Ivan Chajda; Radomír Halaš

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ALGEBRAIC PROPERTIES OF PRE-LOGICS

IVAN CHAJDA — RADIMÍR HALAŠ

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ABSTRACT. We introduce the concept of a pre-logic which is an algebra weaker than a Hilbert algebra (an algebraic counterpart of intuitionistic logic) but strong enough to have deductive systems. On every such a pre-logic $\mathcal{A}$ a quasiorder Q can be defined and a Hilbert algebra can be reached as a quotient algebra of $\mathcal{A}$ by the congruence induced by Q. We study algebraic properties of pre-logics and of lattices of their deductive systems.

In early 50-ties, L. Henkin and T. Skolem have introduced the so-called Hilbert algebras to describe algebraically properties of the logic connective implication in intuitionistic logics. The concept of deductive system in Hilbert algebras was introduced by A. Diego [8]. Properties of these deductive systems were systematically treated by A. Diego, W. A. Dudek [9], Y. B. Jun [10] and the authors [3], [4], [5]. However, we feel that the concept of Hilbert algebra is relatively too strong for deductive systems; in other words, they can be introduced and treated in a more general setting. It was the reason we introduce a concept of so-called pre-logic where deductive systems have desired properties and we show that Hilbert algebras rise as quotient algebras of pre-logics by a congruence induced by a natural quasiorder. Moreover, we show that in fact every quasiordered set can be equipped with a suitable binary and nullary operation to become a pre-logic. The paper is intended as a systematical approach to aforementioned concepts where connections with other algebraic concepts as ideals and pseudocomplements are described.

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1. Preliminaries

Hilbert algebras form important tools in algebraic logic because they can be considered as fragments of any intuitionistic propositional logic containing only a logical connective implication and the constant 1 which is considered as the value TRUE. As usually, we denote the binary operation "$\cdot$" (or by juxtaposition, if possible) instead of "$\Rightarrow$" although it has the same meaning.

We recall the formal definition:

DEFINITION 1. A *Hilbert algebra* is a triplet $\mathcal{H} = (H; \cdot, 1)$ where H is a non-empty set, $\cdot$ is a binary operation on H and $1 \in H$ is a fixed element (i.e. a nullary operation) such that the following axioms hold:

(H1) $x \cdot (y \cdot x) = 1$,
(H2) $\big(x \cdot (y \cdot z)\big) \cdot \big((x \cdot y) \cdot (x \cdot z)\big) = 1$,
(H3) $x \cdot y = 1$ and $y \cdot x = 1$ imply $x = y$.

It was proved by A. Diego [8] that the class of all Hilbert algebras forms a variety, i.e. it is determined by a set of identities. The following result is also adopted from [8]:

LEMMA 1. *Every Hilbert algebra satisfies the following identities:*

$$\begin{aligned} x \cdot x &= 1\,, \\ 1 \cdot x &= x\,, \\ x \cdot 1 &= 1\,, \\ x \cdot (y \cdot z) &= y \cdot (x \cdot z)\,, \\ x \cdot (y \cdot z) &= (x \cdot y) \cdot (x \cdot z)\,. \end{aligned}$$

It can be easily checked that the binary relation $\leq$ introduced in a Hilbert algebra $\mathcal{H} = (H; \cdot, 1)$ by setting

$$x \leq y \quad \text{if and only if} \quad x \cdot y = 1$$

is a partial order on H with 1 as the greatest element.

For our next considerations, let us recall several general algebraic concepts. Let $\mathcal{A} = (A, F)$ be an algebra. Denote by $\operatorname{Con}\mathcal{A}$ the set of all congruences of $\mathcal{A}$; of course, $\operatorname{Con}\mathcal{A}$ is an algebraic lattice (i.e. complete and compactly generated with respect to set inclusion. The identity relation ω_A on the set A is the least and the square $A \times A$ the greatest element of $\operatorname{Con}\mathcal{A}$. For an element $a \in A$ and a congruence $\Theta \in \operatorname{Con}\mathcal{A}$ denote by $[a]_\Theta = \{x \in A :\ \langle x, a\rangle \in \Theta\}$ the *class of Θ containing a*. If $\mathcal{A}$ has a constant (i.e. nullary operation) 1, the class $[1]_\Theta$ is called a *kernel* of Θ.

By a *quasiorder* on a set A is meant a reflexive and transitive binary relation on A. In particular, every partial order and every equivalence relation on A are quasiorders on A. If Q is a quasiorder on A, the couple (A, Q) is called a *quasiordered set*. Denote by E_Q the binary relation on A defined as follows:

$$\langle x, y\rangle \in E_Q \quad \text{if and only if} \quad \langle x, y\rangle \in Q \quad \text{and} \quad \langle y, x\rangle \in Q .$$

It is evident that E_Q is an equivalence relation on A; it is called an *equivalence induced by* Q. If (A, Q) is a quasiordered set, one can introduce a binary relation $\leq_Q$ on a quotient set A/E_Q by

$$[a]_{E_Q} \leq_Q [b]_{E_Q} \quad \text{if and only if} \quad \langle a, b\rangle \in Q .$$

It is well known and easy to see that $\leq_Q$ is a partial order on a quotient set A/E_Q; we call the couple $(A/E_Q, \leq_Q)$ an *ordered set assigned to the quasiordered set* (A, Q).

2. Basic properties of pre-logics

At first, we define the concept of a pre-logic formally.

DEFINITION 2. By a *pre-logic* it is meant a triplet $\mathcal{A} = (A; \cdot, 1)$ where A is a non-empty set, $\cdot$ is a binary operation on A and $1 \in A$ is a nullary operation such that the following identities hold:

(P1) $x \cdot x = 1$,
(P2) $1 \cdot x = x$,
(P3) $x \cdot (y \cdot z) = (x \cdot y) \cdot (x \cdot z)$,
(P4) $x \cdot (y \cdot z) = y \cdot (x \cdot z)$.

Comparing Definition 2 with Lemma 1, we conclude that every Hilbert algebra is a pre-logic, i.e. Hilbert algebras are stronger systems than pre-logics. Moreover, pre-logics are determined by identities thus the class of all pre-logics forms a variety.

LEMMA 2. *Let* $\mathcal{A} = (A; \cdot, 1)$ *be a pre-logic. Then*

(a) $x \cdot 1 = 1$;
(b) $x \cdot (y \cdot x) = 1$;
(c) *a binary relation* Q_A *on* A *defined by*

$$\langle x, y\rangle \in Q_A \quad \textit{if and only if} \quad x \cdot y = 1$$

is a quasiorder on A;
(d) $\langle a, 1\rangle \in Q_A$ *for each* $a \in A$;
(e) $\langle 1, a\rangle \in Q_A$ *for* $a \in A$ *implies* $a = 1$.

P r o o f . Applying (P3) for $x = y = z$ we obtain $x \cdot (x \cdot x) = (x \cdot x) \cdot (x \cdot x)$; by (P1) this yields

$$x \cdot 1 = 1 \cdot 1 = 1$$

proving (a).

If we consider (P3) once more with $x = z$, then, by (P1) and (a), we conclude

$$x \cdot (y \cdot x) = (x \cdot y) \cdot (x \cdot x) = (x \cdot y) \cdot 1 = 1$$

proving (b).

Introduce Q_A on A as in (c). Due to (P1), Q_A is reflexive. If $\langle x, y\rangle \in Q_A$ and $\langle y, z\rangle \in Q_A$, then $x \cdot y = 1$ and $y \cdot z = 1$ and, by (a), (P2) and (P3),

$$1 = x \cdot 1 = x \cdot (y \cdot z) = (x \cdot y) \cdot (x \cdot z) = 1 \cdot (x \cdot z) = x \cdot z$$

which yields $\langle x, z\rangle \in Q_A$, i.e. Q_A is also transitive and hence a quasiorder on A.

