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EXISTENCE THEOREMS FOR A CERTAIN NON-LINEAR BOUNDARY VALUE PROBLEM OF THE THIRD ORDER

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Introduction

In the paper we shall investigate a boundary value problem (abbr. BVP) of the form

$$x''' = f(t, x, x', x''), \quad (t, x, x', x'') \in [a_1, a_3] \times R^3, \tag{1}$$

$$\begin{aligned} \alpha_2 x'(a_1) - \alpha_3 x''(a_1) &= A_1 \\ x(a_2) &= A_2 \\ \gamma_2 x'(a_3) + \gamma_3 x''(a_3) &= A_3, \end{aligned} \tag{2}$$

$$\alpha_2, \alpha_3, \gamma_2, \gamma_3 \geqq 0, \quad \alpha_2 + \alpha_3 > 0, \quad \gamma_2 + \gamma_3 > 0, \quad \alpha_2 + \gamma_2 > 0,$$
$$a_1 < a_2 < a_3.$$

Denote $I = [a_1, a_3]$, $I_1 = [a_1, a_2]$, $I_2 = [a_2, a_3]$.

This BVP is a special case of BVP with general linear boundary conditions at boundary points investigated in [3, Theorem 1] where under assumptions that f is continuous and bounded on $I \times R^3$ there was deduced that BVP (1) and (2) has at least one solution.

We shall prove existence theorems for a solution of (1) and (2) which lies between its lower and upper solutions and its derivative also lies between the derivative of lower and upper solutions, all this under hypotheses that both f is bounded and f need not be bounded. Similarly as in [3] there are utilized methods of proofs from [1]. In [1] K. Schmitt considers a non-linear two point BVP of the second order with general linear conditions at boundary points.

A function $\alpha \in C_3(I)$ ($C_3(I)$ is the space of all functions with continuous derivatives of order $\leqq 3$, on I) will be called a *lower solution* of BVP (1) and (2) if there holds

$$\alpha''' \geqq f(t, \alpha, \alpha', \alpha''), \tag{3}$$

$$\alpha_2\alpha'(a_1) - \alpha_3\alpha''(a_1) \leqq A_1, \quad \alpha(a_2) = A_2,$$
$$\gamma_2\alpha'(a_3) + \gamma_3\alpha''(a_3) \leqq A_3. \tag{4}$$

Similarly $\beta \in C_3(I)$ will be an *upper solution* of BVP (1) and (2) if

$$\beta''' \leqq f(t, \beta, \beta', \beta''), \tag{5}$$

$$\alpha_2\beta'(a_1) - \alpha_3\beta''(a_1) \geqq A_1, \quad \beta(a_2) = A_2,$$
$$\gamma_2\beta'(a_3) + \gamma_3\beta''(a_3) \geqq A_3. \tag{6}$$

Moreover let there hold for α and β (from above)

$$\alpha'(t) \leqq \beta'(t), \quad \forall t \in I. \tag{7}$$

Denote

$$\omega_1 = \{(t, x, x'): t \in I_1, \beta(t) \leqq x \leqq \alpha(t), \alpha'(t) \leqq x' \leqq \beta'(t)\},$$
$$\omega_2 = \{(t, x, x'): t \in I_2, \alpha(t) \leqq x \leqq \beta(t), \alpha'(t) \leqq x' \leqq \beta'(t)\},$$
$$\omega = \omega_1 \cup \omega_2,$$

$$\delta(y_1, y_2, y_3) = \begin{cases} y_1, & y_2 < y_1 \leqq y_3 \\ y_2, & y_1 \leqq y_2 \leqq y_3, \\ y_3, & y_1 \leqq y_3 < y_2 \end{cases} \qquad y_1, y_2, y_3 \in R.$$

Let the function f be continuous and bounded on $I \times R^3$ and let there exist functions α, $\beta \in C_3(I)$ that are lower and upper solutions of BVP (1) and (2), respectively. Let us modify the function f to F as follows:

for $t \in I_1$:

$$F(t, x, x', x'') = f(t, \delta(\beta(t), x, \alpha(t)), \delta(\alpha'(t), x', \beta'(t)), x'') + \\ + \frac{x' - \delta(\alpha'(t), x', \beta'(t))}{1 + x'^2}, \tag{8}$$

for $t \in I_2$ F has the same form as for $t \in I_1$, we just must formally interchange the symbols $\alpha(t)$ and $\beta(t)$.

