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ON PROPERTY K IN F-SPACES

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ABSTRACT. The following property of K-subspaces in an arbitrary F-space X of dimension $\mathfrak{c}$ is obtained without the Continuum Hypothesis by means of a transfinite construction: given an arbitrary F_σ-subspace E of X of dimension $\mathfrak{c}$ and infinite codimension and an arbitrary K_σ-subspace F of X such that $E \cap F = \{0\}$ there exist dense K-subspaces Y_1 and Y_2 such that $E \oplus Y_1 = E \oplus Y_2 = X$ and $Y_1 \cap Y_2 = F$. A generalization of this result and various consequences are proved in this paper.

1. Introduction

Let us start with two definitions. We say that a topological linear space X

1° *has property K* or *is a K-space* if every sequence convergent to 0 in X contains a subsequence which is summable in X,

2° *has property N* or *is an N-space* if every sequence convergent to 0 in X contains a subsequence every subsequence of which is summable in X.

Clearly, property N implies property K. The converse is not true; an appropriate example is given in [15]. Every F-space (i.e. a complete metrizable linear space) is an N-space, but there exist non-complete metrizable linear spaces which are K-spaces (see [14] and [15]) and even N-spaces (see [4]).

Property K was introduced by S. Mazur and W. Orlicz [17] (see also [1]) as a substitute for completeness in various theorems of functional analysis. In the seventies, it was considered (under the present name) at the seminar of Professors J. Mikusiński and P. Antosik in Katowice and subsequently studied by various authors (see e.g. [14], [15], [2], [13], [16], [3], [4], [12], [6]). Property N was studied by A. Alexiewicz in [1] and then in [9], [19], [4] and [6].

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According to the known theorem of J. Lindenstrauss and L. Tzafriri, in every infinite-dimensional Banach space nonisomorphic to a Hilbert space, there exist closed subspaces without closed algebraic complements. In contrast, the following nice property was proved in [6] (see [6; Theorem 1(ii)]) for F-spaces X of dimension $\mathfrak{c}$: given an F_σ-subspace (i.e. a countable union of closed sets) E of infinite codimension and a K_σ-subspace (i.e. a countable union of compact sets) F in X such that $E \cap F = \{0\}$ there exists an enlargement of F to a dense algebraic complement of E with property K.

In the case $\dim E = \mathfrak{c}$, the above result can be strengthened as follows: if the space X and subspaces E and F satisfy the above mentioned assumptions, then there exist two dense algebraic complements Y_1 and Y_2 of E with property K such that $Y_1 \cap Y_2 = F$ (see Theorem 2 in Section 2).

On the other hand, the following observation in a negative direction was made in [16]: the intersection of two K-spaces is not generally a K-space, i.e. property K is not multiplicative (the same is true for the product of two K-spaces).

This result will also be strengthened in the paper as follows: given a subspace E of X, where X and E are as before, and a subspace H such that $E \cap H = \{0\}$ and $\dim H < \mathfrak{c}$ there are two dense algebraic complements Y_1 and Y_2 of E with property K such that $Y_1 \cap Y_2 = H$ (see Theorem 3 in Section 2). Thus the structure of K-spaces is not preserved under intersection (see also Propositions 2 and 3 in Section 2 and [7; Corollary]).

The results mentioned have a common generalization which is the main assertion of this paper (see Theorem 1 in Section 2).

For precise formulations of the results of the paper see the next section and for their proofs see Section 4.

The proof of Theorem 1 is based on a transfinite construction and several lemmas given in Section 3. We follow the ideas of [6]. It should be emphasized that the Continuum Hypothesis is not assumed in the paper.

2. Formulation of the results

By an F-*space* we mean a complete metrizable topological linear space (over either the real or complex field) or, equivalently, a linear space equipped with a complete F-*norm*; let us recall that for a given F-norm in a linear space there exists an equivalent non-decreasing F-norm (see e.g. [18; pp. 1–8]).

Suppose that two F-norms are given in a linear space X and that the topologies generated by them satisfy the property K (resp. property N), introduced in Section 1. One may ask whether the topology generated by the sum of these F-norms satisfies property K (resp. property N) as well.

The following straightforward assertion (Proposition 1) gives a partial answer to this question under the assumption that the given F-norms are compatible.

We say that two F-norms $\|\cdot\|_1$ and $\|\cdot\|_2$ in a linear space X are *compatible* whenever the following implication holds: if $\|\xi_n - \xi\|_1 \to 0$ and $\|\xi_n - \eta\|_2 \to 0$ as $n \to \infty$, then $\xi = \eta$ for arbitrary $\xi_n, \xi, \eta \in X$. In particular, if there exists a Hausdorff topology in X weaker than each of the topologies generated by given F-norms $\|\cdot\|_1$ and $\|\cdot\|_2$, then these F-norms are compatible.