(d) and (e) follows immediately from (a) and (P2). □

Remark. The quasiorder Q_A of Lemma 2(c) will be called the *induced quasiorder of a pre-logic* $\mathcal{A}$.

LEMMA 3. *Let* Q_A *be the induced quasiorder of a pre-logic* $\mathcal{A} = (A; \cdot, 1)$ *and* $x, y, z \in A$. *If* $\langle x, y\rangle \in Q_A$, *then* $\langle z \cdot x, z \cdot y\rangle \in Q_A$ *and* $\langle y \cdot z, x \cdot z\rangle \in Q_A$.

P r o o f . Suppose $\langle x, y\rangle \in Q_A$. Then $x \cdot y = 1$ and

$$(z \cdot x) \cdot (z \cdot y) = z \cdot (x \cdot y) = z \cdot 1 = 1$$

giving $\langle z \cdot x, z \cdot y\rangle \in Q_A$. Further,

$$\begin{aligned}(y \cdot z) \cdot (x \cdot z) &= x \cdot \big((y \cdot z) \cdot z\big) = \big(x \cdot (y \cdot z)\big) \cdot (x \cdot z)\\ &= \big((x \cdot y) \cdot (x \cdot z)\big) \cdot (x \cdot z) = \big(1 \cdot (x \cdot z)\big) \cdot (x \cdot z)\\ &= (x \cdot z) \cdot (x \cdot z) = 1\end{aligned}$$

proving $\langle y \cdot z, x \cdot z\rangle \in Q_A$. □

We conclude this section by the following essential result:

THEOREM 1. *Let* $\mathcal{A} = (A; \cdot, 1)$ *be a pre-logic,* Q_A *its induced quasiorder and* $\Theta = E_{Q_A}$ *the equivalence induced by* Q_A. *Then*

(1) Θ *is a congruence on* $\mathcal{A}$ *with kernel* $[1]_\Theta = \{1\}$;

(2) *the quotient algebra* $\mathcal{A}/\Theta = \big(A/\Theta; \cdot, [1]_\Theta\big)$ *is a Hilbert one.*

P r o o f . Since Θ is an equivalence on a set A, we need only to show that it has the substitution property with respect to $\cdot$. Suppose $\langle x, y\rangle \in \Theta$ and $\langle z, v\rangle \in \Theta$. Then $\langle x, y\rangle \in Q_A$, $\langle y, x\rangle \in Q_A$, $\langle v, z\rangle \in Q_A$ and $\langle z, v\rangle \in Q_A$.

Applying Lemma 3, we obtain $\langle x \cdot z, x \cdot v\rangle \in Q_A$ and $\langle x \cdot v, y \cdot v\rangle \in Q_A$. Due to transitivity of Q_A, we have $\langle x \cdot z, y \cdot v\rangle \in Q_A$. Analogously it can be shown $\langle y \cdot v, x \cdot v\rangle \in Q_A$ and $\langle x \cdot v, x \cdot z\rangle \in Q_A$ thus also $\langle y \cdot v, x \cdot z\rangle \in Q_A$. Together, we obtain $\langle x \cdot z, y \cdot v\rangle \in \Theta$, i.e. $\Theta \in \operatorname{Con} \mathcal{A}$. Moreover, (e) of Lemma 2 immediately yields $[1]_\Theta = \{1\}$.

The quotient algebra $\mathcal{A}/\Theta$ clearly satisfies all the identities of $\mathcal{A}$. Hence, by (b) of Lemma 2, $\mathcal{A}/\Theta$ satisfies (H1) and, by (P1) and (P3), it satisfies also (H2). Finally, let $x, y \in \mathcal{A}/\Theta$ and $x \cdot y = [1]_\Theta$ and $y \cdot x = [1]_\Theta$. Clearly $x = [a]_\Theta$ and $y = [b]_\Theta$ for some $a, b \in A$. This means $\langle a, b\rangle \in Q_A$ and $\langle b, a\rangle \in Q_A$ thus $\langle a, b\rangle \in \Theta$, i.e. $x = [a]_\Theta = [b]_\Theta = y$ proving (H3), i.e. $\mathcal{A}/\Theta$ is a Hilbert algebra. □

EXAMPLE 1. Let $A = \{a, b, c, 1\}$ and the binary operation is defined by the table

$\cdot$	a	b	c	1
a	1	b	c	1
b	a	1	1	1
c	a	1	1	1
1	a	b	c	1

Then $\mathcal{A} = (A; \cdot, 1)$ is a pre-logic which is not a Hilbert algebra: we have $b \cdot c = c \cdot b = 1$, but $c \neq b$.

3. Deductive systems

The concept of a deductive system of a pre-logic can be induced formally in the same way as for Hilbert algebras (cf. [8]):

DEFINITION 3. Let $\mathcal{A} = (A; \cdot, 1)$ be a pre-logic. A subset $D \subseteq A$ is called a *deductive system of* $\mathcal{A}$ if the following conditions hold:

(d1) $1 \in D$;
(d2) if $x \in D$ and $x \cdot y \in D$, then $y \in D$.

EXAMPLE 2. A pre-logic from Example 1 has the following deductive systems: $\{1\}$, $\{1, a, b, c\}$, $\{1, a\}$ and $\{1, b, c\}$.

Also the concept of an *ideal* was introduced for Hilbert algebras in [4] formally by the same way as for pre-logics:

DEFINITION 4. Let $\mathcal{A} = (A; \cdot, 1)$ be a pre-logic. A nonempty subset I of A is called an *ideal of* $\mathcal{A}$ if the following conditions are satisfied:

(I1) $x \in A$ and $y \in I$ imply $x \cdot y \in I$,
(I2) $x \in A$ and $y_1, y_2 \in I$ imply $\big(y_2 \cdot (y_1 \cdot x)\big) \cdot x \in I$.

It was recently shown by W. A. Dudek [9] that for a Hilbert algebra $\mathcal{H}$, ideals and deductive systems coincide. In what follows we prove the same also for pre-logics:

THEOREM 2. *Let $\mathcal{A} = (A; \cdot, 1)$ be a pre-logic. Then every ideal of $\mathcal{A}$ is a deductive system on $\mathcal{A}$ and, conversely, every deductive system of $\mathcal{A}$ is an ideal of $\mathcal{A}$.*

Proof. Let I be an ideal of a pre-logic $\mathcal{A}$. We need only to verify (d2). For this, let $x \in I$ and $x \cdot y \in I$. Denote $a_1 = x \cdot y$. By (P2) and (I2) we have $a_2 = (x \cdot y) \cdot y = \big(1 \cdot (x \cdot y)\big) \cdot y \in I$ and hence

$$y = 1 \cdot y = \big[\big((x \cdot y) \cdot y\big) \cdot \big((x \cdot y) \cdot y\big)\big] \cdot y = \big[a_2 \cdot (a_1 \cdot y)\big] \cdot y \in I,$$

thus I is a deductive system of $\mathcal{A}$.

Conversely, let D be a deductive system of $\mathcal{A}$. If $y \in D$ and $x \in A$, then, by (b) of Lemma 2 and (d1), (d2), $y \cdot (x \cdot y) = 1 \in D$ and hence $x \cdot y \in D$ proving (I2). We need only to show (I3).

At first, if $y \in D$, then $y \cdot \big((y \cdot x) \cdot x\big) = (y \cdot x) \cdot (y \cdot x) = 1 \in D$ thus, by (d2), also $(y \cdot x) \cdot x \in D$.

