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INTEGRATING BOUNDED FUNCTIONS FOR THE DOBRAKOV INTEGRAL

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In [5] and [6], I. Dobrakov has developed a theory for the integration of vector-valued functions with respect to operator-valued measures which is much more general than the, perhaps-better-known, integration theory developed by R. Bartle in [1] (cf. [4] II. 4). Due to the generality of the Dobrakov integral, it is even non-trivial to integrate bounded mesurable functions as is evidenced by Example 7′ of [5]; this should be contrasted with Theorem 3 of [1] and the restrictive definition of measurability employed in [1]. In Theorem 5 of [5], Dobrakov shows that under suitable restrictions on the measure μ, it is indeed true that all bounded measurable functions are μ-integrable. In this note, we point out that in a certain sense Dobrakov's result in Theorem 5 is best possible. We then use this result to make several remarks pertaining to various other results of [5].

Let X, Y be (real) B-spaces and $L(X, Y)$ the space of bounded linear operators from X into Y. If Σ is a σ-algebra of subsets of a set S and $\mu: \Sigma \to L(X, Y)$ is finitely additive, the semi-variation of μ is defined by

$$\hat{\mu}(E) = \sup \left\| \sum_{k=1}^{n} \mu(E_k) x_k \right\|,$$

where the supremum is taken over all partitions $\{E_k\}$ of E and all $x_k \in X$, $\|x_k\| \leq 1$. Let $bca(\Sigma, L(X, Y))$ be the space of all vector measures $\mu: \Sigma \to L(X, Y)$ which have bounded semi-variation and are countably additive in the uniform operator topology of $L(X, Y)$.

The finitely additive set function $\mu: \Sigma \to L(X, Y)$ is strongly bounded (continuous in [5]) if $\hat{\mu}(E_j) \to 0$ whenever $\{E_j\}$ from Σ decreases to the empty set. If μ is strongly bounded, then μ has finite semi-variation ([6] Th. 5), and since $\|\mu(E)\| \leq \hat{\mu}(E)$, $\mu \in bca(\Sigma, L(X, Y))$. The converse is false, i. e., μ may belong to $bca(\Sigma, L(X, Y))$ and fail to strongly bounded (Example 7 of [5]; see also the example constructed in Theorem 1 below). However, if the space Y contains no subspace isomorphic to c_0 then every $\mu \in bca(\Sigma, L(X, Y))$ is strongly bounded ([5], *-Theorem).

Throughout this paper, the term integral will refer to the integral of Dobrakov developed in [5], [6]. Let $\mu: \Sigma \to L(X, Y)$ have bounded semi-variation and be countably additive with respect to the strong operator topology of $L(X, Y)$. A measurable function $f: S \to X$ is said to be scalarly μ-integrable if f is $y'\mu$-integrable for each $y' \in Y'$; $y'\mu: \Sigma \to X'$ is the measure defined by $\langle y'\mu(E), x \rangle = \langle y', \mu(E)x \rangle$ (the term weakly integrable is used in [5]). If f is scalarly μ-integrable, then $y' \to \int_E f \, dy'\mu$ defines an element of Y'' for each $E \in \Sigma$, and we denote this element by $\int_E f \, d\mu$. If f is scalarly μ-integrable, then f is μ-integrable iff $\int_E f \, d\mu \in Y$ for each $E \in \Sigma$ ([8]). It follows from Theorem 5 of [5] that every bounded measurable function is scalarly μ-integrable; however, a bounded measurable function may fail to be integrable (Example 7′ of [5]).

From Theorem 5 of the *-Theorem of [5], it follows that if Y contains no subspace isomorphic to c_0, then every bounded measurable function is μ-integrable for every $\mu \in bca(\Sigma, L(X, Y))$. The following theorem shows that in a very real sense this result is best possible.

In what follows $\mathcal{P}$ will denote the power set of the positive integers N.

Theorem 1. *Let X be infinite dimensional. Then there exist $\mu \in bca(\mathcal{P}, L(X, c_0))$ and a bounded measurable function $f: N \to X$ such that f is not μ-integrable.*

Proof: By Corollary 2.3 of [7], there is a bounded sequence $\{x'_j\} \subseteq X'$ such that $x_j \to 0$ weak* and $\inf_j \|x'_j\| > 0$. Let $\sigma_j = \{2^{j-1}, 2^{j-1}+1, \dots 2^j - 1\}$ and note σ_j contains 2^{j-1} integers. Let $\{e_j\}$ be the canonical basis vertors in c_0, $e_j = \{\delta_{jk}\}_{k=1}^{\infty}$ and set $y_j = (1/2^{j-1})e_j$. For $k \in N$ define $T_k \in L(X, c_0)$ by $T_k x = \langle x'_j, x \rangle y_j$ where $k \in \sigma_j$. The series ΣT_k is subseries convergent in the strong operator topology since for $x \in X$, $\sum_{k=1}^{\infty} T_k x = \sum_{j=1}^{\infty} \langle x'_j, x \rangle e_j$, and, moreover, since $\|T_k\| \to 0$, the series ΣT_k is norm-subseries convergent (see the proof of Theorem IV.1.1 of [3]). Define $\mu: \mathcal{P} \to L(X, c_0)$ by $\mu(\sigma) = \sum_{k \in \sigma} T_k$. By the observations above μ is countably additive in the norm topology and $\hat{\mu}(N) \leq \sup_j \|x'_j\|$ so $\mu \in bca(\mathcal{P}, L(X, c_0))$.