PROPOSITION 1. *Let X be an arbitrary linear space, let $\|\cdot\|_1$ and $\|\cdot\|_2$ be compatible F-norms in X and let $\|\cdot\|_3 := \|\cdot\|_1 + \|\cdot\|_2$. The following implications hold:*

(a_1) *If $(X, \|\cdot\|_1)$ and $(X, \|\cdot\|_2)$ are N-spaces, then $(X, \|\cdot\|_3)$ is an N-space;*

(a_2) *If $(X, \|\cdot\|_1)$ is a K-space and $(X, \|\cdot\|_2)$ is an N-space, then $(X, \|\cdot\|_3)$ is a K-space.*

The next statement shows that the assumption of compatibility of the F-norms in Proposition 1 is essential and cannot be omitted even under the stronger assumption that the norms are complete.

PROPOSITION 2. *Let $(X, \|\cdot\|_1)$ be an arbitrary F-space of infinite dimension. Then there exists an F-norm $\|\cdot\|_2$ in X such that*

(b_1) *$(X, \|\cdot\|_2)$ is an F-space;*

(b_2) *$(X, \|\cdot\|_3)$ is not a K-space, where $\|\cdot\|_3 := \|\cdot\|_1 + \|\cdot\|_2$.*

On the other hand, we observe that the assumption in the implication (a_2) of Proposition 1 cannot be relaxed by assuming that $(X, \|\cdot\|_i)$ are K-spaces for $i \in \{1, 2\}$. In fact, the following assertion is true:

PROPOSITION 3. *Suppose that $(X, \|\cdot\|)$ is an arbitrary F-space such that $\dim X = \mathfrak{c}$ and there exists a closed subspace of X of infinite dimension and infinite codimension. Then there exist F-norms $\|\cdot\|_1$ and $\|\cdot\|_2$ which generate stronger topologies than $\|\cdot\|$ (and so $\|\cdot\|_1$ and $\|\cdot\|_2$ are compatible) such that*

(c_1) *$(X, \|\cdot\|_i)$ are K-spaces for $i \in \{1, 2\}$;*

(c_2) *$(X, \|\cdot\|_3)$ is not a K-space, where $\|\cdot\|_3 := \|\cdot\|_1 + \|\cdot\|_2$.*

Proposition 1 is obvious and its proof will be omitted. The proof of Proposition 2 is based on a simple construction of an automorphism in an arbitrary linear space (see Lemma 9) and is given in the second part of Section 4. The third proposition is not at all obvious. We shall obtain it as a corollary of Theorem 3 which is a particular case of Theorem 1, the main result of this note.

Before formulating the main theorems let us introduce Assumption A which will be in force throughout the paper.

Recall (cf. [6]) that a subspace E of an F-space X is called an *F_σ-subspace* of X if $E = \bigcup_{k=1}^{\infty} E_k$, where the E_k are closed subsets of X for $k \in \mathbb{N}$, and a *K_σ-subspace* of X if $E = \bigcup_{k=1}^{\infty} F_k$, where the F_k are compact subsets of X for $k \in \mathbb{N}$.

ASSUMPTION A. *Let X be an F-space and let E and F be its subspaces such that the following conditions are satisfied:*

(a) $\dim X = \mathfrak{c}$;
(b) *E is an F_σ-subspace of X;*
(c) *F is a K_σ-subspace of X;*
(d) $E \cap F = \{0\}$;
(e) $\operatorname{codim} E \geq \aleph_0$;
(f) $\dim E = \mathfrak{c}$.

The following is the main theorem of the paper:

THEOREM 1. *Suppose that X, E, F satisfy Assumption* A *and H is a subspace of X such that*

$$\dim H < \mathfrak{c}, \qquad (E \oplus F) \cap H = \{0\}. \tag{1}$$

Then there exist dense K-subspaces Y_1 and Y_2 of X satisfying the identities:

$$E \oplus Y_1 = E \oplus Y_2 = X, \tag{2}$$

$$Y_1 \cap Y_2 = F \oplus H. \tag{3}$$

Let us formulate two particular cases of Theorem 1. First, taking $H := \{0\}$ in Theorem 1, we get:

THEOREM 2. *Suppose that X, E, F satisfy Assumption* A. *Then there exist dense K-subspaces Y_1 and Y_2 of X such that* (2) *holds and*

$$Y_1 \cap Y_2 = F.$$

Observe that Theorem 1 easily follows from Theorem 2 under the Continuum Hypothesis. In fact, if X, E, F satisfy Assumption A and H satisfies (1), then $E \cap (F \oplus H) = \{0\}$ and, since $\dim H \leq \aleph_0$, $F \oplus H$ is a K_σ-subspace of X. Hence Assumption A is satisfied if F is replaced by $F \oplus H$, so Theorem 1 follows from Theorem 2.

When the subspace E satisfies condition (f), Theorem 2 strengthens the assertion (ii) of Theorem 1 in [6]. This and other results of [6] are proved under the system of five assumptions called Hypothesis H. Due to [15; Corollary 2], Hypothesis H is equivalent to the system of conditions (a) – (e) above. Assumption A is thus stronger than Hypothesis H, because of the additional condition (f) assumed above.

