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NOTE ON THE INTEGRAL WITH RESPECT TO THE PRE-MEASURE

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In [1] the integration process was defined with respect to the pre-measure (non-negative, monotone, in an empty set vanishing set function) and it was shown that the integrability of the function $|f|$ implies the integrability of f. In [2] it was proved that the integrability of f and $|f|$ is equivalent for a wide class of pre-measures, namely for strong submeasures (for definitions see below). The question arises whether this equivalence holds in the case of the general pre-measure too. We give an example which shows that the answer to this question is in general in the negative. Our pre-measure will be a continuous strong supermeasure.

First we recall the definition of the integral given in [1]. Let $(X, \mathscr{S})$ be a measurable space and let μ be a pre--measure on $\mathscr{S}$. Let $\mathscr{F}$ be a family of all finite subsets of $\langle -\infty, \infty\rangle$ which contain zero. Let $F\in\mathscr{F}$ with

$$F=\{b_m\leqq b_{m-1}\leqq\ldots\leqq b_0=0=a_0\leqq a_1\leqq\ldots\leqq a_{n-1}\leqq a_n\}$$

and let f be an $\mathscr{S}$-measurable function.
We put

$$S(f, F)=\sum_{1\leqq i\leqq n}(a_i-a_{i-1})\mu(\{x\,;f(x)\geqq a_i\})+$$
$$+\sum_{1\leqq j\leqq m}(b_j-b_{j-1})\mu(\{x\,;f(x)\leqq b_j\})$$

if the right-hand side expression contains no expression of the type $\infty-\infty$.

Since $\mathscr{F}$ is directed by inclusion, the triple $(S(f, F), \mathscr{F}, \supset)$ is a net. We put

$$I\mu f=\int f\,\mathrm{d}\mu=\lim_{F\in\mathscr{F}} S(f, F)$$

if the limit exists. The function f is called integrable iff $I\mu f$ is finite.

The properties of $I\mu f$ which we shall mainly use are:
(1) $I\mu$ is a monotone functional.
(2) If f^+ and f^- are integrable, then f is integrable and $I\mu f = I\mu f^+ + I\mu f^-$.
(3) If the function $|f|$ is integrable, then f is also integrable.

Now we recall the definitions of a continuous pre-measure, a strong submeasure and a strong supermeasure.

Let $(X, \mathcal{S})$ be a measurable space. The pre-measure μ defined on $\mathcal{S}$ is

(a) a strong submeasure if

$$\mu(A \cap B) + \mu(A \cup B) \leqq \mu(A) + \mu(B)$$

(b) a strong supermeasure if

$$\mu(A \cap B) + \mu(A \cup B) \geqq \mu(A) + \mu(B)$$

for every A and B in $\mathcal{S}$.

We say that the pre-measure μ defined on $\mathcal{S}$ is continuous if it has the following two properties

(1) $A_n \nearrow A \Rightarrow \mu(A_n) \nearrow \mu(A)$
(2) $A_n \searrow A,\ \mu(A_1) < \infty \Rightarrow \mu(A_n) \searrow \mu(A)$
for $A \in \mathcal{S}$ and $A_n \in \mathcal{S}$, $n = 1, 2, 3, \ldots$

Further we shall need the following lemma, which is an easy consequence of Lemma 1 proved in [2].

Lemma 1. *Let μ be a finite measure on $\mathcal{S}$. Let f be a real valued, increasing, convex, continuous function with $f(0) = 0$. Then the set function v defined on $\mathcal{S}$ by $v(A) = f(\mu(A))$ is a continuous strong supermeasure.*

Before we give the promised example we shall prove this lemma.

Lemma 2. *Let $X = \left\langle -\frac{1}{4}, \frac{1}{4} \right\rangle$, let $\mathcal{B}(X)$ be the family of all Borel subsets of X and μ the Lebesgue measure on X. Let g be a function defined by $g(x) = \exp\left(-\frac{1}{x}\right)$ for $x \in (0, \infty)$ and $g(0) = 0$. Then the set function v defined on $\mathcal{B}(X)$ by $v(A) = g(\mu(A))$ is a continuous strong supermeasure on $\mathcal{B}(X)$.*

Proof. It is clear that $0 \leqq \mu(A) \leqq \frac{1}{2}$ for every $A \in \mathcal{B}(X)$. Since g is a continuous, convex, increasing real function on the interval $\left\langle 0, \frac{1}{2} \right\rangle$ with $g(0) = 0$, by Lemma 1 we get that $v = g(\mu)$ is a continuous strong supermeasure on $\mathcal{B}(X)$.

Now we give an example which shows that in the case of the integral with respect to the pre-measure the integrability of f and $|f|$ is in general not equivalent.