Hypotheses of f, α and β imply that F is continuous and bounded on $I \times R^3$.

Existence theorems

Theorem 1. *Let the function f be continuous and bounded on $I \times R^3$ and let there exist functions α, $\beta \in C_3(I)$ which are lower and upper solutions of BVP* (1) *and* (2), *respectively.*

Let f be non-decreasing in x for $t \in I_1$ and x: $\beta(t) \leqq x \leqq \alpha(t)$ and be non-increasing in x for $t \in I_2$ and x: $\alpha(t) \leqq x \leqq \beta(t)$.

Then there exists at least one solution x of BVP (1) *and* (2) *such that*

$$(t, x(t), x'(t)) \in \omega, \quad \forall t \in I. \tag{9}$$

Proof. Consider a modified equation to (1):

$$x''' = F(t, x, x', x''). \tag{10}$$

From properties of F we have that there exists at least one solution $x(t)$ of BVP (10) and (2). Further we prove that for this solution $x(t)$ the property (9) is fulfilled and therefore with respect to the definition of F $x(t)$ is also the solution of BVP (1) and (2).

Let us form a function $u = x - \alpha$, $v = x - \beta$. From (2), (4) and (6) there holds

$$\begin{gathered} \alpha_2 u'(a_1) - \alpha_3 u''(a_1) \geqq 0, \quad u(a_2) = 0, \\ \gamma_2 u'(a_3) + \gamma_3 u''(a_3) \geqq 0, \\ \alpha_2 v'(a_1) - \alpha_3 v''(a_1) \leqq 0, \quad v(a_2) = 0, \\ \gamma_2 v'(a_3) + \gamma_3 v''(a_3) \leqq 0. \end{gathered} \tag{11}$$

We are to prove

$$\begin{aligned} u(t) &\leqq 0, & v(t) &\geqq 0, & &\forall t \in I_1, \\ u(t) &\geqq 0, & v(t) &\leqq 0, & &\forall t \in I_2, \\ u'(t) &\geqq 0, & v'(t) &\leqq 0, & &\forall t \in I. \end{aligned} \tag{12}$$

First we prove that $u'(t) \geqq 0$, $\forall t \in I$.

Suppose that $u'(a_1) < 0$. Then from the first condition of (11) we have $u''(a_1) \leqq 0$. First, consider the case $u''(a_1) < 0$. Hence there exists such $t_0 \in (a_1, a_3]$ that $\forall t \in [a_1, t_0]$ $u'(t) < 0$ and $\forall t \in [a_1, t_0)$ $u''(t) < 0$. Suppose that $u''(t_0) = 0$ and investigate the value $u'''(t_0)$. Let there, e.g., for values $x(t_0)$, $\alpha(t_0)$, $\beta(t_0)$ hold $\beta(t_0) - \alpha(t_0) \leqq u(t_0) \leqq 0$ and $t_0 \in I_1$. Then by (3), (8) and the hypothesis that f is non-decreasing in x we get

$$\begin{aligned} u'''(t_0) = x'''(t_0) - \alpha'''(t_0) &\leqq f(t_0, x(t_0), \alpha'(t_0), \alpha''(t_0)) + \\ &+ \frac{x'(t_0) - \alpha'(t_0)}{1 + x'^2(t_0)} - f(t_0, \alpha(t_0), \alpha'(t_0), \alpha''(t_0)) < 0. \end{aligned}$$