Now, taking $F := \{0\}$ in Theorem 1, we obtain:

THEOREM 3. *Suppose that X and E satisfy conditions* (a), (b), (e), (f) *of Assumption* A *and H is a subspace of X such that $E \cap H = \{0\}$. Then there exist dense K-subspaces Y_1 and Y_2 of X such that* (2) *holds and*

$$Y_1 \cap Y_2 = H. \tag{4}$$

The above theorem is related to Theorem 3 of [16], which says that in every topological linear space X of dimension $\mathfrak{c}$ there exist κ-subspaces Y_1 and Y_2 satisfying equation (4). Recall that a subspace Y of an F-space X is said to be a *κ-subspace* if every linearly independent sequence convergent to 0 in X contains a subsequence which is summable in Y; for the general definition in topological linear spaces see [10] or [16].

In general, the assertions of Theorems 1 – 3 cannot be extended to the case of κ-subspaces. However, they can under the additional assumption that both E and F are aN-subspaces and this result can be proved in a way similar to that demonstrated in this paper. Recall that a subspace E of an F-space X is said to be an *aN-subspace* if every linearly independent sequence convergent to 0 in E contains a subsequence which is summable in $X \setminus E$ (see [6]).

The proofs of Theorem 1 and Propositions 2 and 3 are given in Section 4 and all auxiliary results are collected in Section 3.

3. Lemmas

Let us introduce the basic notation. In what follows X denotes a fixed F-space. Elements of X will be denoted by Greek letters ξ, υ, ζ, etc. (with or without indices) and sequences in X by the corresponding Latin letters, i.e. $x := \{\xi_n\}$, $y := \{\upsilon_n\}$, $z := \{\zeta_n\}$, etc. According to the context, the symbol 0 will mean the zero element of X or the zero sequence in X.

Given a sequence $x = \{\xi_n\}$ and elements $\eta_1, \dots, \eta_n \in X$ in X, we shall denote by (x), $[x]$ and $[\eta_1, \dots, \eta_n]$ the set of all elements of the sequence x, the linear subspace generated by this set and the linear subspace generated by the set $\{\eta_1, \dots, \eta_n\}$, respectively, i.e.

$$(x) := \{\xi_n : n \in \mathbb{N}\}, \quad [x] := \operatorname{lin}\{\xi_n : n \in \mathbb{N}\}, \quad [\eta_1, \dots, \eta_n] := \operatorname{lin}\{\eta_1, \dots, \eta_n\}.$$

By m, we mean the Banach space of all bounded numerical sequences. Following I. Labuda and Z. Lipecki [15], we say that a sequence $\{\xi_n\}$ of elements of X is *m-independent* if $\sum_{n=1}^{\infty} \lambda_n \xi_n = 0$ implies $\lambda_n = 0$ for $n \in \mathbb{N}$ whenever $\{\lambda_n\} \in m$. Clearly, every m-independent sequence in X is linearly independent in X (i.e. the set of all its elements is linearly independent). A relationship in the converse direction was proved in [15] (see Lemma 5 below).

By *Fin* and *Inf* we denote the families of all finite and of all infinite subsets of the set $\mathbb{N}$ of all positive integers, respectively. If $x = \{\xi_n\}$ is a given sequence in X, then its subsequence $\{\xi_{p_n}\}$, where $\{p_n\}$ is an increasing sequence of positive integers, will also be denoted briefly by $x|_A$, where $A := \{p_1, p_2, \dots\} \in Inf$. Given an arbitrary sequence $x = \{\xi_n\}$ in X satisfying the condition:

$$\sum_{n=1}^{\infty} \|\xi_n\| < \infty, \tag{5}$$

we shall use the notation (see [6] and [12])

$$Z(x) := \left\{ \sum_{n\in A} \xi_n : A \in Inf \right\}, \qquad m(x) := \left\{ \sum_{n=1}^{\infty} \lambda_n \xi_n : \{\lambda_n\} \in m \right\}.$$

Due to completeness of the space X, elements of $Z(x)$ and $m(x)$ are well defined.

We need several lemmas. We begin by quoting some lemmas proved in [6] under Hypothesis H, which is weaker than Assumption A. The statements and proofs of results in [6] contain a number of misprints which are, however, easily spotted and corrected by the reader. We state lemmas from [6] (Lemmas 1 – 4 below), removing inaccuracies and using the notation just introduced.

