$ of BVP (10) and (2). Further we prove that for this solution $x(t)$ the property (9) is fulfilled and therefore with respect to the definition of $F$ $x(t)$ is also the solution of BVP (1) and (2).

Let us form a function $u = x - \alpha, v = x - \beta$. From (2), (4) and (6) there holds

\[a_2u'(a_1) - a_3u''(a_1) \geq 0, \quad u(a_2) = 0,\]
\[\gamma_2u'(a_3) + \gamma_3u''(a_3) \geq 0,\]

\[a_2v'(a_1) - a_3v''(a_1) \leq 0, \quad v(a_2) = 0,\]
\[\gamma_2v'(a_3) + \gamma_3v''(a_3) \leq 0.\]

We are to prove

\[u(t) \leq 0, \quad v(t) = 0, \ \forall \ t \in I_1,\]
\[u(t) \geq 0, \quad v(t) \leq 0, \ \forall \ t \in I_2,\]
\[u'(t) \geq 0, \quad v'(t) \leq 0, \ \forall \ t \in I.\]

First we prove that $u'(t) \geq 0, \ \forall \ t \in I$.

Suppose that $u'(a_i) < 0$. Then from the first condition of (11) we have $u''(a_i) \leq 0$. First, consider the case $u''(a_i) < 0$. Hence there exists such $t_0 \in (a_i, a_3]$ that $\forall \ t \in [a_i, t_0]$ $u'(t) < 0$ and $\forall \ t \in [a_i, t_0]$ $u''(t) < 0$. Suppose that $u''(t_0) = 0$ and investigate the value $u''(t_0)$. Let there, e.g., for values $x(t_0), \alpha(t_0), \beta(t_0)$ hold $\beta(t_0) - \alpha(t_0) \leq u(t_0) \leq 0$ and $t_0 \in I_1$. Then by (3), (8) and the hypothesis that $f$ is non-decreasing in $x$ we get

\[u''(t_0) = x''(t_0) - \alpha''(t_0) \leq f(t_0, x(t_0), \alpha'(t_0), \alpha''(t_0)) + \frac{x'(t_0) - \alpha'(t_0)}{1 + x'(t_0)} - f(t_0, \alpha(t_0), \alpha'(t_0), \alpha''(t_0)) < 0.\]

From this result there follows that $u''(t) > 0$ in some left pure neighbourhood of $t_0$, which is a contradiction; hence we have $u''(t_0) < 0$. Similarly we proceed in further relations among $x(t_0), \alpha(t_0)$ and $\beta(t_0)$. Let us form the set $M = \{t_0 \in (a_i, a_3]: u'(t) < 0 \text{ on } [a_i, t_0] u''(t) < 0 \text{ on } [a_i, t_0]\}$. Let $t^*_0 = \sup M$. From above we have $u''(t_0) < 0$, for any $t_0 \in M$, and since $t^*_0 \in M, u''(t^*_0) < 0$. Let $t^*_0 < a_3$. Then there exists such a $t_1 \in (t^*_0, a_3]$ that has the property of elements of $M$, which is a contradiction. Therefore we can put $t^*_0 = a_3$. When $u'(a_3) < 0, u''(a_3) < 0$, we obtain a contradiction with the third condition in (11). Similarly we also can
delete the case $u'(a_i) < 0, u''(a_i) = 0$. Hence $u'(a_i) \geq 0$. Finally we have that both $u'(t_0) < 0$ and $u''(t_0) \leq 0$ cannot hold at any $t_0 \in [a_1, a_3]$.

Let at some $t_2 \in (a_1, a_3)$ $u'(t_2) < 0$ and $u''(t_2) \geq 0$. Then in the same way as above we can prove that $\forall t \in [a_1, t_2) u'(t) < 0$ and $u''(t) > 0$ from which for $t = a_1$ we get a contradiction with the first condition in (11) and also with the result $u'(a_1) \geq 0$.

Thus we have just proved that $\forall t \in I$ $u'(t) \geq 0$ holds. The proof of a fact that $u'(t) \leq 0 \forall t \in I$ is analogical. From the 2nd and 5th conditions in (11) other inequalities in (12) follow. This completes the proof.

**Lemma**: Let the function $f$ be continuous on $I \times \mathbb{R}^3$.

Let there exist functions $\alpha(t), \beta(t) \in C^3(I)$ such that $\alpha(a_2) = \beta(a_2) = A_2$ and let (7) hold.