Now, let $y_1, y_2 \in D$ and $x \in A$. Applying the previous fact, we obtain by (P4

$$y_2 \cdot \big(\big(y_1 \cdot (y_2 \cdot x)\big) \cdot x\big) = \big(y_1 \cdot (y_2 \cdot x)\big) \cdot (y_2 \cdot x) \in D$$

and, using (d2), we obtain $\big(y_1 \cdot (y_2 \cdot x)\big) \cdot x \in D$. Altogether, we have shown that D is an ideal of $\mathcal{A}$. □

We are going to show that ideals and congruence kernels on pre-logics coincide:

THEOREM 3. *Let $\mathcal{A} = (A; \cdot, 1)$ be a pre-logic, let $\Theta \in \operatorname{Con} \mathcal{A}$ and I be an ideal of $\mathcal{A}$. Then*

(1) *the kernel $[1]_\Theta$ is an ideal of $\mathcal{A}$;*
(2) *I is the kernel of $\Theta_I \in \operatorname{Con} \mathcal{A}$ defined by setting*

$$\langle x, y \rangle \in \Theta_I \qquad \text{if and only if} \qquad x \cdot y \in I \quad \text{and} \quad y \cdot x \in I.$$

Θ_I is the greatest congruence on $\mathcal{A}$ whose kernel is I.

Proof.

(1) Let $I = [1]_\Theta$ for $\Theta \in \operatorname{Con} \mathcal{A}$. The condition (I1) is satisfied trivially. Let $x \in A$ and $y \in I$. Then $\langle y, 1 \rangle \in \Theta$ and

$$\langle x \cdot y, 1 \rangle = \langle x \cdot y, x \cdot 1 \rangle \in \Theta$$

proving $x \cdot y \in I$, i.e. also (I2) holds. Now, let $x \in A$ and $y_1, y_2 \in I$. Then $\langle y_1, 1 \rangle \in \Theta$, $\langle y_2, 1 \rangle \in \Theta$ and hence

$$\langle (y_2 \cdot (y_1 \cdot x)) \cdot x, 1 \rangle = \langle (y_2 \cdot (y_1 \cdot x)) \cdot x, (1 \cdot (1 \cdot x)) \cdot x \rangle \in \Theta$$

proving $(y_2 \cdot (y_1 \cdot x)) \cdot x \in I$, i.e. (I2) holds.

(2) Of course, Θ_I is both reflexive and symmetric. Suppose $\langle x, y \rangle \in \Theta_I$ and $\langle y, z \rangle \in \Theta_I$. Then $x \cdot y$, $y \cdot x$, $y \cdot z$, $z \cdot y \in I$ and, by (P3) and (I1), also

$$(x \cdot y) \cdot (x \cdot z) = x \cdot (y \cdot z) \in I .$$

However, I is a deductive system of $\mathcal{A}$ by Theorem 2 and $x \cdot y \in I$ thus, by (d2), also $x \cdot z \in I$. Analogously, $(z \cdot y) \cdot (z \cdot x) = z \cdot (y \cdot x) \in I$ by (I2) and, due to (d2), also $z \cdot x \in I$. We have shown $\langle x, z \rangle \in \Theta_I$, i.e. Θ_I is transitive.

It remains to check the substitution property of Θ_I. Suppose $\langle x, y \rangle \in \Theta_I$ and $\langle u, v \rangle \in \Theta_I$. Hence $x \cdot y$, $y \cdot x$, $u \cdot v$, $v \cdot u \in I$. We obtain

$$(x \cdot u) \cdot (x \cdot v) = x \cdot (u \cdot v) \in I$$

and

$$(x \cdot v) \cdot (x \cdot u) = x \cdot (v \cdot u) \in I$$

by (I2), i.e. $\langle x \cdot u, x \cdot v \rangle \in \Theta_I$. Further, by (I3)

$$\begin{aligned} (x \cdot v) \cdot (y \cdot v) &= y \cdot ((x \cdot v) \cdot v) = (y \cdot (x \cdot v)) \cdot (y \cdot v) \\ &= ((y \cdot x) \cdot (y \cdot v)) \cdot (y \cdot v) = (1 \cdot ((y \cdot x) \cdot (y \cdot v))) \cdot (y \cdot v) \in I . \end{aligned}$$

Analogously,

$$(y \cdot v) \cdot (x \cdot v) = x \cdot ((y \cdot v) \cdot v) = (1 \cdot ((x \cdot y) \cdot (x \cdot v))) \cdot (x \cdot v) \in I .$$

We have shown $\langle x \cdot v, y \cdot v \rangle \in \Theta_I$. Due to transitivity of Θ_I, this yields $\langle x \cdot u, y \cdot v \rangle \in \Theta_I$ whence $\Theta_I \in \operatorname{Con} \mathcal{A}$. Since $x \cdot 1 = 1$ and $1 \cdot x = x$, we conclude immediately $[1]_{\Theta_I} = I$.

Finally, let $\Psi \in \operatorname{Con} \mathcal{A}$ and suppose $[1]_\Psi = I$. Then for $\langle x, y \rangle \in \Psi$ we have

$$\langle x \cdot y, 1 \rangle = \langle x \cdot y, y \cdot y \rangle \in \Psi$$

and

$$\langle y \cdot x, 1 \rangle = \langle y \cdot x, y \cdot y \rangle \in \Psi$$

giving $x \cdot y, y \cdot x \in [1]_\Psi = I$ and hence $\langle x, y \rangle \in \Theta_I$. Thus Θ_I is the greatest congruence on $\mathcal{A}$ having the kernel I. □

COROLLARY 1. *In every pre-logic $\mathcal{A}$, ideals, deductive systems and congruence kernels coincide.*

We can compare deductive systems of pre-logics with quasiorder-filters of the induced quasiorder. For this, let us first state a technical lemma:

LEMMA 4. *Let $\mathcal{A} = (A; \cdot, 1)$ be a pre-logic and Q_A its induced quasiorder.*

(a) *For every $x, y \in A$, $\langle y, (y \cdot x) \cdot x \rangle \in Q_A$;*

(b) *for every $x, y, z \in A$, $\langle y \cdot z, (x \cdot y) \cdot (x \cdot z) \rangle \in Q_A$;*

(c) *if D is a deductive system of $\mathcal{A}$ and $a \in D$, $\langle a, b \rangle \in Q_A$, then $b \in D$.*

Proof.

(a) By (P1) and (P4), we compute $y \cdot [(y \cdot x) \cdot x] = (y \cdot x) \cdot (y \cdot x) = 1$, i.e $\langle y, (y \cdot x) \cdot x \rangle \in Q_A$.