Lemma 1. (see [6; Lemma 1] and [12; Theorem 1]) *Suppose that X, E, F satisfy Assumption* A *and that sequences x_1, x_2 in X such that $(x_1) \subset E$ and $(x_2) \subset F$ satisfy condition* (5). *Then there are subsequences y_1 and y_2 of x_1 and x_2, respectively, such that*

$$\big(E + m(y_1)\big) \cap \big(F + m(y_2)\big) = \{0\}.$$

Lemma 2. (see [6; Lemma 2']) *Suppose that X, E, F satisfy Assumption* A *and that a sequence x in X such that $(x) \subset F$ satisfies condition* (5). *If $Z(x) \cap F = \emptyset$, then there is a subsequence y of x such that*

$$Z(y) \cap (E \oplus F) = \emptyset.$$

Lemma 3. (see [6; Lemma 3]) *Suppose that X and E satisfy conditions* (a), (b), (e) *of Assumption* A *and that a sequence x in X such that $(x) \subset E$ satisfies condition* (5). *If $Z(x) \cap E = \emptyset$, then for every subspace H of X such that* $\dim H < \mathfrak{c}$ *we have*

$$Z(x) \cap \big(X \setminus (E + H)\big) \neq \emptyset.$$

LEMMA 4. (see [6; Lemmas 4 and 5] and [12; Theorem 1]) *Suppose that X, E, F satisfy Assumption* A. *Then*

$$\operatorname{codim} E = \operatorname{codim}(E \oplus F) = \mathfrak{c}\,.$$

The following result, proved for Hausdorff topological linear spaces by I. Labuda and Z. Lipecki in [15; Proposition 3] (see also [5; Theorem 3]), will be used in what follows.

LEMMA 5. (see [15; Proposition 3]) *Every linearly independent sequence in X satisfying* (5) *contains an m-independent subsequence.*

Applying the above lemmas, we now are going to prove three lemmas (Lemmas 6–8), needed in the proof of Theorem 1. For the sake of homogeneity, all three lemmas are formulated and proved under Assumption A, though Lemma 6 will be used in the proof of Theorem 1 only when E is a K_σ-subspace.

LEMMA 6. *Assume that X and E satisfy conditions* (a), (b), (e), (f) *of Assumption* A. *If H is another subspace of X such that $\aleph_0 \le \dim H < \mathfrak{c}$ and $E \cap H = \{0\}$, then $E \oplus H$ is not a K-space.*

Proof. Let x be a fixed m-independent sequence in X such that $(x) \subset H$ and (5) holds. Such a sequence exists by Lemma 5.

Setting $F := [x]$, $x_1 := 0$ and $x_2 := x$ in Lemma 1, we conclude that there exists a subsequence y of x such that

$$E \cap m(y) = E \cap \big([x] + m(y)\big) = \{0\}\,. \tag{6}$$

Let $\mathcal{S} := \{S_\alpha :\ \alpha < \mathfrak{c}\}$ be a Sierpiński family of sets in *Inf*, i.e.,

$$S_\alpha \cap S_\beta \in Fin \qquad \text{whenever} \quad \alpha \neq \beta\,. \tag{7}$$

Let us denote $y_\alpha := y|_{S_\alpha}$ for $\alpha < \mathfrak{c}$.

Suppose that $E \oplus H$ is a K-space. This means that $Z(y_\alpha) \cap (E \oplus H) \neq \emptyset$ for $\alpha < \mathfrak{c}$, i.e. there exist

$$\zeta^\alpha \in Z(y_\alpha) \tag{8}$$

of the form:

$$\zeta^\alpha = \varepsilon^\alpha + \eta^\alpha\,, \qquad \varepsilon^\alpha \in E\,, \qquad \eta^\alpha \in H\,, \tag{9}$$

for $\alpha < \mathfrak{c}$. Since $\dim H < \mathfrak{c}$, there exist $n \in \mathbb{N}$, non-zero scalars λ_i and ordinals $\alpha_i < \mathfrak{c}$ for $i \in \{1, 2, \dots, n\}$ such that

$$\sum_{i=1}^{n} \lambda_i \eta^{\alpha_i} = 0 \tag{10}$$

and thus, by (9),

$$\sum_{i=1}^{n} \lambda_i \zeta^{\alpha_i} = \sum_{i=1}^{n} \lambda_i \varepsilon^{\alpha_i} . \tag{11}$$

The right hand side of equation (11) is an element of E and the left hand side, by (8), is a member of $m(y)$. In view of (6), it follows that $\sum_{i=1}^{n} \lambda_i \zeta^{\alpha_i} = 0$ and, by (7), this contradicts the assumption that x is m-independent. □

Lemma 7. *Assume that X, E, F satisfy Assumption* A, *and that the subspace H of X satisfies assumption* (1) *of Theorem* 1, *and let $\tilde{F} := F \oplus H$. Let H_1, H_2 be subspaces of X and let x be a linearly independent sequence in X satisfying condition* (5) *such that*

$$\dim H_i < \mathfrak{c}, \quad (E \oplus \tilde{F}) \cap H_i = \{0\} \qquad (i \in \{1,2\}); \tag{12}$$

$$(\tilde{F} \oplus H_1) \cap (\tilde{F} \oplus H_2) = \tilde{F}; \tag{13}$$

$$[x] \subset \tilde{F} \oplus H_1, \qquad Z(x) \cap (\tilde{F} \oplus H_1) = \emptyset. \tag{14}$$

