0. Abstract

Extremal graph theory was begun by Paul Turán when he determined the maximum size of an extremal graph $G$ not containing a prescribed complete graph $K_n$. Paul Erdős and colleagues later found the minimum size of a maximal $G$ not containing $K_n$. The corresponding questions are studied here when the forbidden subgraph $F$ is a path or a star or a matching.

1. Introduction

Given graphs $F$ and $G$, we say that $G$ is $F$-maximal if $F \not\subseteq G$ but $F \subseteq G + e$ for each line $e$ of $G$. The original extremal result of Turán [11] gave for each $n \geq 3$ an exact formula for the maximum size (number of lines) $t(K_n, p)$ in a $K_n$-maximal graph $G$ of given order (number of points) $p$. Erdős, Hajnal and Moon [3] developed a corresponding formula for the size $b(K_n, p)$ of a minimum $K_n$-maximal graph $G$. The most celebrated problem area of extremal graph theory has been to extend Turán's formula to other forbidden subgraphs $F$, by determining the value of $t(F, p)$. The author of the definitive treatise [1] on extremal graph theory, Béla Bollobás (verbal communication) asserted to one of us that even for the simple (in appearance only) case when $F = C_4$, the quadrilateral, the problem is utterly intractable.

Since the general problem is so difficult, we consider for the forbidden subgraph $F$ three simple families of forests: the stars $K_{1,n}$, the matchings $nK_2$ and the paths $P_n$. For each of these families we investigate not only the Turán problem of determining the maximum size $t(F, p)$ of a maximal graph not containing $F$, but also the corresponding minimum size $b(F, p)$ as introduced in [5]. Values of $b(K_{1,n}, p)$, $b(nK_2, p)$ and $b(P_n, p)$ are obtained when $p$ is sufficiently large with respect to $n$. In general, determining $b(F, p)$ seems to be a much more tractable problem than finding $t(F, p)$.
The notation of [4] is followed, in particular \( \Delta(G) \) is the maximum degree and \( \langle U \rangle \) is the subgraph induced by \( U \subset V(G) \). Also \( p(G) \) is the order and \( q(G) \) the size of the graph.

2. Extremal numbers for forbidden stars

Determining \( t(K_{1,n}, p) \) and \( b(K_{1,n}, p) \) are the two easiest extremal problems of this type. Note that if \( G \) is a \( K_{1,n} \)-maximal graph, then \( \Delta(G) \leq n - 1 \) but \( \Delta(G + e) = n \) for any line e in \( G \). For the remainder of this section, we assume that \( p \geq n + 1 \); if not, then obviously \( t(K_{1,n}, p) = b(K_{1,n}, p) = p(p - 1)/2 \) and the unique extremal graph is \( K_p \).

**Theorem 1.** Let \( \zeta \) be 1 if both \( p \) and \( n - 1 \) are odd, and 0 otherwise, so \( \zeta = p(n - 1) \mod 2 \). Then

\[
t(K_{1,n}, p) = ((n - 1)p - \zeta)/2.
\]

**Proof.** Let \( G \) be a graph of order \( p \) and size \( q > ((n - 1)p - \zeta)/2 \). Then the average degree of the points of \( G \) is at least \(((n - 1)p - \zeta + 2)/p > n - 1 \), so that \( \Delta(G) \geq n \), i.e., \( K_{1,n} \subset G \). Therefore, \( t(K_{1,n}, p) \leq ((n - 1)p - \zeta)/2 \).

To see that \( t(K_{1,n}, p) \geq ((n - 1)p - \zeta)/2 \), we merely note [4, p. 89] that \( K_p \) is the sum of \((p - 1)/2\) spanning cycles for \( p \) odd and the sum of a 1-factor and \((p - 2)/2\) spanning cycles for \( p \) even. Using these facts, it is trivial to construct a graph \( G \) of order \( p \) and size \(((n - 1)p - \zeta)/2 \) such that \( \Delta(G) = n - 1 \).

Given \( n \) and \( p \), let \( \mathcal{M}(n, p) \) be the family of \( K_{1,n} \)-maximal graphs of order \( p \) and size \( t(K_{1,n}, p) \). Obviously all points of any graph in \( \mathcal{M}(n, p) \) have degree \( n - 1 \) unless both \( p \) and \( n - 1 \) are odd, in which case there is a unique point \( w \) with degree \( n - 2 \). Somewhat surprisingly, we shall use these graphs to obtain the minimum \( K_{1,n} \)-maximal graphs.