From this result there follows that $u''(t) > 0$ in some left pure neighbourhood of t_0, which is a contradiction; hence we have $u''(t_0) < 0$. Similarly we proceed in further relations among $x(t_0)$, $\alpha(t_0)$ and $\beta(t_0)$. Let us form the set $M = \{t_0 \in \in (a_1, a_3] : u'(t) < 0 \text{ on } [a_1, t_0]\ u''(t) < 0 \text{ on } [a_1, t_0)\}$. Let $t_0^* = \sup M$. From above we have $u''(t_0) < 0$, for any $t_0 \in M$, and since $t_0^* \in M$, $u''(t_0^*) < 0$. Let $t_0^* < a_3$. Then there exists such a $t_1 \in (t_0^*, a_3]$ that has the property of elements of M, which is a contradiction. Therefore we can put $t_0^* = a_3$. When $u'(a_3) < 0$, $u''(a_3) < 0$, we obtain a contradiction with the third condition in (11). Similarly we also can delete the case $u'(a_1) < 0, u''(a_1) = 0$. Hence $u'(a_1) \geqq 0$. Finally we have that both $u'(t_0) < 0$ and $u''(t_0) \leqq 0$ cannot hold at any $t_0 \in [a_1, a_3)$.

Let at some $t_2 \in (a_1, a_3]$ $u'(t_2) < 0$ and $u''(t_2) \geqq 0$. Then in the same way as above we can prove that $\forall t \in [a_1, t_2)$ $u'(t) < 0$ and $u''(t) > 0$ from which for $t = a_1$ we get a contradiction with the first condition in (11) and also with the result $u'(a_1) \geqq 0$.

Thus we have just proved that $\forall t \in I$ $u'(t) \geqq 0$ holds. The proof of a fact that $v'(t) \leqq 0$ $\forall t \in I$ is analogical. From the 2nd and 5th conditions in (11) other inequalities in (12) follow. This completes the proof.

Lemma: *Let the function f be continuous on $I \times R^3$.*

Let there exist functions $\alpha(t), \beta(t) \in C_3(I)$ such that $\alpha(a_2) = \beta(a_2) = A_2$ and let (7) hold.

Finally let there exist a positive constant L so that

$$|f(t, x, x', x'') - f(t, x, x', y'')| \leqq L|x'' - y''|, \quad \forall (t, x, x') \in \omega, \quad x'', y'' \in R. \tag{13}$$

Then there exists such a positive constant R_2 that for any solution $x(t) \in C_3(I)$ of (1) on I which satisfies $(t, x(t), x'(t)) \in \omega$ $\forall t \in I$ we have

$$|x''(t)| \leqq R_2, \quad \forall t \in I. \tag{14}$$

Proof. Consider a solution $x(t)$ of (1) which fulfils the hypotheses of the Lemma. Then by (13) we obtain

$$|f(t, x(t), x'(t), x''(t)) - f(t, x(t), x'(t), \alpha''(t))| \leqq L|x''(t) - \alpha''(t)|,$$

$$\begin{aligned} |x'''(t)| &= |f(t, x(t), x'(t), x''(t))| \leqq \\ &\leqq L|x''(t) - \alpha''(t)| + |f(t, x(t), x'(t), \alpha''(t))| \leqq \\ &\leqq L|x''(t)| + L|\alpha''(t)| + |f(t, x(t), x'(t), \alpha''(t))| \leqq \\ &\leqq L|x''(t)| + \max_{\omega} (L|\alpha''(t)| + |f(t, x, x', \alpha''(t))|). \end{aligned}$$

Denote by m the maximum value in the last expression and put $\varphi(s) = Ls + m$. Further, if we denote $R_1 = \max(\max_I |\alpha'(t)|, \max_I |\beta'(t)|$, then by [3, Lemma 5] there is warranted the existence of a constant R_2 which depends only on $\varphi(s)$, R_1 and $h = a_3 - a_1$.