(b) By (b) of Lemma 2 we have $z \cdot (x \cdot z) = 1$ thus also $\langle z, x \cdot z \rangle \in Q_A$. By Lemma 3 we conclude $\langle y \cdot z, y \cdot (x \cdot z) \rangle \in Q_A$. However,

$$y \cdot (x \cdot z) = x \cdot (y \cdot z) = (x \cdot y) \cdot (x \cdot z),$$

i.e.

$$\langle y \cdot z, (x \cdot y) \cdot (x \cdot z) \rangle \in Q_A.$$

(c) Let D be a deductive system of $\mathcal{A}$ and $a \in D$, $\langle a, b \rangle \in Q_A$. Then $a \cdot b = 1 \in D$ thus also $b \in D$. □

Let (A, Q) be a quasiordered set. For a subset $M \subseteq A$ we denote by

$$U_Q(M) = \{x \in A : \langle m, x \rangle \in Q \text{ for each } m \in M\}.$$

A subset $F \subseteq A$ is called a *Q-filter* if $\bigcup\{U_Q(a) : a \in F\} \subseteq F$. In other words, F is a Q-filter of (A, Q) if $a \in F$ and $\langle a, b \rangle \in Q$ imply $b \in F$. In account of Lemma 4, we have:

COROLLARY. *Every deductive system of a pre-logic $\mathcal{A} = (A; \cdot, 1)$ is a Q_A-filter of (A, Q_A) where Q_A is the induced quasiorder of $\mathcal{A}$.*

4. The lattice of deductive systems

For a pre-logic $\mathcal{A} = (A; \cdot, 1)$, we denote by $\operatorname{Ded} \mathcal{A}$ the set of all deductive systems of $\mathcal{A}$. Of course, $\{1\} \in \operatorname{Ded} \mathcal{A}$ and $A \in \operatorname{Ded} \mathcal{A}$. It is almost evident by Definition 4 that the set theoretical intersection of an arbitrary set of ideals of $\mathcal{A}$ is an ideal of $\mathcal{A}$ again. Hence, due to Theorem 2, the set $\operatorname{Ded} \mathcal{A}$ forms a complete lattice with respect to set inclusion where the operation meet coincides with set intersection; the least (or greatest) element of $\operatorname{Ded} \mathcal{A}$ is $\{1\}$ (or A, respectively).

Hence, given a subset $X \subseteq A$, there exists the least deductive system containing X, the so-called *deductive system of $\mathcal{A}$ generated by X*. It will be denoted by $D_{\mathcal{A}}(X)$. Of course,

$$D_{\mathcal{A}}(X) = \bigcap\{D \in \operatorname{Ded}\mathcal{A} : \ X \subseteq D\}.$$

In particular, $D_{\mathcal{A}}(\emptyset) = \{1\}$. It is almost trivial to check that $X \subseteq D_{\mathcal{A}}(X)$, $D_{\mathcal{A}}(D_{\mathcal{A}}(X)) = D_{\mathcal{A}}(X)$ and $X \subseteq Y \implies D_{\mathcal{A}}(X) \subseteq D_{\mathcal{A}}(Y)$ thus $D_{\mathcal{A}}$ is a *closure operator* on the power set $\operatorname{Exp} A$. This yields immediately that for the operation join in the lattice $\operatorname{Ded}\mathcal{A}$ it holds that

$$D_1 \vee D_2 = D_{\mathcal{A}}(D_1 \cup D_2)$$

or, more generally

$$\bigvee\{D_\lambda : \lambda \in \Lambda\} = D_{\mathcal{A}}\Big(\bigcup\{D_\lambda : \ \lambda \in \Lambda\}\Big). \tag{A}$$

If X is a singleton, say $X = \{b\}$, we will write briefly $D_{\mathcal{A}}(b)$ instead of $D_{\mathcal{A}}(\{b\})$. From the foregoing formula, one can derive

$$D = \bigvee\{D_{\mathcal{A}}(b) : \ b \in D\} \tag{B}$$

for every $D \in \operatorname{Ded}\mathcal{A}$.

THEOREM 4. *The lattice* $\operatorname{Ded}\mathcal{A}$ *of all deductive systems of a pre-logic* $\mathcal{A} = (A; \cdot, 1)$ *is an algebraic lattice whose compact elements are just finitely generated deductive systems. Let $X \subseteq A$. If $X = \emptyset$, then $D_{\mathcal{A}}(X) = \{1\}$; if $X \neq \emptyset$, then*

$$D_{\mathcal{A}}(X) = \Big\{a \in A : \ x_1 \cdot (x_2 \cdot (\cdots (x_n \cdot a) \cdots)) = 1 \ \textit{for} \ x_1, \dots, x_n \in X\Big\}.$$

P r o o f. It is immediately clear that $\operatorname{Ded}\mathcal{A}$ is a complete lattice and that $D_{\mathcal{A}}(\emptyset) = \{1\}$. Let $\emptyset \neq X \subseteq A$. Denote by

$$H = \Big\{a \in A : \ x_1 \cdot (x_2 \cdot (\cdots (x_n \cdot a) \cdots)) = 1 \ \text{for} \ x_1, \dots, x_n \in X\Big\}.$$

Suppose $a \in H$ and $a \cdot b \in H$. This means

$$x_1 \cdot (x_2 \cdot (\cdots (x_n \cdot a) \cdots)) = 1 \qquad \text{for some} \quad x_1, \dots, x_n \in X$$

and

$$x_1' \cdot (x_2' \cdot (\cdots (x_m' \cdot (a \cdot b)) \cdots)) = 1 \qquad \text{for some} \quad x_1', \dots, x_m' \in X.$$

Then

$$\begin{aligned} 1 &= x_n \cdot 1 \\ &= x_n \cdot [x_1' \cdot (x_2' \cdot (\cdots (x_m' \cdot (a \cdot b)) \cdots))] \\ &= \cdots \\ &= x_1' \cdot (x_2' \cdot (\cdots (x_m' \cdot (x_n \cdot (a \cdot b)) \cdots)) \\ &= x_1' \cdot (x_2' \cdot (\cdots (x_m' \cdot ((x_n \cdot a) \cdot (x_n \cdot b)) \cdots))). \end{aligned}$$

Analogously we can show

$$\begin{aligned}
1 &= x_{n-1} \cdot 1 \\
&= x_{n-1} \cdot (x_1' \cdot (x_2' \cdot (\cdots (x_m' \cdot ((x_n \cdot a) \cdot (x_n \cdot b))) \cdots))) \\
&= \cdots \\
&= x_1' \cdot (x_2' \cdot (\cdots (x_m' \cdot (x_{n-1} \cdot ((x_n \cdot a) \cdot (x_n \cdot b)))) \cdots))) \\
&= x_1' \cdot (x_2' \cdot (\cdots (x_m' \cdot ([x_{n-1} \cdot (x_n \cdot a)] \cdot [x_{n-1} \cdot (x_n \cdot b)])) \cdots)) \\
&= \cdots \\
&= x_1' \cdot (x_2' \cdot (\cdots (x_m' \cdot [x_1 \cdot (x_2 \cdot (\cdots (x_{n-1} \cdot (x_n \cdot a)) \cdots))] \cdot \\
&\qquad\qquad \cdot [x_1 \cdot (x_2 \cdot (\cdots x_n \cdot b) \cdots)]) \cdots)) \\
&= x_1' \cdot (x_2' \cdot (\cdots (x_m' \cdot (x_1 \cdot (\cdots (x_n \cdot b) \cdots)) \cdots))
\end{aligned}$$

proving $b \in H$. Hence, $H \in \operatorname{Ded} \mathcal{A}$.

Evidently, $X \subseteq H$ because $x \cdot x = 1$ for each $x \in X$. Suppose $D \in \operatorname{Ded} \mathcal{A}$ and $X \subseteq D$. Let

$$x_1 \cdot (x_2 \cdot (\cdots (x_n \cdot a) \cdots)) = 1 \qquad \text{for some} \quad x_1, \dots, x_n \in X \subseteq D .$$

Since D is a deductive system, this implies

$$x_2 \cdot (\cdots (x_n \cdot a) \cdots) \in D$$

and, after n steps, we derive $a \in D$. We have shown $H = D$.

From the above construction it is immediately clear that for each element $b \in A$, the one-generated deductive system $D_A(b)$ is a compact element of $\operatorname{Ded} \mathcal{A}$. With respect to the previous formula (B), the lattice $\operatorname{Ded} \mathcal{A}$ is compactly generated and hence algebraic. □

By Corollary 1, every deductive system is a congruence kernel and vice versa, hence, it makes sense to compare the lattices $\operatorname{Con} \mathcal{A}$ and $\operatorname{Ded} \mathcal{A}$.