**Theorem 2.** Let \( \sigma \) equal one if both \( n - 1 \) and \( p - n/2 \) are odd integers, and zero otherwise, so that \( \sigma = (n - 1)(p - n/2) \mod 2 \). Then if \( p \geq [3n/2] \),

\[
b(K_{1,n}, p) = (p(n - 1) - [n/2] + [n/2]^2 + \sigma)/2.
\]

**Proof.** Let \( G \) be a minimum \( K_{1,n} \)-maximal graph of order \( p \) and as usual let \( V(G) = \{v_1, \ldots, v_p\} \). Let \( r \) be the number of points of \( G \) with degree less than \( n - 1 \), and assume without loss of generality that these points are \( v_1, \ldots, v_r \) (we may regard \( r > 0 \), for otherwise \( G \) has size \( t(K_{1,n}, p) \), the maximum size among all maximal graphs). Because \( G \) is maximal, its induced subgraph \( \langle v_1, \ldots, v_r \rangle \) is \( K_r \).

Also, \( \deg(v_i) = n - 1 \) for \( r + 1 \leq i \leq p \), and since \( G \) is a minimum maximal graph, this implies that \( \langle v_{r+1}, \ldots, v_p \rangle \in \mathcal{M}(n, p - r) \). If both \( n - 1 \) and \( p - r \) are odd, then \( \langle v_{r+1}, \ldots, v_p \rangle \) contains just one point, say \( v_{r+1} \), with degree \( n - 2 \), so that \( G \) has
If a line joining \( v_{r+1} \) and \( v_i \) for some \( i \leq r \). If they are not both odd, then \( G \) has no lines between point sets \( \{v_1, \ldots, v_r\} \) and \( \{v_{r+1}, \ldots, v_p\} \).

Consequently, \( G \) has size

\[
q = (r(r-1) + (p-r)(n-1) + \sigma)/2. \tag{1}
\]

If the \( \sigma \) term is disregarded, equation (1) is a quadratic in the variable \( r \), and then it follows that its minimum among integers \( r \) is attained at both \( r = [n/2] \) and \( [p/n] \). However, \( \sigma = 1 \) only if \( n \) is even, in which case \( r = n/2 \) still gives a minimum for (1), since it previously was the unique minimum. Therefore, \( r = [n/2] \) always yields a minimum for \( q \) (in (1), giving the value in the statement of the theorem.

3. Extremal values for forbidden matchings

The evaluation of \( t(nK_2, p) \) is not very difficult. In fact, Simonovits [10] determined the maximum \( nK_2 \)-maximal graphs for any positive integer \( s \), after Erdős [2] and Moon [8] obtained partial solutions. We state the result only for the case \( s = 2 \).

**Theorem A** [10]. If \( 2n \leq p \), then the unique maximum \( nK_2 \)-maximal graph of order \( p \) is \( K_{n-1} + K_{p-n+1} \).

The value of \( t(nK_2, p) \) now follows immediately.

**Corollary.** If \( 2n \leq p \), then \( t(nK_2, p) = (n-1)(2p-n)/2 \).

The value of \( b(nK_2, p) \) is also easy to find when \( p \) is large enough relative to \( n \).

**Theorem 3.** If \( p \geq 3(n-1) \) then \( b(nK_2, p) \geq 3(n-1) \).

Proof. Obviously \( (n-1)K_3 \cup (p-3n+3)K_1 \) is \( nK_2 \)-maximal, so \( b(nK_2, p) \leq 3(n-1) \).

Now let \( G \) be an \( nK_2 \)-maximal graph of order \( p \), and let \( S = \{u_1v_1, \ldots, u_{n-1}v_{n-1}\} \) be a set of \( n-1 \) independent lines in \( G \). Let \( G_i = G - [V(S) - u_i - v_i] \) for \( i = 1, \ldots, n-1 \). Because \( G \) is \( nK_2 \)-maximal, the lines of \( G_i \) must form either a triangle or a star, so that \( q(G_i) = 3 \) or \( p - 2n + 3 \) for each \( i \). Since \( E(G_j) \cap E(G_i) = \emptyset \) for \( j \neq i \), it follows that \( q(G) \geq 3(n-1) \).

In fact, the \( nK_2 \)-maximal graphs have been characterized by Mader.

**Theorem B** [7]. The \( nK_2 \)-maximal graphs are the join graphs \( K_{k_0} + (K_{k_1} \cup K_{k_2} \cup \ldots \cup K_{k_t}) \) where \( k_0 \geq 0 \), \( t \geq k_0 + 2 \), every \( k_j \) is odd for \( j \geq 1 \), and \( n-1 = k_0 + \sum_{j=1}^{t} (k_j - 1)/2 \).

