Zbigniew Grande  
On Borsík's problem concerning quasiuniform limits of Darboux quasicontinuous functions


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ON BORSÍK’S PROBLEM CONCERNING
QUASIUNIFORM LIMITS OF DARBOUX
QUASICONTOINUOUS FUNCTIONS

ZBIGNIEW GRANDE

(Communicated by Ladislav Mišik)

ABSTRACT. It is proved that every cliquish function \( f : \mathbb{R} \to \mathbb{R} \) is a quasiuniform limit of a sequence of Darboux quasicontinuous functions.

Let \( \mathbb{R} \) be the set of all reals. A function \( f : \mathbb{R} \to \mathbb{R} \) is said to be \textit{quasicontinuous} (cliquish) at a point \( x \in \mathbb{R} \) if for every \( \varepsilon > 0 \) and every neighbourhood \( U \) of \( x \) there is a nonempty open set \( V \subset U \) such that \( |f(t) - f(x)| < \varepsilon \) for each \( t \in V \) (osc \( f < \varepsilon \) on \( V \)).

A function \( f \) is \textit{quasicontinuous} (cliquish) if it is such at each point of its domain \( [2] \). A sequence \( (f_n)_n : \mathbb{R} \to \mathbb{R} \), quasiuniformly converges to \( f : \mathbb{R} \to \mathbb{R} \) ([3]) if \( (f_n) \) pointwise converges to \( f \) and

\[
\forall \varepsilon > 0 \ \forall m \ \exists p \ \forall x \in \mathbb{R} : \min\{|f_{m+1}(x) - f(x)|, \ldots, |f_{m+p}(x) - f(x)|\} < \varepsilon.
\]

In the article [1], Borsík proved that every cliquish function \( f : \mathbb{R} \to \mathbb{R} \) is a quasiuniform limit of a sequence of quasicontinuous functions and he puts the following problem:

PROBLEM. ([1]) Let \( f : \mathbb{R} \to \mathbb{R} \) be a cliquish function. Is the function \( f \) a quasiuniform limit of a sequence of Darboux quasicontinuous functions?

In this article, I prove that the answer to the above Borsík’s question is affirmative.

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THEOREM. Let \( f : \mathbb{R} \to \mathbb{R} \) be a cliquish function. There is a sequence of Darboux quasicontinuous functions \( f_n : \mathbb{R} \to \mathbb{R} \) which quasiuniformly converges to \( f \).

In the proof of this theorem, we use the following lemmata:

**Lemma.** If a continuous function \( f : [a, b] \to \mathbb{R} \) and a closed interval \([c, d]\) are such that \( f([a, b]) \subset [c, d] \), then there is a continuous function \( g : [a, b] \to \mathbb{R} \) such that \( g(a) = f(a) \), \( g(b) = f(b) \), and \( g([a, b]) = [c, d] \).

The proof of this lemma is obvious.

**Lemma.** Let \( \varepsilon > 0 \) and let \( f : (a, b) \to \mathbb{R} \) be a function such that for every \( x \in (a, b) \) we have \( \text{osc} f(x) < \varepsilon \). There is a continuous function \( g : (a, b) \to \mathbb{R} \) such that \( |f(x) - g(x)| < 2\varepsilon \) for each \( x \in (a, b) \).

**Proof.** It suffices to prove that for every closed interval \([c, d] \subset (a, b)\) there is a continuous function \( h : [c, d] \to \mathbb{R} \) such that \( h(c) = f(c) \), \( h(d) = f(d) \) and \( |h(x) - f(x)| < 2\varepsilon \) for every \( x \in [c, d] \). Let \([c, d] \subset (a, b)\) be a closed interval. Since \( \text{osc} f(x) < \varepsilon \) for every \( x \in [c, d] \), there are open intervals \( J_i = (a_i, b_i) \), \( i = 1, \ldots, k \), such that \( a_1 < c < a_2 < b_1 < a_3 < b_2 < \cdots < a_k < b_{k-1} < d < b_k \), and \( \text{osc} f < \varepsilon \) on every \( J_i \), \( i = 1, \ldots, k \). In every interval \( (a_{i+1}, b_i) \), \( i = 1, \ldots, k-1 \), we find a point \( x_i \). Let \( x_0 = c \), \( x_k = d \). Put \( h(x_i) = f(x_i) \) for \( i = 0, 1, \ldots, k \) and let \( h \) be linear in every interval \([x_i, x_{i+1}] \), \( i = 0, 1, \ldots, k-1 \). Obviously \( h \) is continuous and \( h(c) = f(c) \) and \( h(d) = f(d) \). Let \( x \in (c, d) \). Then \( x \in (x_i, x_{i+1}) \) for some \( i < k \). Since \([x_i, x_{i+1}] \subset (a_{i+1}, b_{i+1}) \), we have \( \text{osc} f < \varepsilon \) on \([x_i, x_{i+1}] \). Consequently, \( |f(x) - f(x_i)| < \varepsilon \), \( |f(x) - f(x_{i+1})| < \varepsilon \) and \( |h(x) - f(x)| \leq |h(x) - h(x_i)| + |h(x_i) - f(x)| \leq |h(x_{i+1}) - h(x_i)| + |f(x) - f(x_i)| + |f(x_i) - f(x)| < \varepsilon + \varepsilon = 2\varepsilon \). Thus the proof is completed.