LEMMA 5. *Let $\mathcal{A} = (A; \cdot, 1)$ be a pre-logic and $\Theta, \Phi \in \operatorname{Con} \mathcal{A}$. Denote by $\Theta \vee \Phi$ the join of Θ, Φ in $\operatorname{Con} \mathcal{A}$. Then in $\operatorname{Ded} \mathcal{A}$ we have*

$$[1]_\Theta \vee [1]_\Phi = [1]_{\Theta \vee \Phi} .$$

Proof. Of course, $[1]_\Theta \vee [1]_\Phi$ is a deductive system of $\mathcal{A}$ and, due to Corollary 1, there is a $\Psi \in \operatorname{Con} \mathcal{A}$ such that $[1]_\Theta \vee [1]_\Phi = [1]_\Psi$. Without loss of generality we suppose that Ψ is the greatest congruence on $\mathcal{A}$ having the kernel $[1]_\Theta \vee [1]_\Phi$. Let $\langle x, y \rangle \in \Theta$. Then, by Theorem 3, $x \cdot y$, $y \cdot x \in [1]_\Theta$, thus also $x \cdot y$, $y \cdot x \in [1]_\Psi$. Since Ψ is the greatest congruence with this kernel, by (2) of

Theorem 3 this yields $\langle x, y\rangle \in \Psi$, hence $\Theta \subseteq \Psi$. Analogously we obtain $\Phi \subseteq \Psi$ thus also $\Theta \vee \Phi \subseteq \Psi$. This yields immediately

$$[1]_\Theta \vee [1]_\Phi \subseteq [1]_{\Theta\vee\Phi} \subseteq [1]_\Psi = [1]_\Theta \vee [1]_\Phi$$

proving the desired equality. □

THEOREM 5. *For a pre-logic $\mathcal{A}$, the lattice* Ded $\mathcal{A}$ *is distributive.*

Proof. It is trivial to see that for every $\alpha, \beta \in \operatorname{Con}\mathcal{A}$ it holds

$$[1]_\alpha \cap [1]_\beta = [1]_{\alpha\cap\beta}\,.$$

To prove distributivity of Ded $\mathcal{A}$, we need only to show

$$[1]_{\Theta\cap(\Phi\vee\Psi)} \subseteq [1]_{(\Theta\cap\Phi)\vee(\Theta\cap\Psi)}$$

for every $\Theta, \Phi, \Psi \in \operatorname{Con}\mathcal{A}$ (with respect to Corollary 1). Let $x \in [1]_{\Theta\cap(\Phi\vee\Psi)}$. Thus $\langle x, 1\rangle \in \Theta \cap (\Phi \vee \Psi)$, i.e. there exist elements $c_1, \dots, c_n \in A$ such that

$$\langle x, c_1\rangle \in \Phi\,,\ \ \langle c_1, c_2\rangle \in \Psi\,,\ \ \langle c_2, c_3\rangle \in \Phi\,,\ \dots\ ,\ \langle c_n, 1\rangle \in \Phi$$

(we can suppose that n is even with respect to reflexivity of congruences).

Since $\langle x, 1\rangle \in \Theta$ thus also

$$\langle c_i \cdot x,\, 1\rangle = \langle c_i \cdot x,\, c_i \cdot 1\rangle \in \Theta \qquad \text{for} \quad i = 1, \dots, n\,,$$

which yields (with respect to symmetry and transitivity)

$$\langle c_i \cdot x, c_{i+1} \cdot x\rangle \in \Theta \qquad \text{for} \quad i = 1, \dots, n-1\,.$$

Hence,

$$\begin{aligned} \langle c_1 \cdot x,\, 1\rangle = \langle c_1 \cdot x,\, c_1 \cdot 1\rangle &\in \Theta \cap \Phi \\ \langle c_1 \cdot x,\, c_2 \cdot x\rangle &\in \Theta \cap \Psi \\ \langle c_2 \cdot x,\, c_3 \cdot x\rangle &\in \Theta \cap \Phi \\ &\vdots \\ \langle c_n \cdot x,\, x\rangle = \langle c_n \cdot x,\, 1 \cdot x\rangle &\in \Theta \cap \Phi \end{aligned}$$

giving $\langle x, 1\rangle \in (\Theta \cap \Phi) \vee (\Theta \cap \Psi)$. □

Remark. Although we have shown that the lattice of all congruence kernels of $\mathcal{A}$ is distributive, it does not mean that $\operatorname{Con}\mathcal{A}$ has the same property. (See the following example.)

EXAMPLE 3. Let $A = \{a, b, c, 1\}$ and the binary operation is defined by the table

$\cdot$	a	b	c	1
a	1	1	1	1
b	1	1	1	1
c	1	1	1	1
1	a	b	c	1

It is an exercise to check that $\mathcal{A} = (A; \cdot, 1)$ is a pre-logic. The lattice of congruences is as depicted in Fig. 1, where Θ is given by the partition $\{a, b, c\}, \{1\}$.

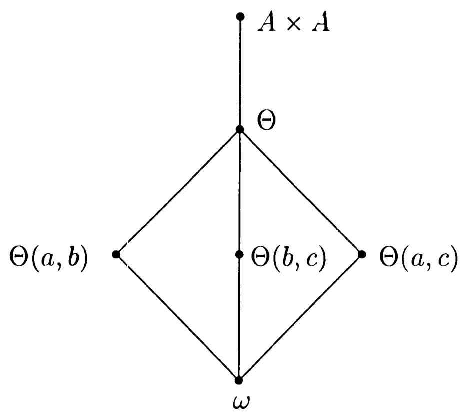

FIGURE 1.

Of course, $\operatorname{Con}\mathcal{A}$ is not distributive however $\operatorname{Ded}\mathcal{A}$ is isomorphic to the two-element chain.

It is well known that every distributive and algebraic lattice is also infinitely distributive, i.e. $\operatorname{Ded}\mathcal{A}$ satisfies the equality

$$D \cap \left(\bigvee\{D_\lambda :\ \lambda \in \Lambda\}\right) = \bigvee\{D \cap D_\lambda :\ \lambda \in \Lambda\}$$

for each $D, D_\lambda \in \operatorname{Ded}\mathcal{A}$ and an arbitrary index-set Λ. This yields immediately:

COROLLARY 3. *For every pre-logic $\mathcal{A}$ the lattice $\operatorname{Ded}\mathcal{A}$ is relatively pseudocomplemented.*

5. Annihilators of pre-logics

In this section we will describe the (relative) pseudocomlpements of $\operatorname{Ded}\mathcal{A}$ explicitly. At first, we describe the intersection (i.e. the meet in $\operatorname{Ded}\mathcal{A}$ in terms of the language of pre-logics:

LEMMA 6. *Let $\mathcal{A} = (A; \cdot, 1)$ be a pre-logic and $C, D \in \operatorname{Ded}\mathcal{A}$. Then*

(a) $C \cap D = \{(d \cdot c) \cdot c : c \in C, d \in D\}$;

(b) $C \cap D = \{1\}$ *if and only if* $\langle d \cdot c, c \rangle \in E_{Q_A} = \Theta$ *(the congruence induced by the quasiorder Q_A) for each $c \in C$ and $d \in D$.*

Proof.

(a) Denote by $M = \{(d \cdot c) \cdot c : c \in C, d \in D\}$ for $C, D \in \operatorname{Ded}\mathcal{A}$. If $y \in M$, then $y = (d \cdot c) \cdot c$ for $c \in C$, $d \in D$, and due to (I2) also $y \in C$ and $y = (1 \cdot (d \cdot c)) \cdot c$ yields by (I3), $y \in D$, i.e. $M \subseteq C \cap D$. Conversely, let $y \in C \cap D$. Take $c = y = d$. Then $(y \cdot y) \cdot y = 1 \cdot y = y \in M$, i.e. $M = C \cap D$.