For each j pick $x_j \in X$, $\|x_j\| = 1$, such that $\langle x'_j, x_j \rangle + 1/j > \|x'_j\|$. Define $f: N \to X$ by $f(k) = x_j$ where $k \in \sigma_j$. Then f is bounded and $\mathcal{P}$-measurable, but f is not μ-integrable since

$$\langle x'_j, x_j \rangle \not\to 0 \text{ and } \int_N f \, d\mu = \sum_{j=1}^{\infty} \sum_{k \in \sigma_j} \mu(k) x_j = \sum_{j=1}^{\infty} \langle x'_j, x_j \rangle e_j \in l^{\infty} \backslash c_0.$$

Remark 2. The construction of the measure μ in Theorem 1 is motivated by Example 7 of [5]. Note that the function f constructed above is actually an elementary function, where $f: S \to X$ is Σ-elementary if $f = \sum_{j=1}^{\infty} C_{E_j} x_j$ with the $\{E_j\}$ disjoint from Σ and $x_j \in X$. (Here C_E denotes the characteristic function of E.)

Using Theorem 1 we obtain the following Corollary which gives several characterizations of B-spaces not containing a copy of c_0 in terms of integrability for the Dobrakov integral.

Corollary 3. *Let X be infinite dimensional. The following are equivalent:*

(i) *Y contains no subspace isomorphic to c_0,*
(ii) *every bounded function $f: N \to X$ is integrable with respect to each $\mu \in bca(\mathscr{P}, L(X, Y))$,*
(iii) *every bounded elementary $f: N \to X$ is integrable with respect to each $\mu \in bca(\mathscr{P}, L(X, Y)$,*
(iv) *every $\mu \in bca(\mathscr{P}, L(X, Y))$ is strongly bounded,*
(v) *every scalarly μ-integrable function is μ-integrable for each $\mu \in bca(\mathscr{P}, L(X, Y))$.*

Proof: (i) implies (ii) follows from the *-Theorem and Theorem 5 of [5]; (ii) clearly implies (iii) and (iii) implies (i) by Theorem 1. (i) implies (iv) by the *-Theorem of [5]; if (iv) holds, then (ii) holds by Theorem 5 of [5]. (v) implies (ii) since a bounded measurable function is always scalarly integrable ([5], Theorem 5), and (i) implies (v) by Theorem 17 of [5].

Remark 4. Note the equivalence of (ii) and (iv) above shows that the *-Theorem of [5] is also best possible.

Theorem 1 also gives the following characterization of finite dimensional spaces.

Corollary 5. *X is finite dimensional iff every bounded function $f: N \to X$ is μ-integrable with respect to every $\mu \in bca(\mathscr{P}, L(X, c_0))$.*

Proof: If X is finite dimensional, we may assume $X = R$ by treating the coordinate functions of f. Then $L(X, c_0) = c_0$ and $\int_E f \, d\mu = \sum_{k \in E} \mu(k) f(k)$, where the series is norm-subseries convergent in c_0 since the $\{f(k)\}$ are bounded and $\mu: \mathscr{P} \to c_0$ is norm countably additive ([3], p. 59).

The converse follows from Theorem 1.

As noted above Dobrakov shows in Theorem 5 of [5] that if $\mu: \Sigma \to L(X, Y)$ is strongly bounded, than any bounded measurable function is μ-integrable. We show below in Theorem 6 that this result is also best possible.

Theorem 6. *Let $\mu: \Sigma \to L(X, Y)$ have bounded semi-variation and be countably additive in the strong operator topology. If every bounded (elementary) Σ-measurable function is μ-integrable, then μ is strongly bounded.*

Proof: Let $\{E_j\} \subseteq \Sigma$ be disjoint and $\|x_j\| \leq 1$, $x_j \in X$. Set $f = \sum_{j=1}^{\infty} C_{E_j} x_j$, where C_E

denotes the characteristic function of E. Then f is scalarly μ-integrable with $\int_S f\,\mathrm{d}\mu = \sum_{j=1}^{\infty} \mu(E_j)x_j \in Y''$ (Theorems 10 and 17 of [6] applies to the measure $y'\mu$ for each $y' \in Y'$.) By hypothesis $\int_S f\,\mathrm{d}\mu \in Y$ and $\left\langle y', \int_S f\,\mathrm{d}\mu \right\rangle = \sum_{j=1}^{\infty} y'\mu(E_j)x_j$ for $y' \in Y'$ (Theorem 17 of [6]). Thus, $\Sigma\mu(E_j)x_j$ is weak-subseries convergent and, therefore, norm-subseries convergent by the Orlicz—Pettis Theorem ([3] p. 60). By [2], Lemma 3.1, μ is strongly bounded.

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ИНТЕГРИРОВАНИЕ ОГРАНИЧЕННЫХ ФУНКЦИЙ ДЛЯ ИНТЕГРАЛА ДОБРАКОВА

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Резюме

В статье рассматривается интегрирование векторных функций по операторной мере в смысле Добракова. При некоторых удобных ограничениях наложенных на меру, все ограниченные измеримые функции интегрируемы. Показано, что, в некотором смысле, этот результат лучший возможный, и приводятся также некоторие дальнейшие результаты.