**Proof of Theorem.** Put \( A_n = \{ x \in \mathbb{R} ; \text{osc} f(x) \geq 1/n \} \), \( n = 1, 2, \ldots \). Then all sets \( A_n \), \( n = 1, 2, \ldots \), are closed and nowhere dense. Fix a positive integer \( n \). For every component \((a, b)\) of the set \( \mathbb{R} - A_n \), we have \( \text{osc} f(x) < 1/n \) for every \( x \in (a, b) \). So, by Lemma 2, there is a continuous function \( g_{(a, b)} : (a, b) \to \mathbb{R} \) such that \( |f(x) - g_{(a, b)}(x)| < 2/n \) for every \( x \in (a, b) \).

Let

\[
g_n(x) = f(x), \quad \text{for} \quad x \in A_n,
\]

and

\[
g_n(x) = g_{(a, b)}(x)
\]

if \( x \) belongs to some component \((a, b)\) of the set \( \mathbb{R} - A_n \). If \( a > -\infty \), \( i \leq n \), and \( \text{dist}(a, A_i) = \inf \{ |a - x| ; \ x \in A_i \} < 1/n \), then there is a sequence \((I_{i, k})\)
of closed intervals (which depends on \((a, b)\)) such that:

- \( I_{i,k} = [a_{i,k}, b_{i,k}] \subset (a, a + \min(1/n, (b - a)/2)) \cap \{ x \in (a, b) ; \) \( \text{dist}(x, A_i) < 1/n \} \) for \( k = 1, 2, \ldots \);
- \( \lim_{k \to \infty} b_{i,k} = a \);
- \( I_{i,k} \cap I_{j,l} = \emptyset \) if \((i, k) \neq (j, l), i, j \leq n, k, l = 1, 2, \ldots \);
- \( \text{osc } g_n < 1/n \) on every \( I_{i,k}, k = 1, 2, \ldots \).

Similarly, if \( b < \infty, i \leq n, \) and \( \text{dist}(b, A_i) < 1/n \), then we find a sequence of closed intervals \( J_{i,k} = [c_{i,k}, d_{i,k}], k = 1, 2, \ldots \), such that:

- \( J_{i,k} \subset (b - \min(1/n, (b - a)/2), b) \cap \{ x \in (a, b) ; \) \( \text{dist}(x, A_i) < 1/n \} \) for \( k = 1, 2, \ldots \);
- \( \lim_{k \to \infty} c_{i,k} = b \);
- \( J_{i,k} \cap J_{j,l} = \emptyset \) if \((i, k) \neq (j, l), i, j \leq n, k, l = 1, 2, \ldots \);
- \( \text{osc } g_n < 1/n \) on every \( J_{i,k}, k = 1, 2, \ldots \).