(b) If $C \cap D = \{1\}$, then, by (a), we obtain $(d \cdot c) \cdot c = 1$ for each $c \in C$, $d \in D$. By Lemma 2 we have $\langle d \cdot c, c \rangle \in Q_A$. However, (d) of Lemma 2 and Lemma 3 give $\langle c, d \cdot c \rangle = \langle 1 \cdot c, d \cdot c \rangle \in Q_A$ whence $\langle d \cdot c, c \rangle \in E_{Q_A} = \Theta$. Conversely, if $\langle d \cdot c, c \rangle \in \Theta$ for each $c \in C$, $d \in D$, then (c) of Lemma 2 yields $(d \cdot c) \cdot c = 1$ and, by (a), we have $C \cap D = \{1\}$. □

DEFINITION 5. Let $\mathcal{A} = (A; \cdot, 1)$ be a pre-logic: for $C, B \subseteq A$ we denote by

$$\langle C \rangle = \{x \in A : \langle x \cdot c, c \rangle \in E_{Q_A} \text{ for each } c \in C\},$$
$$\langle C, B \rangle = \{x \in A : (x \cdot c) \cdot c \in B \text{ for each } c \in C\}.$$

If $C = \{c\}$, we will write briefly $\langle c \rangle$ instead of $\langle \{c\} \rangle$. The set $\langle C \rangle$ is called an *annihilator of a set C*. The set $\langle C, B \rangle$ is called a *relative annihilator of C with respect to B*.

The following results are easy observations:

- if $C_1 \subseteq C_2$, then $\langle C_1 \rangle \supseteq \langle C_2 \rangle$;
- for each $C \subseteq A$ we have $\langle C \rangle = \bigcap\{\langle c \rangle : c \in C\}$.

THEOREM 6. *For every element c of a pre-logic $\mathcal{A} = (A; \cdot, 1)$, the annihilator $\langle c \rangle$ is a deductive system of $\mathcal{A}$.*

Proof. Denote by $\Theta = E_{Q_A}$. As shown by Theorem 1, $\Theta \in \operatorname{Con}\mathcal{A}$. Suppose now $x \in \langle c \rangle$ and $x \cdot y \in \langle c \rangle$ for some $c \in A$. By Definition 5, $\langle x \cdot c, c \rangle \in \Theta$ and $\langle (x \cdot y) \cdot c, c \rangle \in \Theta$. Then

$$(x \cdot y) \cdot (x \cdot c) = x \cdot (y \cdot c) = y \cdot (x \cdot c),$$

further $\langle x\cdot c,\, c\rangle \in \Theta$ implies

$$\big\langle y\cdot(x\cdot c),\, y\cdot c\big\rangle \in \Theta\,,$$
$$\big\langle (x\cdot y)\cdot(x\cdot c),\, (x\cdot y)\cdot c\big\rangle \in \Theta\,,$$

i.e. $\big\langle (x\cdot y)\cdot c,\, y\cdot c\big\rangle \in \Theta$. Together with $\big\langle (x\cdot y)\cdot c,\, c\big\rangle \in \Theta$ we conclude $\langle y\cdot c,\, c\rangle \in \Theta$ thus $y \in \langle c\rangle$ directly by Definition 5. We have checked (d2). Since (d1) holds trivially, $\langle c\rangle$ is a deductive system of $\mathcal{A}$. □

THEOREM 7. *For every deductive system D of a pre-logic $\mathcal{A} = (A;\cdot,1)$, its annihilator $\langle D\rangle$ is a pseudocomplement of D in the lattice* $\operatorname{Ded}\mathcal{A}$.

Proof. If $d \in D \cap \langle D\rangle$, then $d \in \langle d\rangle$ since $\langle D\rangle \subseteq \langle d\rangle$ thus $\langle 1,d\rangle$ $\langle d\cdot d,\, d\rangle \in E_{Q_A}$ by Definition 5, i.e. $\langle 1,d\rangle \in Q_A$ and, by (e) of Lemma 2, $d = 1$. Thus $D \cap \langle D\rangle = \{1\}$. Suppose now $F \in \operatorname{Ded}\mathcal{A}$ and $D\cap F = \{1\}$. Then $\langle f\cdot d,\, d\rangle \in E_{Q_A}$ for each $f\in F$ and $d\in D$ by (b) of Lemma 6, i.e. $f \in \langle d\rangle$ for each $d \in D$ thus also $f \in \bigcap\big\{\langle d\rangle :\ d\in D\big\} = \langle D\rangle$ proving $F \subseteq \langle D\rangle$. Altogether, $\langle D\rangle$ is the greatest deductive system of $\mathcal{A}$ with $D\cap\langle D\rangle = \{1\}$ and hence the pseudocomplement of D in the lattice $\operatorname{Ded}\mathcal{A}$. □

We can ask whether the annihilator of a given subset coincides with the annihilator of a deductive system generated by this set:

THEOREM 8. *For a pre-logic $\mathcal{A} = (A;\cdot,1)$, the following conditions are equivalent:*

(1) *$\langle M\rangle = \langle D(M)\rangle$ for each subset $M \subseteq A$;*
(2) *$\langle b\cdot c,\, c\rangle \in E_{Q_A}$ if and only if $\langle c\cdot b,\, b\rangle \in E_{Q_A}$ for every two elements b, c of A.*

Proof.

(1) $\implies$ (2): Let $c, b \in A$. By the assumption (1), $\langle c\rangle = \langle D(c)\rangle$, i.e. $b \in \langle c\rangle$ implies $\langle b\cdot c,\, c\rangle \in E_{Q_A}$. Applying (I3) we get $(c\cdot x)\cdot x = \big(1\cdot(c\cdot x)\big)\cdot x \in D(c$ for each $x\in A$ thus also $b \in \langle (c\cdot x)\cdot x\rangle$. Taking $x = b$ we obtain $b \in \langle (c\cdot b)\cdot b\rangle$, i.e.

$$\big\langle b\cdot\big((c\cdot b)\cdot b\big),\, (c\cdot b)\cdot b\big\rangle \in E_{Q_A}\,.$$

By (b) of Lemma 2 we have $b\cdot\big((c\cdot b)\cdot b\big) = 1$, i.e. $\big\langle 1,\, (c\cdot b)\cdot b\big\rangle \in E_{Q_A}$ and, by (e) of Lemma 2 again, also $(c\cdot b)\cdot b = 1$ proving (2).

(2) $\implies$ (1): Let $b, c \in A$. Then $\langle c\rangle = \big\{x\in A:\ \langle x\cdot c,\, c\rangle \in E_{Q_A}\big\}$ and $\langle b\rangle = \big\{x \in A:\ \langle x\cdot b,\, b\rangle \in E_{Q_A}\big\}$. By (2) we have $b\in\langle c\rangle$ if and only if $c \in \langle b\rangle$. Prove $\langle c\rangle \subseteq \langle D(c)\rangle$: let $z \in \langle c\rangle$. As shown this gives w.r.t. (2) also $c \in \langle z\rangle$ whence $D(c) \subseteq \langle z\rangle$. Suppose $x \in D(c)$. Then $x \in \langle z\rangle$ and hence $z \in \langle x\rangle$, i.e.

$$z \in \bigcap\big\{\langle x\rangle :\ x \in D(c)\big\} = \langle D(c)\rangle\,.$$

Now, let $M \subseteq A$. We have

$$\langle M \rangle = \bigcap \{ \langle m \rangle : \ m \in M \} = \bigcap \{ \langle D(m) \rangle : \ m \in M \} .$$