Finally let there exist a positive constant $L$ so that

$$|f(t, x, x', x'', t_0, x', x'', t_0)| \leq L|x'' - y''|,$$

$$\forall (t, x, x') \in \omega, \quad x'', y'' \in \mathbb{R}.$$

Then there exists such a positive constant $R_2$ that for any solution $x(t) \in C^3(I)$ of (1) on $I$ which satisfies $(t, x(t), x'(t)) \in \omega \forall t \in I$ we have

$$|x''(t)| \leq R_2, \forall t \in I.$$

**Proof.** Consider a solution $x(t)$ of (1) which fulfils the hypotheses of the Lemma. Then by (13) we obtain

$$|f(t, x(t), x'(t), x''(t)) - f(t, x(t), x'(t), A''(t))| \leq L|x''(t) - A''(t)|,$$

$$|x'''(t)| = |f(t, x(t), x'(t), x''(t))| \leq L|x''(t) - A''(t)| \leq L|A''(t)| + |f(t, x(t), x'(t), A''(t))| \leq L|A''(t)| + \max_{\omega}(L|A''(t)| + |f(t, x, x', A''(t))|).$$

Denote by $m$ the maximum value in the last expression and put $\phi(s) = Ls + m$. Further, if we denote $R_1 = \max_{\omega}(\max_I |\alpha'(t)|, \max_I |\beta'(t)|)$, then by [3, Lemma 5] there is warranted the existence of a constant $R_2$ which depends only on $\phi(s), R_1$ and $n = a_3 - a_1$.

We note that [3, Lemma 5], the so-called Nagumo lemma, is derived from Lemma 5.1 in [2, pg. 503].

**Theorem 2.** Let all assumptions of Theorem 1 be fulfilled except the boundedness of the function $f$ which need not be satisfied. Further let the hypotheses of Lemma be satisfied and let for the constant $R_2$ from Lemma $R_2 \geq |\alpha''(t)|, |\beta''(t)|, \forall t > I$ hold.
Then there exists at least one solution \( x \) of BVP (1) and (2) for which (9) is true.

Proof. Define a function \( \Phi(t, x, x', x'') \) on \( I \in \mathbb{R}^3 \) by means of \( f \) as follows: let there hold for \( \Phi \) on \( \omega \in \mathbb{R}^3 \)

\[
\Phi(t, x, x', x'') = f(t, x, x', \delta(-R_2, x'', R_2))
\]

and let there be an extension to the whole domain of definition \( I \times \mathbb{R}^3 \)

\[
\Phi(t, x, x', x'') =
\begin{cases}
\Phi(t, \delta(\beta(t), x, a(t)), \delta(\alpha'(t), x', \beta'(t)), x''), & \forall t \in I_1 \\
\Phi(t, \delta(\alpha(t), x, \beta(t)), \delta(\alpha'(t), x', \beta'(t)), x''), & \forall t \in I_2.
\end{cases}
\]

Consider a modified equation to (1)

\[
(15) \quad x''' = \Phi(t, x, x', x'').
\]

The function \( \Phi \) is (on \( I \times \mathbb{R}^3 \)) continuous and bounded. Further, \( \Phi \) is non-decreasing in \( x \) for \( t \in I_1 \) and \( x : \beta(t) \leq x \leq \alpha(t) \) and is non-increasing in \( x \) for \( t \in I_2 \) and \( x : \alpha(t) \leq x \leq \beta(t) \). The functions \( \alpha \) and \( \beta \) are also the lower and the upper solutions of BVP (15) and (2), respectively. Thus all assumptions of Theorem 1 are fulfilled for BVP (15) and (2) considering instead of (1) the equation (15). Therefore there exists at least one solution \( x(t) \) of BVP, for which (9) holds. From the definition of \( \Phi \) it follows that \( x(t) \) is also a solution of BVP (1) and (2).

Remark 1. If the definitions of the lower and the upper solutions of BVP (1) and (2) are replaced by new stronger definitions which arise from the original definition after replacing (3) and (5) by the following conditions:

\[
\begin{align*}
\alpha''' &\geq f(t, x, \alpha', \alpha''), \\
\beta''' &\leq f(t, x, \beta', \beta''),
\end{align*}
\]

\( \forall t \in I_1, \ x : \beta(t) \leq x \leq \alpha(t) \) and \( \forall t > I_2, \ x : \alpha(t) \leq x \leq \beta(t), \)

then from the existence Theorems 1 and 2 we can delete the hypotheses that \( f \) is non-increasing and non-decreasing in \( x \).

Remark 2. From the proof of Theorem 2, we get for the above found solution \( x(t) \) of (1) and (2) the estimate of the absolute value of its second derivative, which is expressed by (14). By [3, Lemma 5], the delimiting constant \( R_2 \) can be determined from the equation

\[
\int_{\frac{2R_1}{h}}^{R_2} \frac{s \, ds}{Ls + m} = 2R_1.
\]
REFERENCES


Received September 7, 1984

Katedra matematiky
Vysokej školy dopravy a spojov
ul. Marxa-Engelsa 15
01088 Žilina

ТЕОРЕМЫ СУЩЕСТВОВАНИЯ ДЛЯ НЕКОТОРОЙ НЕЛИНЕЙНОЙ КРАЕВОЙ ЗАДАЧИ ТРЕТЬЕГО ПОРЯДКА

Ján Rusnáč

Резюме

В статье рассматривается краевая задача: \( x''' = f(t, x, x', x'') \), \( a_2 x'(a_2) - a_1 x''(a_1) = A_1 \), \( x(a_2) = A_2 \), \( \gamma_x x'(a_2) + \gamma_x x''(a_2) = A_3 \). Доказаны теоремы существования для решения этой задачи, которое находится между нижним и верхним решением задачи.