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ON THE OSCILLATION OF NONLINEAR DIFFERENTIAL SYSTEMS WITH RETARDED ARGUMENTS

PAVOL MARUŠIÁK

1. Introduction

We consider systems of nonlinear differential inequalities with retarded arguments of the form

\[ y'_i(t) - f_i(t, y_{i+1}(t), y_{i+1}(h_i(t))) = 0, \quad i = 1, 2, ..., n-1, \]  
\[ \{y'_i(t) + f_i(t, y_i(t), y_i(h_i(t)))\} \text{ sgn } y_i(h_i(t)) \leq 0. \]  

where the following conditions are always assumed:

(a) \( h_i : [a, \infty) \rightarrow \mathbb{R} \) (\( i = 1, 2, ..., n \)) are continuous and \( h_i(t) \leq t \text{ for } t \geq a, \lim_{t \to \infty} h_i(t) = \infty, \) (\( i = 1, 2, ..., n \));

(b) \( f_i : [a, \infty) \times \mathbb{R}^2 \rightarrow \mathbb{R} \) (\( i = 1, 2, ..., n \)) are continuous, \( v f_i(t, u, v) \geq 0 \) (\( i = 1, 2, ..., n \)) for \( uv > 0 \)

and not identically zero on any subinterval of \([a, \infty); f(t, u, v) \) (\( i = 1, 2, ..., n-1 \)) are nondecreasing in \( u \) and \( v \) for each fixed \( t \in [a, \infty) \).

Denote by \( W \) the set of all solutions \( y(t) = (y_1(t), ..., y_n(t)) \) of the system (S) which exist on some ray \([T, \infty) \subset [a, \infty)\) and satisfy \( \sup_{t \geq T} \left\{ \sum_{i=1}^{n} |y_i(t)| : t \geq T \right\} > 0 \) for any \( T \geq T_y \).

**Definition 1.** A solution \( y \in W \) is called oscillatory (resp. weakly oscillatory) if each component (resp. at least one component) has arbitrarily large zeros. A solution \( y \in W \) is called nonoscillatory (resp. weakly nonoscillatory) if each component (resp. at least one component) is eventually of a constant sign.

**Definition 2.** We shall say that the system (S) has the property A, if every solution \( y \in W \) is oscillatory for \( n \), even, while for \( n \) odd it is either oscillatory or \( y_i \) (\( i = 1, 2, ..., n \)) tend monotonically to zero as \( t \to \infty \).
The oscillatory properties of solutions of two-dimensional differential systems with deviating arguments are studied in the following papers: Kitamura and Kusano [2, 3], Varech, Gritsai and Ševalo [4], Ševalo and Varech [5, 6]. The oscillation results for the system \( x'_k(t) = f_k(t, x(q_1(t)), \ldots, x(q_n(t))) \), \( k = 1, 2, \ldots, n \) were studied, followed by Foltynska and Werbowski [1].

In the present paper we proceed further in this direction to extend the theory developed in [4—6] to the systems of the form (S). Our results in lude some of the results in [1, 5, 6] and they do not follow from Theorem 1 in [1].

### 2. Oscillation Theorems

We introduce the notation:

\[
\gamma_i(t) = \sup \{ s > 0; h_i(s) < t \} \quad \text{for} \quad t \geq a, i - 1, 2 \ldots n,
\]

\[
\gamma(t) - \max \{ \gamma_i(t), \ldots, \gamma_n(t) \}.
\]

**Lemma 1.** Let \( \mathbf{y} = (y_1, \ldots, y_n) \in W \) be a weakly nonoscillatory solution of (S), then \( \mathbf{y} \) is nonoscillatory.

**Proof.** Suppose that \( y_k \) is a nonoscillatory component of solution \( \mathbf{y} = (y_1, \ldots, y_k, \ldots, y_n) \in W \) and \( y_k(t) \neq 0 \) for \( t \geq t_i, a \).

i) Let \( 1 < k \leq n \). With the help of (a), (b), the system (S) implies that either

\[
y'_k(t) \geq 0 \quad \text{or} \quad y'_k(t) \leq 0 \quad \text{for} \quad t \geq \gamma(t_i) - t_i,
\]

and not identically zero on any infinite subinterval of \([t_i, \infty)\). We remark that \( y_k(t) \neq 0 \) for all \( t \geq t_2 \geq t_i \). If \( y_k(t) \equiv 0 \) for \( t \geq t_2 \), then \( y'_k(t) \equiv 0 \) for \( t \geq t_2 \) and the \((k-1)\)-st equation of (S) gives that \( f_k(t, y_k(t), y_k(h_k))) = 0 \) for all \( t \geq t \), which contradicts assumption (b). From (S) we get that \( y_k(t) \) is the monotone function and thus there exists a \( t_i \geq t_i \) such that \( y_k(t) \neq 0 \) for \( t \geq t_i \). We have proved that \( y_k \) is the nonoscillatory component of \( \mathbf{y} \). Analogously we can prove that \( y_{k-1}(t), \ldots, y_1(t) \) are also nonoscillatory components of \( \mathbf{y} \).