If $y \in \langle D(m) \rangle$ for each $m \in M$, then also $y \in \langle m \rangle = \langle D(m) \rangle$ and hence $m \in \langle y \rangle$ giving $D(M) \subseteq \langle y \rangle$. This implies $\langle D(m) \rangle \supseteq \langle \langle y \rangle \rangle$. It remains to show $y \in \langle \langle y \rangle \rangle$. We have $\langle y \rangle = \{ x \in A : \ \langle x \cdot y, \, y \rangle \in E_{Q_A} \}$ and

$$\langle \langle y \rangle \rangle = \{ z \in A : \ \langle z \cdot x, \, x \rangle \text{ for each } x \in \langle y \rangle \} .$$

But $x \in \langle y \rangle$ yields $y \in \langle x \rangle$, i.e. $\langle y \cdot x, \, x \rangle \in E_{Q_A}$ for each $x \in \langle y \rangle$ proving $y \in \langle \langle y \rangle \rangle$. In the summary, we conclude $\langle M \rangle = \langle D(M) \rangle$. □

We are ready to describe relative pseudocomplements of $\operatorname{Ded} \mathcal{A}$ in terms of relative annihilators:

THEOREM 9. *Let B, C be deductive systems of a pre-logic $\mathcal{A} = (A; \cdot, 1)$. Then $\langle C, B \rangle$ is the relative pseudocomplement of C with respect to B in the lattice* $\operatorname{Ded} \mathcal{A}$.

P r o o f. It is almost evident that if $x \in C \cap \langle C, B \rangle$, then $x = 1 \cdot x = (x \cdot x) \cdot x \in B$, i.e. $C \cap \langle C, B \rangle \subseteq B$. Moreover, if $F \in \operatorname{Ded} \mathcal{A}$ and $C \cap F \subseteq B$, then for each $c \in C$ and $f \in F$ we have by Lemma 6, $(f \cdot c) \cdot c \in B$ thus, by Definition 5, $F \subseteq \langle C, B \rangle$. It remains to prove that $\langle C, B \rangle$ is a deductive system of $\mathcal{A}$.

Suppose $x \in \langle C, B \rangle$ and $x \cdot y \in \langle C, B \rangle$. Then $(x \cdot c) \cdot c \in B$ and $\big((x \cdot y) \cdot c \big) \cdot c \in B$ for each $c \in C$. Since C is an ideal of $\mathcal{A}$, we have $x \cdot c \in C$ and hence also $\big((x \cdot y) \cdot (x \cdot c) \big) \cdot (x \cdot c) \in B$. Then

$$u = (y \cdot c) \cdot (x \cdot c) = x \cdot \big((y \cdot c) \cdot c \big) = \big(x \cdot (y \cdot c) \big) \cdot (x \cdot c) = \big((x \cdot y) \cdot (x \cdot c) \big) \cdot (x \cdot c) \in B$$

for each $c \in C$. Set $v = (y \cdot c) \cdot c$. Then

$$\big((x \cdot c) \cdot c \big) \cdot \big((y \cdot c) \cdot c \big) = (y \cdot c) \cdot \big(\big((x \cdot c) \cdot c \big) \cdot c \big) = \big((y \cdot c) \cdot \big((x \cdot c) \cdot c \big) \big) \cdot \big((y \cdot c) \cdot c \big) = (u \cdot v) \cdot v .$$

Since B is an ideal of $\mathcal{A}$ and $u \in B$, also $(u \cdot v) \cdot v \in B$, i.e.

$$\big((x \cdot c) \cdot c \big) \cdot \big((y \cdot c) \cdot c \big) \in B .$$

However, $(x \cdot c) \cdot c \in B$ and B is a deductive system of $\mathcal{A}$, thus also $(y \cdot c) \cdot c \in B$ giving $y \in \langle C, B \rangle$. We have shown $\langle C, B \rangle \in \operatorname{Ded} \mathcal{A}$. □

6. Principal deductive systems

It was shown in Section 4 that every deductive system D in a pre-logic $\mathcal{A} = (A; \cdot, 1)$ is a join of one-generated deductive systems, namely

$$D = \bigvee \{D_{\mathcal{A}}(b) : \ b \in D\}.$$

Hence, these one-generated deductive systems play a crucial role. In what follows, we call a deductive system $D \in \operatorname{Ded} \mathcal{A}$ *principal* if $D = D_{\mathcal{A}}(b)$ for some $b \in D$. On the other hand, the description of a deductive system generated by a given set as shown in Section 4 is rather complex. Moreover, by Corollary 1, every deductive system is a Q_A-filter of $\mathcal{A}$ where Q_A is the induced quasiorder on $\mathcal{A}$. We can ask if also a principal deductive system is a principal Q_A-filter. Both the questions are answered by the following theorem.

THEOREM 10. *Let $\mathcal{A} = (A; \cdot, 1)$ be a pre-logic and $c \in A$. Then*

$$D_{\mathcal{A}}(c) = \{(c \cdot x) \cdot x : \ x \in A\} = U_{Q_A}(c)$$

where Q_A is the induced quasiorder of $\mathcal{A}$.

Proof. Since $D_{\mathcal{A}}(c)$ is a Q_A-filter of $\mathcal{A}$, by Corollary 1 and $c \in D_{\mathcal{A}}(c)$, it is immediately clear that $U_{Q_A}(c) = \{y \in A : \ \langle c, y\rangle \in Q_A\} \subseteq D_{\mathcal{A}}(c)$. To prove the converse inclusion it is enough to show that $U_{Q_A}(c)$ is a deductive system. By Theorem 2 we only need to show that $U_{Q_A}(c)$ is an ideal of $\mathcal{A}$.

Let $z \in U_{Q_A}(c)$, i.e. $\langle c, z\rangle \in Q_A$. By Lemma 3 we conclude $\langle c, x \cdot c\rangle \in Q_A$ and $\langle x \cdot c, x \cdot z\rangle \in Q_A$ for each $x \in A$ thus also $x \cdot z \in U_{Q_A}(c)$. Hence $U_{Q_A}(c)$ satisfies (I2). The condition (I1) is evident. Prove (I3).

Suppose $c_1, c_2 \in U_{Q_A}(c)$. Hence $\langle c, c_2\rangle \in Q_A$ thus $\langle c_2 \cdot x, c \cdot x\rangle \in Q_A$ and

$$\langle c_1 \cdot (c_2 \cdot x),\ c_1 \cdot (c \cdot x)\rangle \in Q_A,$$

moreover $c_1 \cdot (c \cdot x) = (c \cdot c_1) \cdot (c \cdot x) = c \cdot x$ because $\langle c, c_1\rangle \in Q_A$ implies $c \cdot c_1 = 1$ by Lemma 2. Hence, $\langle c_1 \cdot (c_2 \cdot x),\ c \cdot x\rangle \in Q_A$ which yields $\langle (c \cdot x) \cdot x,\ (c_1 \cdot (c_2 \cdot x)) \cdot x\rangle \in Q_A$ and also

$$\langle c \cdot ((c \cdot x) \cdot x),\ c \cdot ((c_1 \cdot (c_2 \cdot x)) \cdot x)\rangle \in Q_A.$$

However, $1 = (c \cdot x) \cdot (c \cdot x) = c \cdot ((c \cdot x) \cdot x)$ gives $\langle c \cdot ((c_1 \cdot (c_2 \cdot x)) \cdot x),\ 1\rangle \in E_{Q_A}$ and hence $c \cdot ((c_1 \cdot (c_2 \cdot x)) \cdot x) = 1$ giving $\langle c, (c_1 \cdot (c_2 \cdot x)) \cdot x\rangle \in E_{Q_A}$, i.e. $(c_1 \cdot (c_2 \cdot x)) \cdot x \in U_{Q_A}(c)$ which proves (I3).