For every \( k = 1, 2, \ldots \) and \( i \leq n \) there are closed intervals \( K_{i,k} \supset g_n(I_{i,k}) \) and \( L_{i,k} \supset g_n(J_{i,k}) \) such that \( K_{i,k} \) has the same center as \( g_n(I_{i,k}) \), \( L_{i,k} \) has the same center as \( g_n(J_{i,k}) \), and the diameters \( d(K_{i,k}), d(L_{i,k}) \) are equal to \( 25/(i - 1) \) for \( i > 1 \) and \( K_{1,k} \cap L_{1,k} \supset [-k, k] \) for \( k = 1, 2, \ldots \). By Lemma 1, for every \( k = 1, 2, \ldots \) and \( i \leq n \) there are continuous functions \( s_{n,i,k}, t_{n,i,k} : I_{i,k} \to K_{i,k} \), and \( s_{n,i,k}, t_{n,i,k} : J_{i,k} \to L_{i,k} \) such that \( s_{n,i,k}(I_{i,k}) = K_{i,k}, t_{n,i,k}(J_{i,k}) = L_{i,k}, s_{n,i,k}(a_{i,k}) = g_n(a_{i,k}), s_{n,i,k}(b_{i,k}) = g_n(b_{i,k}), t_{n,i,k}(c_{i,k}) = g_n(c_{i,k}), t_{n,i,k}(d_{i,k}) = g_n(d_{i,k}) \). If \( x \in (a, b) \), then let \( f_{2n-1}(x) = s_{n,i,2k-1}(x) \) for \( x \in I_{i,2k-1}, f_{2n-1}(x) = t_{n,i,2k-1}(x) \) for \( x \in J_{i,2k-1}, i \leq n, k = 1, 2, \ldots \) and let \( f_{2n-1}(x) = g_n(x) \) at other points of \((a, b)\). Moreover, let \( f_{2n-1}(x) = f(x) \) for \( x \in A_n \). Similarly, let \( f_{2n}(x) = s_{n,i,2k}(x) \) for \( x \in I_{i,2k}, f_{2n}(x) = t_{n,i,2k}(x) \) for \( x \in J_{i,2k}, i \leq n, k = 1, 2, \ldots \), \( f_{2n}(x) = g_n(x) \) otherwise in \((a, b)\), and \( f_{2n}(x) = f(x) \) for \( x \in A_n \). Now, we shall prove that the sequence \((f_n)\) pointwise converges to \( f \). If \( x \in A_n \) for some \( n = 1, 2, \ldots \), then \( f_k(x) = f(x) \) for every \( k > 2n - 1 \) and \( \lim_{k \to \infty} f_k(x) = f(x) \). Suppose that \( x \) is not in any \( A_n, n = 1, 2, \ldots \). Fix a positive \( \varepsilon \). There is a positive integer \( n \) such that \( 15/n < \varepsilon \), and a positive integer \( m > n \) such that \( \text{dist}(x, A_n) > 1/m \). Then for \( k > m \) we have \( \text{dist}(x, A_n) > 1/k \), and if \( x \in I_{i,2p-1} \cup J_{i,2p-1} \) for some \( i \) and \( p \), then \( i > n \). Since for \( k > m \) and \( i > n \) we have \( f_{2k-1}(I_{i,2p-1}) = K_{i,2p-1}, f_{2k-1}(J_{i,2p-1}) = L_{i,2p-1}, K_{i,2p-1}(L_{i,2p-1}) \) has the same center as \( g_k(I_{i,2p-1}) \), \( g_k(J_{i,2p-1}) \), \( d(K_{i,2p-1}) = d(L_{i,2p-1}) = 25/(i - 1) \leq 25/n \), and \( d(g_k(I_{i,2p-1})) < 1/k < 1/n \), \( d(g_k(J_{i,2p-1})) < 1/k < 1/n \), we may observe that \( |f_{2k-1}(x) - g_k(x)| < 13/n \). Consequently, for \( k > m \) we have \( |f_{2k-1}(x) - f(x)| \leq |f_{2k-1}(x) - g_k(x)| + |g_k(x) - f(x)| < 13/n + 2/k < 15/n < \varepsilon \), and similarly, \( |f_{2k}(x) - f(x)| < \varepsilon \). So, the sequence \((f_n)\) pointwise converges.
to \( f \). Since \( \min \{ |f_{2n-1}(x) - f(x)|, |f_{2n}(x) - f(x)| \} \leq |g_n(x) - f(x)| < 2/n \) for every \( x \in \mathbb{R} \) and \( n = 1, 2, \ldots \), the above convergence of the sequence \( (f_n) \) is quasicontinuous. We will show that every function \( f_{2n}, \ n = 1, 2, \ldots \), is quasicontinuous. Fix a positive integer \( n \). Since \( f_{2n} \) is continuous at every point \( x \in \mathbb{R} - A_n \), it suffices to prove that it is quasicontinuous at each point \( x \in A_n \).

Fix \( x \in A_n \) and \( \varepsilon > 0 \). If \( x \in A_1 \), then there is an interval \( I_{1,2k} \subset (x - \varepsilon, x + \varepsilon) \) such that \( f_{2n}(x) \in (-k, k) \). Consequently, there is an open interval \( I \subset I_{1,2k} \) such that \( f_{2n}(I) \subset (f_{2n}(x) - \varepsilon, f_{2n}(x) + \varepsilon) \). So, in this case, \( f_{2n} \) is quasicontinuous at \( x \).