ii) Let \( k = 1 \). From the \( n \)-th inequality of (S) we obtain \( y'_i(t) \text{ sgn } y_i(h_i(t)) \leq 0 \) for \( t > t_i \) and not identically zero on any subinterval of \([t_i, \infty)\). Thus there exists \( a t_i \geq t_i \) such that \( y_n(t) \neq 0 \) for \( t \geq t_i \). If we consider now the case i) for \( k = n \), we get that all components of \( \mathbf{y} \) are nonoscillatory.

The proof of Lemma 1 is complete.

**Lemma 2.** Suppose that

\[
y - (y_1, \ldots, y_n) \in W
\]

is a nonoscillatory solution of (S) in the interval \([a, \infty)\). If

\[
\int_{T}^{*} |f_k(t, c, c)| \, dt = \infty \quad \text{for all} \quad c \neq 0, k = 1, 2, \ldots, n
\]

then

\[
\text{Lemma 2.} \quad \int_{T}^{*} |f_k(t, c, c)| \, dt = \infty \quad \text{for all} \quad c \neq 0, k = 1, 2, \ldots, n
\]
then there exist an integer \( l \in \{1, 2, ..., n\} \), \( n + l \) even, and a \( t_0 \geq a \) such that
\[
y_i(t)y_i(t) > 0 \text{ on } [t_0, \infty) \text{ for } i = 1, 2, ..., l, \quad (4)
\]
\[
(-1)^{n+l}y_i(t)y_i(t) > 0 \text{ on } [t_0, \infty) \text{ for } i = l + 1, ..., n \quad (5)
\]
hold.

Proof. Without loss of generality we may suppose that \( y_i(t) > 0 \) for \( t \geq a \). Similar arguments hold if \( y_i(t) < 0 \). According to (a) there exists a \( T_1 \geq \gamma(a) \) such that \( y_i(h_1(t)) > 0 \) for \( t \geq T_1 \). Then the \( n \)-th inequality of (S) implies that \( y_n(t') \) is nonincreasing on \([T_1, \infty)\) and not identically zero on any infinite subinterval of \([T_1, \infty)\). We shall show that \( y_n(t) \geq 0 \) for \( t \geq T_1 \). If \( y_n(t) < 0 \) for some \( t_1 \geq T_2 \), then \( y_n(t) \leq y_n(t_1) = c_n < 0 \) for \( t \geq t_1 \). Taking this into account and then integrating the \((n-1)\)st equation of (S) from \( t_2 = \gamma(t_1) \) to \( t \), we have
\[
y_{n-1}(t) = y_{n-1}(t_2) + \int_{t_2}^{t} f_{n-1}(s, y_n(s), y_n(h_n(s))) \, ds \leq \]
\[
\leq y_{n-1}(t_2) + \int_{t_2}^{t} f_{n-1}(s, c_n, c_n) \, ds \to -\infty \text{ as } t \to \infty.
\]

Then there exists a \( t_3 \geq \gamma(t_2) \) such that \( y_{n-1}(t) \leq c_{n-1} < 0 \), \( y_{n-1}(h_{n-1}(t)) \leq c_{n-1} \) for \( t \geq t_3 \). Integrating again the \((n-2)\)nd equation of (S) we prove that \( y_{n-2}(t) \to -\infty \) as \( t \to \infty \). Similarly we shall prove that \( y_i(t) \to -\infty \) as \( t \to \infty \) (\( i = n-3, ..., 2, 1 \)), which contradicts \( y_i(t) > 0 \) for \( t \geq a \). Therefore \( y_n(t) > 0 \) on \([T_2, \infty)\). Thus with the help of the \((n-1)\)st equation we obtain that \( y_{n-1}(t) \) is a nondecreasing function for \( t \geq T_3 = \gamma(T_2) \) and that it is eventually of one sign. a) Let \( y_{n-1}(t) \geq c_{n-1} > 0 \) for \( t \geq T_4 \). Taking this into account and integrating the \((n-2)\)nd equation of (S) from \( T_4 \) to \( t \), we obtain
\[
y_{n-2}(t) \geq y_{n-2}(T_4) + \int_{T_4}^{t} f_{n-2}(s, c_{n-1}, c_{n-1}) \, ds \to -\infty
\]
as \( t \to \infty \). Repeating this method, we prove that \( y_i(t) > 0 \) (\( i = 1, 2, ..., n-1 \)) for \( t \geq T_5 \). Therefore (4) is true for \( l = n \).