Example. *Let $X = \left\langle -\frac{1}{4}, \frac{1}{4} \right\rangle$. Put*

$$f(x) = \begin{cases} \exp\left(\frac{1}{2x}\right) & x \in \left(0, \frac{1}{4}\right\rangle \\ 0 & x = 0 \\ -\exp\left(-\frac{1}{2x}\right) & x \in \left\langle -\frac{1}{4}, 0\right) \end{cases}$$

Let ν be a continuous strong supermeasure on $\mathcal{B}(X)$ from Lemma 2. Then the function f is integrable on X with respect to ν and $|f|$ is not integrable.

Proof. Let h be a function defined on $\left\langle 0, \frac{1}{4} \right\rangle$ by

$$h(x) = \begin{cases} 0 & x = 0 \\ k+1 & x \in \left(\frac{1}{2 \ln (k+1)}, \frac{1}{2 \ln k} \right\rangle \quad k \geqq 8 \\ 8 & x \in \left(\frac{1}{2 \ln 8}, \frac{1}{4} \right\rangle \end{cases}$$

The range of h is the set $H = \{0, 8, 9, \ldots\}$. Since the function h is non-negative, $I_\nu h$ exists. It is easy to see that

$$I_\nu h = \lim_k \left[8\nu\left(\left(0, \frac{1}{4}\right)\right) + \sum_{i=9}^{k} \nu\left(\left(0, \frac{1}{2 \ln (i-1)}\right)\right)\right] =$$

$$= 8 \exp(-4) + \lim_k \sum_{i=9}^{k} \exp\left(- \frac{1}{\frac{1}{2 \ln (i-1)}} \right) =$$

$$= 8 \exp(-4) + \lim_k \sum_{i=9}^{k} \left(\frac{1}{(i-1)^2} \right) =$$

$$= 8 \exp(-4) + \sum_{k=1}^{\infty} \frac{1}{k^2} - \sum_{k=1}^{7} \frac{1}{k^2} =$$

$$= 8 \exp(-4) + \frac{\pi}{6} - \sum_{k=1}^{7} \frac{1}{k^2}$$

Hence h is integrable on $\left\langle 0, \frac{1}{4} \right\rangle$.

Since in the interval $\left\langle 0, \frac{1}{4} \right\rangle f^+ = f \leqq h$ and I_ν is a monotone functional, f^+ is also integrable on $\left\langle 0, \frac{1}{4} \right\rangle$. As $f^+ = 0$ on $\left\langle -\frac{1}{4}, 0 \right\rangle$, we get that f^+ is integrable on X.

The integrability of f^- can be shown similarly.

Since f^+ and f^- are integrable on X, by the property (2) of I_ν we get that f is integrable on X.

To show that $|f|$ is not integrable it is enough to find a function φ defined on X with $0 \leqq \varphi \leqq |f|$ and $I_\nu \varphi = \infty$.

Put

$$\varphi(x)=\begin{cases} 7 & x\in\left\langle -\frac{1}{4}, -\frac{1}{2\ln 8}\right) \cup \left(\frac{1}{2\ln 8}, \frac{1}{4}\right\rangle \\ k & x\in\left\langle -\frac{1}{2\ln k}, -\frac{1}{2\ln(k+1)}\right) \cup \left(\frac{1}{2\ln(k+1)}, \frac{1}{2\ln k}\right\rangle \quad k\geqq 8 \\ 0 & x=0 \end{cases}$$

Since φ is non-negative, $I_\nu\varphi$ exists. It is clear that

$$I_\nu\varphi=\lim_k \sum_{1\leqq i\leqq k}(i-(i-1))\nu(\{x;\varphi(x)\geqq i\})=$$
$$=7\exp(-2)+\lim_k \sum_{8\leqq i\leqq k}\exp\left(-\frac{2}{\frac{1}{2\ln i}}\right)=$$
$$=7\exp(-2)+\lim_k \sum_{8\leqq i\leqq k}\frac{1}{i}=7\exp(-2)+\sum_{k=8}^{\infty}\frac{1}{k}$$

Using the fact that the series $\sum_{k=8}^{\infty}\frac{1}{k}$ is divergent we have $I_\nu\varphi=+\infty$, which means that φ is not integrable on X with respect to ν.

Thus we found a function φ defined on X with the properties

$$0\leqq\varphi\leqq|f| \quad \text{and} \quad I_\nu\varphi=+\infty.$$

Hence we have $I_\nu|f|=+\infty$ because I_ν is a monotone functional. This implies that $|f|$ is not integrable on X with respect to ν.

REFERENCES

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[2] ŠIPOŠ, J.: Non linear integrals. Math. Slovaca 29, 1979, 257—270.

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ЗАМЕЧАНИЕ К ИНТЕГРАЛУ ПО ПРЕДМЕРЕ

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Резюме

В статье приведен пример, кторый показывает, что для интеграла по предмере, введенного в [1], не верно, что функция f интегрируема тогда и только тогда, когда $|f|$ интегрируема. Вышеприведенный пример показывает, что существует функция f и предмера μ, что интеграл от f по μ существует, по функция $|f|$ уже не интегрируема.