We note that [3, Lemma 5], the so-called Nagumo lemma, is derived from Lemma 5.1 in [2, pg. 503].

Theorem 2. *Let all assumptions of Theorem 1 be fulfilled except the boundedness of the function f which need not be satisfied. Further let the hypotheses of Lemma be satisfied and let for the constant R_2 from Lemma $R_2 \geqq |\alpha''(t)|, |\beta''(t)|$, $\forall t > I$ hold.*

Then there exists at least one solution x of BVP (1) *and* (2) *for which* (9) *is true.*

Proof. Define a function $\Phi(t, x, x', x'')$ on $I \in R^3$ by means of f as follows: let there hold for Φ on $\omega \in R$

$$\Phi(t, x, x', x'') = f(t, x, x', \delta(-R_2, x'', R_2))$$

and let there be an extension to the whole domain of definition $I \times R^3$

$$\Phi(t, x, x', x'') =$$

$$= \begin{cases} \Phi(t, \delta(\beta(t), x, \alpha(t)), \delta(\alpha'(t), x', \beta'(t)), x''), & \forall\, t \in I_1 \\ \Phi(t, \delta(\alpha(t), x, \beta(t)), \delta(\alpha'(t), x', \beta'(t)), x''), & \forall\, t \in I_2 . \end{cases}$$

Consider a modified equation to (1)

$$x''' = \Phi(t, x, x', x''). \tag{15}$$

The function Φ is (on $I \times R^3$) continuous and bounded. Fruther, Φ is non-decreasing in x for $t \in I_1$ and x: $\beta(t) \leqq x \leqq \alpha(t)$ and is non-increasing in x for $t \in I_2$ and x: $\alpha(t) \leqq x \leqq \beta(t)$. The functions α and β are also the lower and the upper solutions of BVP (15) and (2), respectively. Thus all assumptions of Theorem 1 are fulfilled for BVP (15) and (2) considering instead of (1) the equation (15). Therefore there exists at least one solution $x(t)$ of BVP, for which (9) holds. From the definition of Φ it follows that $x(t)$ is also a solution of BVP (1) and (2).

Remark 1. If the definitions of the lower and the upper solutions of BVP (1) and (2) are replaced by new stronger definitions which arise from the original definition after replacing (3) and (5) by the following conditions:

$$\alpha''' \geqq f(t, x, \alpha', \alpha''), \quad \beta''' \leqq f(t, x, \beta', \beta''),$$

$$\forall\, t \in I_1,\ x: \beta(t) \leqq x \leqq \alpha(t) \quad \text{and} \quad \forall\, t > I_2,\ x: \alpha(t) \leqq x \leqq \beta(t),$$

then from the existence Theorems 1 and 2 we can delete the hypotheses that f is non-increasing and non-decreasing in x.

Remark 2. From the proof of Theorem 2, we get for the above found solution $x(t)$ of (1) and (2) the estimate of the absolute value of its second derivative, which is expressed by (14). By [3, Lemma 5], the delimiting constant R_2 can be determined from the equation

$$\int_{\frac{2R_1}{h}}^{R_2} \frac{s\, \mathrm{d}s}{Ls + m} = 2R_1 .$$

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ТЕОРЕМЫ СУЩЕСТВОВАНИЯ ДЛЯ НЕКОТОРОЙ НЕЛИНЕЙНОЙ КРАЕВОЙ ЗАДАЧИ ТРЕТЬЕГО ПОРЯДКА

Ján Rusnák

Резюме

В статье рассматривается краевая задача: $x''' = f(t, x, x', x'')$, $\alpha_2 x'(a_1) - \alpha_3 x''(a_1) = A_1$, $x(a_2) = A_2$, $\gamma_2 x'(a_3) + \gamma_3 x''(a_3) = A_3$. Доказаны теоремы существования для решения этой задачи, которое находится между нижним и верхним решением задачи.