Then there exists ζ such that

$$\zeta \in Z(x) \setminus (E \oplus \tilde{F} + \tilde{H}), \tag{15}$$

where $\tilde{H} := H_1 + H_2$, and conditions (12) *and* (13) *are satisfied if the subspace H_1 is replaced by $\tilde{H}_1 := H_1 \oplus [\zeta]$.*

Proof. By Lemma 5, we may assume that x is an m-sequence. Notice that $F' := F + [x]$ is a K_σ-subspace of X, $(x) \subset F'$ and, by (14), $Z(x) \cap F' = \emptyset$. Moreover, $E \cap F' \subset E \cap (\tilde{F} \oplus H_1) = \{0\}$, by the first part of (14) and the second part of (12), so X, E, F' satisfy the assumptions of Lemma 2. Replacing F by F' in Lemma 2, we conclude that there is a subsequence y of x such that $Z(y) \cap (E \oplus F') = \emptyset$. Next, replacing x by y, H by $\tilde{H}$ and E by $E \oplus F'$ in Lemma 3 (all the assumptions of which are satisfied after this replacement; in particular, condition (e) for $E \oplus F'$ follows from Lemma 4), we see that there exists a $\zeta \in Z(y) \subset Z(x)$ satisfying (15).

Due to (15), the subspaces $\tilde{H}_1 := H_1 \oplus [\zeta]$ and H_2 satisfy conditions (12) and (13). □

Lemma 8. *Assume that X, E, F satisfy Assumption* A *and that subspaces H, H_1 and H_2 of X satisfy the assumptions* (1), (12) *and* (13), *respectively, with $\tilde{F} := F \oplus H$. Then the set $A := X \setminus (E \oplus \tilde{F} \oplus H_1)$ is nonempty and for an arbitrary $\xi \in A$, there exists $\eta \in X$ such that $\xi \in E \oplus [\eta]$ and*

$$\eta \notin (E \oplus \tilde{F} \oplus H_1) \cup (\tilde{F} \oplus \tilde{H}), \tag{16}$$

where $\tilde{H} := H_1 + H_2$, *and conditions* (12) *and* (13) *are satisfied if the subspace* H_1 *is replaced by* $\tilde{H}_1 := H_1 \oplus [\eta]$.

P r o o f. Since $\dim H < \mathfrak{c}$ and $\dim H_1 < \mathfrak{c}$, it follows from Lemma 4 that the set A is nonempty.

Fix an arbitrary $\xi \in A$. It is easy to see that every $\eta \in \xi + E$ does not belong to the first summand of the union in (16).

By condition (f) of Assumption A, there exists a system $\{\varepsilon^\alpha\}_{\alpha<\mathfrak{c}}$ of linearly independent elements of E. Suppose that $\xi + E \subset \tilde{F} \oplus \tilde{H}$. Then, for every $\alpha < \mathfrak{c}$, there exist $\phi^\alpha \in \tilde{F}$ and $\eta^\alpha \in \tilde{H}$ such that

$$\xi + \varepsilon^\alpha = \phi^\alpha + \eta^\alpha . \tag{17}$$

Since $\dim \tilde{H} < \mathfrak{c}$, equation (10) holds for some $n \in \mathbb{N}$, certain ordinals $\alpha_i < \mathfrak{c}$ and certain non-zero scalars λ_i, where $i \in \{1, 2, \dots, n\}$. By (17) and (10),

$$\left(\sum_{i=1}^{n} \lambda_i\right) \xi + \sum_{i=1}^{n} \lambda_i \varepsilon^{\alpha_i} = \sum_{i=1}^{n} \lambda_i \phi^{\alpha_i} .$$

Since $E \cap \tilde{F} = \{0\}$, we infer that $\sum_{i=1}^{n} \lambda_i \neq 0$ and so $\xi \in E \oplus \tilde{F} \oplus H_1$. This contradicts the assumption that $\xi \in A$. The contradiction means that the set $B := (\xi + E) \setminus (\tilde{F} \oplus \tilde{H})$ is non-empty. Clearly, (16) is valid for every $\eta \in B$.

By (16), the subspaces $\tilde{H}_1 := H_1 \oplus [\eta]$ and H_2 satisfy conditions (12) and (13), where η is an arbitrary element of the set B. □

For the proof of Proposition 2 we need the following simple lemma (which is generally true in arbitrary linear spaces):

LEMMA 9. *Let H be a proper subspace of X. Then there exists an automorphism $L\colon X \to X$ such that*

(d$_1$) $L(\xi) = \xi$ *for* $\xi \in H$;
(d$_2$) $L(\xi) \neq \xi$ *for* $\xi \in X \setminus H$.

P r o o f. Let Y be a subspace of X such that $X = H \oplus Y$. Since every $\xi \in X$ can be uniquely represented in the form $\xi = \eta + v$ with $\eta \in H$, $v \in Y$, it suffices to put $L(\xi) := \eta + 2v$ for example. □

4. Proofs of the main results

We are now going to prove Theorem 1 and Propositions 2 and 3, formulated in Section 2. Theorems 2 and 3 are particular cases of Theorem 1, and Proposition 1 is obvious and its proof is omitted here.