Finally, $c \cdot ((c \cdot x) \cdot x) = (c \cdot x) \cdot (c \cdot x) = 1$ implies $(c \cdot x) \cdot x \in U_{Q_A}(c)$. Conversely, if $z \in U_{Q_A}(c)$, then $\langle c, z\rangle \in Q_A$, i.e. $c \cdot z = 1$ and hence

$$z = 1 \cdot z = (c \cdot z) \cdot z \in \{(c \cdot x) \cdot x : \ x \in A\}.$$

We have shown $U_{Q_A}(c) = \{(c \cdot x) \cdot x : \ x \in A\}$. □

7 Quasiorder algebras

We are going to show that every quasiordered set can be considered as a pre-logic.

THEOREM 11. *Let* (A, Q) *be a quasiordered set. Suppose* $1 \notin A$ *and set* $A_1 = A \cup \{1\}$. *Define a binary operation* $\cdot$ *on* A_1 *as follows*

$$x \cdot y = \begin{cases} 1 & \textit{if } \langle x, y\rangle \in Q\,, \\ y & \textit{otherwise.} \end{cases}$$

Then $\mathcal{A} = (A_1; \cdot, 1)$ *is a pre-logic.*

Proof. We need to verify the conditions of Definition 2. Of course, (P1) and (P2) are evident.

Prove (P3). If $\langle x, y\rangle \in Q$ and $\langle y, z\rangle \in Q$, then also $\langle x, z\rangle \in Q$ and

$$x \cdot (y \cdot z) = x \cdot 1 = 1 = 1 \cdot 1 = (x \cdot y) \cdot (x \cdot z)\,.$$

If $\langle x, y\rangle \in Q$ and $\langle y, z\rangle \notin Q$, then

$$x \cdot (y \cdot z) = x \cdot z = 1 \cdot (x \cdot z) = (x \cdot y) \cdot (x \cdot z)\,.$$

Suppose $\langle x, y\rangle \notin Q$ and $\langle y, z\rangle \in Q$. Then $x \cdot (y \cdot z) = x \cdot 1 = 1$.

If $\langle x, z\rangle \in Q$, then $(x \cdot y) \cdot (x \cdot z) = y \cdot 1 = 1$; if $\langle x, z\rangle \notin Q$, then $(x \cdot y) \cdot (x \cdot z) = y \cdot z = 1$. Finally, suppose $\langle x, y\rangle \notin Q$ and $\langle y, z\rangle \notin Q$. Then $x \cdot (y \cdot z) = x \cdot z$. If $\langle x, z\rangle \in Q$, then $x \cdot (y \cdot z) = x \cdot z = 1$ and

$$(x \cdot y) \cdot (x \cdot z) = y \cdot 1 = 1 = x \cdot (y \cdot z)\,.$$

If $\langle x, z\rangle \notin Q$, then $x \cdot (y \cdot z) = x \cdot z = z$ and

$$(x \cdot y) \cdot (x \cdot z) = y \cdot z = z = x \cdot (y \cdot z)\,.$$

It remains to prove (P4). We can compute the term $x \cdot (y \cdot z)$ as follows:

$$x \cdot (y \cdot z) = \begin{cases} x \cdot 1 = 1 & \text{for } \langle y, z\rangle \in Q\,, \\ x \cdot z = \begin{cases} 1 & \text{for } \langle x, z\rangle \in Q\,,\ \langle y, z\rangle \notin Q\,, \\ z & \text{for } \langle x, z\rangle \notin Q\,,\ \langle y, z\rangle \notin Q\,. \end{cases} \end{cases}$$

Analogously, we have

$$y \cdot (x \cdot z) = \begin{cases} y \cdot 1 = 1 & \text{for } \langle x, z\rangle \in Q\,, \\ y \cdot z = \begin{cases} 1 & \text{for } \langle y, z\rangle \in Q\,,\ \langle x, z\rangle \notin Q\,, \\ z & \text{for } \langle y, z\rangle \notin Q\,,\ \langle x, z\rangle \notin Q\,. \end{cases} \end{cases}$$

Hence, $x \cdot (y \cdot z) = z = y \cdot (x \cdot z)$ for $\langle x, z\rangle \notin Q$, $\langle y, z\rangle \notin Q$ and $x \cdot (y \cdot z) = 1 = y \cdot (x \cdot z)$ in all other possible cases. □

Congruences on quasiorder algebras have very special properties:

THEOREM 12. *Let (A, Q) be a quasiordered set and $\mathcal{A} = (A_1; \cdot, 1)$ its assigned quasiorder algebra. Suppose $\Phi \in \operatorname{Con} \mathcal{A}$. If $\langle x, y\rangle \in \Phi$, then either $x, y \in [1]_\Phi$ or $\langle x, y\rangle \in E_Q$ (where E_Q is the equivalence induced by Q).*

Proof. Suppose $\Phi \in \operatorname{Con} \mathcal{A}$ and $\langle x, y\rangle \in \Phi$. Let $\langle x, y\rangle \notin E_Q$. Due to reflexivity, also $\langle x, x\rangle \in \Phi$ and $\langle y, y\rangle \in \Phi$ thus

$$\langle x \cdot y, x \cdot x\rangle \qquad \text{and} \qquad \langle y \cdot x, y \cdot y\rangle \in \Phi .$$

Since $x \cdot x = 1 = y \cdot y$ and $\langle x, y\rangle \notin E_Q$, then either $x \cdot y = y$ or $y \cdot x = x$ so we have either $\langle 1, y\rangle \in \Phi$ or $\langle 1, x\rangle \in \Phi$. Applying transitivity, we conclude $x, y \in [1]_\Phi$. □

As mentioned in Section 1, a partial order and an equivalence relation are particular cases of a quasiorder. We conclude our paper by the example of prelogics which are quasiorder algebras in these cases.

EXAMPLES.

(a) If Q is a partial order on a set A, then the quasiordered algebra assigned to (A, Q) is just a Hilbert algebra since the induced equivalence E_Q is the identity relation ω_A due to antisymmetry of Q.

(b) If Q is an equivalence relation on a set A, then $Q = E_Q$, i.e. Q forms a partition of A. With respect to (e) of Lemma 2, $[1]_{E_Q} = \{1\}$ in this partition and hence the quasiordered algebra assigned to (A, Q) is a semi-implication algebra (see [2] for details). It can be visualised as shown in Fig. 2. Moreover, the quotient Hilbert algebra $\mathcal{A}/\Theta$ for $\Theta = E_Q$ existing by Theorem 1 is just an implication algebra (defined by J. C. Abott in [1] as a fragment of a classical logic containing only the implication and the constant value 1). It was shown by A. Diego that implication algebras are especial case of Hilbert algebras.

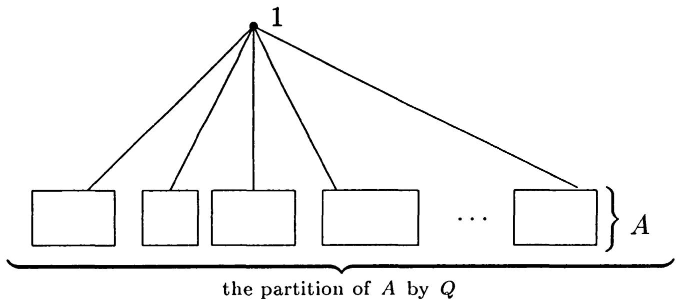

FIGURE 2.

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Department of Algebra and Geometry
Palacký University Olomouc
Tomkova 40
CZ-779 00 Olomouc
CZECH REPUBLIC
E-mail: chajda@risc.upol.cz
halas@risc.upol.cz