If \( x \in A_i - A_{i-1}, \ 1 < i \leq n \), then there is a positive number \( \delta < \varepsilon \) such that \( f < 1/(i - 1) \) on \( (x - \delta, x + \delta) \subset (x - \varepsilon, x + \varepsilon) \). There is an interval \( I_{i,2k} \subset (x - \delta, x + \delta) \). Let \( z \in I_{i,2k} \) be a point. Then \( f(x) < 1/(i - 1) \), and \( |g_n(z) - f(z)| < 2/n < 2/(i - 1) \). Consequently, \( f(x) \in (g_n(z) - 3/(i - 1), g_n(z) + 3/(i - 1)) \), and there is a point \( u \in I_{i,2k} \) such that \( f_{2n}(u) = f(x) \). Since the function \( f_{2n} \) is continuous at \( u \), there is an open interval \( I \subset I_{i,2k} \) such that \( f_{2n}(I) \subset (f(x) - \varepsilon, f(x) + \varepsilon) \). So, \( f_{2n} \) is quasicontinuous at \( x \). The proof of the quasicontinuity of the function \( f_{2n-1} \) is analogous. Now we shall prove that \( f_{2n} \) has the Darboux property. Let \( K \subset \mathbb{R} \) be a closed interval. If \( K \subset \mathbb{R} - A_n \), then \( f_{2n} \) is continuous on \( K \), and \( f_{2n}(K) \) is a connected set in \( \mathbb{R} \). If \( A_1 \cap K \neq \emptyset \), then \( f_{2n}(K) = \mathbb{R} \). Assume that the set \( f_{2n}(K) \) is not connected. Let \( c \in \mathbb{R} \) be such that

\[
A = \{ x \in K; \ f_{2n}(x) < c \} \neq \emptyset, \quad B = \{ x \in K; \ f_{2n}(x) > c \} \neq \emptyset.
\]

and \( f_{2n}(x) \neq c \) for every \( x \in K \). Find a point \( z \in K \cap \text{cl} A \cap \text{cl} B \) (\( \text{cl} \) denotes the closure operation). Evidently, \( z \in A_n \). Since \( z \) is not in \( A_1 \), there is \( i \leq n, \ i > 1 \), such that \( z \in A_i - A_{i-1} \). Assume that \( f_{2n}(z) = f(z) > c \). Since osc \( f(z) < 1/(i - 1) \) and \( |g_n(u) - g_n(v)| \leq |g_n(u) - f(u)| + |f(u) - f(v)| + |f(v) - g_n(v)| < 2/n + |f(u) - f(v)| + 2/n = |f(u) - f(v)| + 4/n \) for all points \( u, v \in \mathbb{R} \), we may observe that osc \( g_n(z) \leq 1/(i - 1) + 4/n < 5/(i - 1) \). Let \( U \) be an open set containing \( z \) such that osc \( g_n < 5/(i - 1) \) on \( U \) and osc \( f < 1/(i - 1) \) on \( U \). Assume that \( g_n(u) < c \) at a point \( u \in U \). Then for every \( x \in U \) we have \( |g_n(x) - c| \leq |g_n(x) - g_n(u)| + |g_n(u) - c| < 5/(i - 1) + (c - g_n(u)) < 5/(i - 1) + (f(z) - g_n(u)) = 5/(i - 1) + |g_n(z) - g_n(u)| < 5/(i - 1) + 5/(i - 1) = 10/(i - 1) \). There is \( I_{i,2k} \subset U \cap \text{int} K \) (or \( J_{i,2k} \subset U \cap \text{int} K \) ). If \( I_{i,2k} \subset U \cap \text{int} K \), then \( g_n(I_{i,2k}) \subset (c - 10/(i - 1), c + 10/(i - 1)) \), and consequently \( c \in K_{i,2k} = f_{2n}(I_{i,2k}) \). Similarly, if \( J_{i,2k} \subset U \cap \text{int} K \), then also \( c \in f_{2n}(J_{i,2k}) \). This contradiction proves that \( g_n(x) > c \) on the set \( U \). Now we find \( I_{i,2k} \subset U \cap \text{int} K \) (or \( J_{i,2k} \subset U \cap \text{int} K \) ). Since \( c \) is not in \( K_{i,2k} \) (or in \( L_{i,2k} \), we obtain that \( g_n(x) > c + 10/(i - 1) \) for \( x \in I_{i,2k} \) (or for \( x \in J_{i,2k} \)). But osc \( g_n < 5/(i - 1) \) on \( K \). So \( g_n(x) \geq c + 5/(i - 1) \) for \( x \in U \). In particular, \( f(z) = g_n(z) \geq c + 5/(i - 1) \).
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Since $z \in \text{cl} A$, there is a point $w \in A \cap U$. Then $f(w) < c$, in a contradiction with the facts $f(z) \geq 5/(i - 1) + c$ and $\text{osc} f < 1/(i - 1)$ on $U$. In the case where $f(z) < c$, the proof is analogous.

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