P r o o f o f T h e o r e m 1. Since $\operatorname{card} X = \mathfrak{c}$, the family $\mathcal{L}$ of all linearly independent sequences in X satisfying condition (5) also has cardinality $\mathfrak{c}$, and there exists a base $\mathcal{B}$ of cardinality $\mathfrak{c}$ of open sets in X. Arrange all the members of X, $\mathcal{L}$ and $\mathcal{B}$ into transfinite sequences $\{\xi^\alpha : \alpha < \mathfrak{c}\}$, $\{x_\alpha : \alpha < \mathfrak{c}\}$ and $\{\mathcal{W}_\alpha : \alpha < \mathfrak{c}\}$, respectively.

Set $\tilde{F} := F \oplus H$. We shall define, by transfinite induction, the subspaces $H_{\alpha,i}$ satisfying, for all $0 \le \alpha < \mathfrak{c}$ and $i \in \{1,2\}$, the following conditions:

(e$_1$) $\aleph_0 \le \dim H_{\alpha,i} < \aleph_0 + \operatorname{card}\alpha$, $\quad H_{\alpha,1} \cap H_{\alpha,2} = \{0\}$;

(e$_2$) $(E \oplus \tilde{F}) \cap H_{\alpha,i} = \{0\}$;

(e$_3$) $(\tilde{F} \oplus H_{\alpha,1}) \cap (\tilde{F} \oplus H_{\alpha,2}) = \tilde{F}$;

(e$_4$) $\xi^\alpha \in E \oplus \tilde{F} \oplus H_{\alpha,i}$;

(e$_5$) $\mathcal{W}_\alpha \cap H_{\alpha,i} \ne \emptyset$;

(e$_6$) $\tilde{H}_{\alpha,i} \subset H_{\alpha,i}$, where $\tilde{H}_{0,i} := \{0\}$ and $\tilde{H}_{\alpha,i} := \bigcup_{\beta<\alpha} H_{\beta,i}$ for $\alpha > 0$;

(e$_7$) $Z(x_{\kappa(\alpha,i)}) \cap (\tilde{F} \oplus H_{\alpha,i}) \ne \emptyset$,

where

$$\kappa(\alpha,i) := \min\{\beta < \mathfrak{c} :\ (x_\beta) \subset \tilde{F} \oplus \tilde{H}_{\alpha,i},\ Z(x_\beta) \cap (\tilde{F} \oplus \tilde{H}_{\alpha,i}) = \emptyset\}. \tag{18}$$

Clearly, (e$_6$) implies

$$\tilde{H}_{\beta,i} \subset H_{\beta,i} \subset \tilde{H}_{\alpha,i} \subset H_{\alpha,i}, \qquad \text{whenever} \quad 0 \le \beta < \alpha < \mathfrak{c}. \tag{19}$$

Fix $0 \le \alpha_0 < \mathfrak{c}$ and suppose that $H_{\alpha,i}$ are defined for $0 \le \alpha < \alpha_0$ and $i \in \{1,2\}$ so that conditions (e$_1$)–(e$_7$) are satisfied. First it should be noted that $\tilde{F} \oplus \tilde{H}_{\alpha_0,i}$ is not a K-space. This follows from Lemma 6, where E is replaced by F and H by $H \oplus \tilde{H}_{\alpha_0,i}$, due to (e$_1$) and (e$_2$). Hence the set of ordinals on the right hand side of (18) is nonempty and, consequently, $\kappa(\alpha_0,i)$ is well defined for $i \in \{1,2\}$ (in particular, the above is valid for $\alpha_0 = 0$).

Denote $\kappa_i := \kappa(\alpha_0,i)$ for $i \in \{1,2\}$. Notice that the conditions (e$_1$)–(e$_3$) are satisfied if $H_{\alpha,i}$ are replaced by $\tilde{H}_{\alpha_0,i}$. Moreover, by (18), $[x_{\kappa_i}] \subset \tilde{F} \oplus \tilde{H}_{\alpha_0,i}$ and $Z(x_{\kappa_i}) \cap (\tilde{F} \oplus \tilde{H}_{\alpha_0,i}) = \emptyset$, i.e. the assumptions of Lemma 7 are satisfied when x is replaced by x_{κ_i} and H_i by $\tilde{H}_{\alpha_0,i}$. Applying Lemma 7 twice, we find $\zeta^i \in X$ such that

$$\zeta^i \in Z(x_{\kappa_i}) \setminus (E \oplus \tilde{F} \oplus \tilde{H}_{\alpha_0}) \qquad \text{for} \quad i \in \{1,2\}, \tag{20}$$

where $\tilde{H}_{\alpha_0} := \tilde{H}_{\alpha_0,1} \oplus \tilde{H}_{\alpha_0,2}$. Moreover, ζ_1, ζ_2 are linearly independent and (12), (13), (14) are satisfied with H_i replaced by $\tilde{H}_{\alpha_0,i} \oplus [\zeta^i]$ for $i \in \{1,2\}$.