Note that \( k_0 = 0 \) gives the “null graph” \( K_0 \) which was intensively investigated in [6].

If \( p < 3(n-1) \), then the formula given in Theorem 3 for \( b(nK_2, p) \) is not valid. However, the method of the proof of Theorem 3 can be used to show that the
unique minimum $nK_2$-maximal graph $G$ has the form of Theorem B with $k_0 = 0$ and $|k_i - k_j| \leq 2$ for $1 \leq i, j \leq t$, so that $G$ is a union of complete graphs.

4. Extremal numbers for forbidden paths

We turn now to the problem of computing extremal values with a path as the forbidden subgraph. By way of comparison, $b(C_4, p)$ has been determined by Oilman [9], but $b(C_n, p)$ is unknown for $n \geq 5$. Also, as mentioned in the introduction, determining $t(C_n, p)$ is an extremely difficult problem even for $n = 4$.

As might be expected, evaluating $t(P_n, p)$ also appears to be quite hard and no major results are known. However, it is possible to compute $b(P_n, p)$ for values of $p$ that are 'large enough' with respect to $n$. First we require some preliminary results on maximal trees, which are developed in four lemmas.

**Lemma 4.1.** If $T$ is a $P_n$-maximal tree of order $p \geq 3$, then $T$ has no points of degree 2 (is homeomorphically irreducible).

**Proof.** Suppose $v$ is a point of $T$ of degree 2 with neighbors $u$, $w$. Then the line $uw$ is not in $T$, so $T + uw$ contains a path $P^*$ of order $n$ and $uw$ is in $P^*$. Now $P^*$ must also contain either $vu$ or $vw$, for otherwise removing $uw$ from $P^*$ and adding $vu$ and $vw$ to it will yield a path on $n + 1$ points in $T$. So, suppose without loss of generality that line $uv$ is in $P^*$. Since $v$ has degree 2 in $T$, it is an endpoint of $P^*$, which must now have the form $v, u, w, x_4, \ldots, x_n$. But then $P^* - u, v, w, x_4, \ldots, x_n$ is an $n$-point path in $T$, a contradiction.

**Lemma 4.2.** If $T$ is a $P_n$-maximal tree of order $p \geq 3$, then $T \Rightarrow O_{n-1}$.

**Proof.** Let $uv, vw$ be adjacent lines in $T$. Then $T + uw$ contains $P_n$ by the maximality of $T$. But it is easily verified that the maximum length of a path in $T + uw$ is at most one more than its length in $T$, so that $T \Rightarrow P_{n-1}$.

For $n \geq 4$, let $f(n)$ be the minimum order of a $P_n$-maximal tree $T$ other than the trivial cases $K_1$ and $K_2$. Also, we define $f(3) = 2$ and $f(2) = 1$. It will soon be shown that $f(n)$ is finite.

**Lemma 4.3.** If $n \geq 4$, then $f(n) \geq 2f(n-2) + 2$.

**Proof.** Let $T$ be a $P_n$-maximal tree of order $f(n), n \geq 4$. Let $T'$ be the graph obtained from $T$ by deleting all its endpoints. Since $T$ has at least three points, $diam(T') = diam(T) - 2$, so that $P_{n-2} \not\subseteq T'$. Obviously any line $e$ in $T'$ is also in $T$. Since the unicyclic graph $T + e$ contains $P_n$ by the maximality of $T$, it follows that $P_{n-2} \subseteq T' + e$ because by Lemma 4.1, a longest path in $T + e$ begins and ends at endpoints of $T$. Hence $T'$ is a $P_{n-2}$-maximal tree.

By Lemma 4.2, $T'$ can be $K_1$ or $K_2$ only if $n$ is 4 or 5, so

$$p(T') \geq f(n - 2). \quad (4.1)$$

On the other hand, recall that $p(T) = f(n)$. By Lemma 4.1, no points of $T$ have
degree 2 and since $q(T) = f(n) - 1$, it follows that $T$ has at least $1 + f(n)/2$ endpoints. Therefore by the construction of $T'$,

$$p(T') \leq f(n) - (1 + f(n)/2). \quad (4.2)$$

Combining inequalities (4.1) and (4.2) then yields the desired result. □

Now define recursively a family of trees $T_n$ by setting $T_2 = K_1$, $T_3 = K_2$, $T_4 = K_{1,3}$ and for $n > 4$, let $T_n$ be the tree obtained from $T_{n-2}$ by adding to each endpoint $v$ of $T_{n-2}$, two new endpoints adjacent only to $v$, as illustrated in Figure 1.