Fix $i \in \{1,2\}$. Suppose that

$$\xi^{\alpha_0} \in E \oplus \tilde{F} \oplus \tilde{H}_{\alpha_0,i} \oplus [\zeta^i]. \tag{21}$$

By Lemma 4, there exists $\omega^i \in \mathcal{W}_{\alpha_0}$ such that

$$\omega^i \notin E \oplus \tilde{F} \oplus \tilde{H}_{\alpha_0} \oplus [\zeta^1, \zeta^2].$$

In this case, taking an arbitrary $\omega^i \in \mathcal{W}_{\alpha_0}$ as above, we define

$$H_{\alpha_0,i} := \tilde{H}_{\alpha_0,i} \oplus [\zeta^i, \omega^i].$$

If (21) does not hold, then, by Lemma 8, there exists $\eta^i \in X$ such that $\xi^{\alpha_0} \in E \oplus [\eta^i]$ and the conditions (12), (13) and (16) hold with the respective replacements. Then we define

$$H_{\alpha_0,i} := \tilde{H}_{\alpha_0,i} \oplus [\zeta^i, \eta^i, \omega^i],$$

where ω^i denotes an arbitrary element of $\mathcal{W}_{\alpha_0}$ such that

$$\omega^i \notin E \oplus \tilde{F} \oplus \tilde{H}_{\alpha_0} \oplus [\zeta^1, \zeta^2, \eta^1, \eta^2].$$

Notice that η^1, η^2, ω^1, ω^2 satisfying the above requirements can be chosen in $\mathcal{W}_{\alpha_0}$ to be linearly independent elements. The subspaces $H_{\alpha_0,i}$ just defined satisfy (e_1)–(e_6) and

$$\zeta^i \in H_{\alpha_0,i} \subset \tilde{F} \oplus H_{\alpha_0,i}$$

for $i \in \{1,2\}$. By (20), this means that they also satisfy condition (e_7). The induction construction is thus completed.

We define the linear subspaces

$$Y_i := \bigcup_{\alpha<\mathfrak{c}} \tilde{F} \oplus H_{\alpha,i} \qquad \text{for} \quad i \in \{1,2\}. \tag{22}$$

Fix $i \in \{1,2\}$ and suppose that Y_i is not a K-space, i.e. $(x) \subset Y_i$ and $Z(x) \cap Y_i = \emptyset$ for some linearly independent sequence x satisfying (5). By (22),

$$Z(x) \cap (\tilde{F} \oplus H_{\alpha,i}) = \emptyset \tag{23}$$

for all $\alpha < \mathfrak{c}$, and there exists an $\alpha_0 < \mathfrak{c}$ such that $(x) \subset \tilde{F} \oplus H_{\alpha_0,i}$. By (19),

$$(x) \subset \tilde{F} \oplus H_{\alpha,i} \qquad \text{whenever} \quad \alpha_0 \le \alpha < \mathfrak{c}. \tag{24}$$

Evidently, $x = x_{\alpha_1}$ for a certain $\alpha_1 < \mathfrak{c}$. Hence, by (24), (23) and (18),

$$\alpha_0 \le \alpha < \mathfrak{c} \implies \kappa(\alpha, i) \le \alpha_1. \tag{25}$$

On the other hand, it follows from condition (18), (e_7), and (19) that

$$\alpha < \alpha' < \mathfrak{c} \implies \kappa(\alpha, i) < \kappa(\alpha', i),$$

which contradicts (25). Since, in addition, $Y_i \cap \mathcal{W}_\alpha \neq \emptyset$ for $\alpha < \mathfrak{c}$, we conclude that Y_i is a dense K-subspace in X.

To complete the proof notice that equations (2) follow from (e_4) and equation (3) follows from (e_3) and (19). □

P r o o f o f P r o p o s i t i o n 2. Let $x = \{\xi_n\}$ be an arbitrary m-independent sequence in $(X, \|\cdot\|_1)$ satisfying (5) (with $\|\cdot\|_1$ instead of $\|\cdot\|$). Put $H := [x]$ and let $L\colon X \to X$ be an automorphism fulfilling assertions (d_1) and (d_2) of Lemma 9. Define $\|\xi\|_2 := \|L(\xi)\|_1$ for $\xi \in X$. Clearly, $\|\cdot\|_2$ is an F-norm and (b_1) holds true.

To prove (b_2) denote $\|\cdot\|_3 := \|\cdot\|_1 + \|\cdot\|_2$. By (d_1), we have

$$\|\xi_n\|_3 = \|\xi_n\|_1 + \|L(\xi_n)\|_1 = 2\|\xi_n\|_1 \to 0$$

as $n \to \infty$. If $(X, \|\cdot\|_3)$ were a K-space, there would exist a subsequence $\{\xi_{p_n}\}$ of the sequence x and $\eta \in X$ such that $\|\eta_n - \eta\|_3 \to 0$ as $n \to \infty$, where $\eta_n := \sum_{k=1}^{n} \xi_{p_k}$. Then, by the definition of the norm $\|\cdot\|_3$, we would have

$$\|\eta_n - \eta\|_i \to 0 \qquad \text{for} \quad i \in \{1, 2\} \tag{26}$$

and thus, by the definition of $\|\cdot\|_2$,

$$\|\eta_n - L(\eta)\|_1 = \|L(\eta_n) - L(\eta)\|_1 = \|\eta_n - \eta\|_2 \to 0 \tag{27}$$

as $n \to \infty$. Since x is m-independent in $(X, \|\cdot\|_1)$, (26) implies $\eta \notin H$. On the other hand, (26) and (27) imply $\eta = L(\eta)$, which contradicts assertion (d_2) of Lemma 9 and finishes the proof. □

We shall now derive Proposition 3 from Theorem 3.

P r o o f o f P r o p o s i t i o n 3. Let E be a closed subspace of $(X, \|\cdot\|)$ of infinite dimension and infinite codimension and let H, Y_1, Y_2 be as in Theorem 3 with $\dim H \geq \aleph_0$. By (2), every $\xi \in X$ has a unique representation in the form:

$$\xi = \varepsilon^i + v^i, \quad \varepsilon^i \in E, \quad v^i \in Y_i \qquad \text{for} \quad i \in \{1, 2\}. \tag{28}$$

Defining

$$\begin{aligned} &\|\xi\|_i := \|\varepsilon^i\| + \|v^i\| \qquad \text{for} \quad i \in \{1, 2\}, \\ &\|\xi\|_3 := \|\xi\|_1 + \|\xi\|_2. \end{aligned} \tag{29}$$

we have

$$\|\xi\| \leq \|\xi\|_i \leq \|\xi\|_3 \qquad \text{for} \quad i \in \{1, 2\}. \tag{30}$$

Hence, in particular, the topology generated by each of the norms $\|\cdot\|_1$, $\|\cdot\|_2$, is stronger than the original topology in X.

Since Y_i are K-spaces and E, being a closed subspace of X, is an N-space, it follows from (28) and (29) that $(X, \|\cdot\|_i)$ is a K-space for $i \in \{1,2\}$ (cf. (a_2) in Proposition 1), i.e. assertion (c_1) is true.

Before proving (c_2) notice that the subspaces Y_1 and Y_2 of X are closed in the topologies of the norms $\|\cdot\|_1$ and $\|\cdot\|_2$, respectively. In fact, if $v_n \in Y_i$ for $n \in \mathbb{N}$ and $\|v_n - \xi^i\|_i \to 0$ for some $\xi^i \in X$ of the form (28), then

$$\|\varepsilon^i\| + \|v^i - v_n\| = \|\xi^i - v_n\|_i \to 0$$

as $n \to \infty$, i.e. $\varepsilon^i = 0$ and $\xi^i = v^i \in Y_i$ for $i \in \{1,2\}$ (see also [16; Corollary 3]).

Now suppose that $(X, \|\cdot\|_3)$ is a K-space and consider an arbitrary linearly independent sequence $x = \{\xi_n\}$ in X such that $(x) \subset H$ and $\|\xi_n\| \to 0$. By (4), (28) and (29), we have $\|\xi_n\|_1 = \|\xi_n\| = \|\xi_n\|_2$ for $n \in \mathbb{N}$. Consequently, $\|\xi_n\|_3 = 2\|\xi_n\| \to 0$ as $n \to \infty$. Therefore there exists a subsequence y of x which is summable to a certain $\eta \in X$ in the norm $\|\cdot\|_3$ and, by (30), also in $\|\cdot\|_i$ $(i \in \{1,2\})$ and in the original norm $\|\cdot\|$. Since y is a sequence in $H = Y_1 \cap Y_2$, summable to η in the norms $\|\cdot\|_1$ and $\|\cdot\|_2$, we conclude that $\eta \in Y_1 \cap Y_2 = H$ and thus $(H, \|\cdot\|)$ is a K-space. This, however, contradicts the assumption that $\dim H < \mathfrak{c}$ (see [15; Corollary 1]). Consequently, (c_2) holds. □

Added in proof:

Property K was also studied in

CHERESIZ, V. M.: *Equicontinuity of group representations*, Sibirsk. Mat. Ž. **19** (1978), 1381–1385 (Russian).

Proposition 2 is related to the result that the supremum of two non comparable complete norms can be not barrelled, given in

DE WILDE, M.—TSIRULNIKOV, B.: *Barrelledness and the supremum of two locally convex topologies*, Math. Ann. **246** (1980), 241–248.

Every Banach space X of dimension $\mathfrak{c}$ has the property mentioned in the assumption of Proposition 3. It is an open problem whether every F-space of this dimension has the property.

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