It is straightforward to verify that $T_n$ is $P_n$-maximal. Moreover, $p(T_n) = 2p(T_{n-2}) + 2$ for $n \geq 4$, so that by Lemma 4.3, $T_n$ is a minimum order $P_n$-maximal graph and $f(n) = p(T_n)$ for $n \geq 2$. A simple counting argument then shows that

$$f(n) = \begin{cases} 
3 \cdot 2^{(n-2)/2} - 2 & \text{for } n \text{ even} \\
2^{(n+1)/2} - 2 & \text{for } n \text{ odd}.
\end{cases} \quad (4.3)$$

It can also be shown that $T_n$ is the unique $P_n$-maximal tree of order $f(n)$ although the tedious proof will not be included here.

**Lemma 4.4.** Let $u, w$ be endpoints of $T_n$, $n \geq 4$, which are both adjacent to the point $v$. Let $T_n'$ be the graph obtained from $T_n$ by adding $r \geq 1$ new endpoints, each adjacent to $v$. Then $T_n'$ is also $P_n$-maximal.
Proof. Since \( p(T_n) \geq 3 \), \( \text{diam } (T_n^*) = \text{diam } (T_n) \) and so \( P_n \not\subseteq T_n^* \). Given a line \( e \) in \( T_n^* \), let \( \{x_1, \ldots, x_\ell\} \) be \( r \) endpoints in \( T_n^* + e \), which are adjacent to \( v \) but not incident with \( e \). Then \( T_n^* - \{x_1, \ldots, x_\ell\} = T_n \) and it follows that \( P_n \not\subseteq T_n^* + e \).

We are finally ready to return to the problem of determining \( b(P_n, p) \). Obviously, \( b(P_2, p) = 0 \) and \( b(P_3, p) = \lceil p/2 \rceil \). Also \( b(P_4, p) = p/2 \) for \( p \) even and \( (p + 3)/2 \) for \( p \) odd, \( p \geq 3 \). When \( p \) is \textquoteleft large enough\textquoteright, we have the following result.

**Theorem 4.** If \( p \geq f(n) \), then

\[
b(P_n, p) = \begin{cases} p - 1 - \lfloor (p - 2)/f(n) \rfloor & \text{for } n = 5, \\ p - \lceil p/f(n) \rceil & \text{for } n \geq 6, \end{cases}
\]

where \( f(n) \) is as in (4.3).

Proof. Obviously \( b(P_n, p) \leq p - 1 \) since by Lemma 4.4 the tree \( T_n^{f(n)} \) is \( P_n \)-maximal. In general, if \( b(P_n, p) = p - k \), then at least \( k \) of the components of any minimum size \( P_n \)-maximal graph \( G \) must be trees. If \( T \not= K_1 \) is any tree such that \( P_n \not\subseteq T \), then obviously \( T \cup K_1 \) is not \( P_n \)-maximal, so \( K_1 \) cannot be a component of \( G \) when \( k \geq 2 \).

If \( n = 5 \), it is easy to see that \( K_2 \cup T_n \) is \( P_n \)-maximal but \( 2K_2 \) is not. It follows that \( k \) is equal to \( 1 + \lfloor (p - 2)/f(n) \rfloor \).

If \( n \geq 6 \) and \( T \) is any \( P_n \)-maximal tree, then \( K_2 \cup T \) is not \( P_n \)-maximal since adding a line \( e \) between a point of \( K_2 \) and a central point of \( T \) would still leave \( \text{diam } (K_2 \cup T + e) = n - 2 \). Thus in this case \( k = \lceil p/f(n) \rceil \), that is, \( k \) is the maximum integer \( r \) such that \( r \cdot f(n) \leq p \).

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**REFERENCES**


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МАКСИМАЛЬНЫЕ ГРАФЫ С ЗАПРЕЩЕННЫМИ ПОДГРАФАМИ
ОБЛАДАЮЩИЕ МИНИМАЛЬНЫМ ЧИСЛОМ РЕБЕР

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Резюме

П. Туран установил максимальное число ребер графа несодержащего заданный полный
граф $K_n$. П. Эрдёш и его сотрудники нашли минимальное число ребер максимального графа
несодержащего $K_n$. В работе изучаются соответствующие вопросы для случаев, когда запре-
щенным графом является путь, или звезда, или паросочетание.