b) Let \( y_{n-1}(t) < 0 \) on \([T_3, \infty)\). Then the \((n-2)\)nd equation of (S) implies that \( y_{n-2}(t) \) is nonincreasing for \( t \geq T_6 = \gamma(T_3) \) and that it is eventually of one sign. We show that \( y_{n-2}(t) > 0 \) for \( t \geq T_7 \). If \( y_{n-2}(t) < 0 \) for some \( t_4 \geq T_7 \); then \( y_{n-2}(t) \geq y_{n-2}(t_4) = c_{n-1} < 0 \). Similarly as in the assumption \( y_n(t_1) < 0 \) we can prove that \( y_i(t) \to -\infty \) as \( t \to \infty \), which contradicts the assumption \( y_i(t) > 0 \) on \([a, \infty)\). Therefore \( y_{n-2}(t) > 0 \) on \([T_7, \infty)\). According to the \((n-3)\)rd equation of (S) we obtain that \( y_{n-3}(t) \) is nondecreasing for \( t \geq T_8 = \gamma(T_7) \) and \( y_{n-3}(t) \) is either positive for \( t \geq T_9 \) or \( y_{n-3}(t) < 0 \) for \( t \geq T_8 \). a2) If \( y_{n-3}(t) > 0 \) for \( t \geq T_9 \), we can prove that \( y_i(t) > 0 \) (\( i = 1, 2, ..., n-3 \)) for \( t \geq T_{10} \). Then (4) is true for \( l = n-2 \). b2) If \( y_{n-3}(t) < 0 \) for \( t \geq T_8 \), we can proceed as in the case of b1),
only instead of \( n - 1 \) we have \( n - 3 \). So we get that either \( y_i(t) > 0 \) (\( i = 1, 2, \ldots, n - 4 = l \)) or \( y_{n-4}(t) > 0 \) and \( y_{n-5}(t) < 0 \) for sufficiently large \( t \). Proceeding further similarly to the case of \( b_1 \), \( b_2 \) we prove (4) and (5) for \( l = n - 4, \ldots, 4, 2 \) (\( l = n - 4, \ldots, 3, 1 \)) if \( n \) is even (odd). This completes the proof.

**Lemma 3.** Suppose that the assumptions of Lemma 2 hold. If a component \( y_k \) (\( k \in \{1, 2, \ldots, n\} \)) of a solution \( y = (y_1, \ldots, y_n) \in W \) has the property

\[
\liminf_{t \to \infty} |y_k(t)| = L_k,
\]

then

a) \( \lim_{t \to \infty} y_i(t) = +\infty (-\infty), \ (i = 1, 2, \ldots, k - 1) \) when \( L_k > 0, k > 1 \);

b) \( \liminf_{t \to \infty} |y_i(t)| = 0, \ (i = k + 1, \ldots, n) \) when \( L_k < \infty, k < n \).

**Proof.** Lemma 3 may be proved in the same way as Lemma 2 [1] and therefore we omit here the proof.

**Theorem 1.** Suppose that

\[
f_k(t, x, y) \text{ is nondecreasing in } x \text{ and } y \text{ for each fixed } t \geq a.
\]

If, in addition,

\[
\int_{t}^{\infty} |f_k(t, c, c)| \ dt = \infty \ \text{for} \ k = 1, 2, \ldots, n
\]

for every \( c \neq 0 \), then the system \( S \) has the property A.

**Proof.** Suppose that the system \( S \) has a nonoscillatory solution \( y = (y_1, \ldots, y_n) \in W \). Without loss of generality we may suppose that \( y_i(t) > 0 \) for \( t \geq t_0 \geq a \). According to (a), \( y_i(h_i(t)) > 0 \) for \( t \geq t_1 = \gamma(t_0) \). Then the \( n \)-th inequality of \( S \) implies \( y_i(t) \geq 0 \) for \( t \geq t_1 \) and it is not identically zero on any subinterval of \( [t_1, \infty) \). As \( y_i(t) > 0 \), \( y_i(t) \leq 0 \) for \( t \geq t_1 \), by Lemma 2 there exists an integer \( l \in \{1, \ldots, n\} \), \( n + l \) is even and a \( T_0 \geq t_1 \) such that

\[
y_i(t) > 0 \ \text{or} \ [T_0, \infty) \ \text{for} \ i = 1, 2, \ldots, l,
\]

\[
(-1)^{n-l} y_i(t) > 0 \ \text{on} \ [T_0, \infty) \ \text{for} \ i = l + 1, \ldots, n
\]

hold.

1. Let \( l \geq 2 \). In view of (8) and (a) we have \( y_1(t) > 0, y_2(t) > 0 \) for \( t \geq T \). Then by the 1st equation of \( S \), in view of (b) we get \( y_1(t) \geq 0 \) for \( t \geq t_2 = \gamma(T_0) \) and not identically zero on any subinterval of \( [t_2, \infty) \). The function \( y_i(t) \) is nondecreasing and therefore \( y_i(t) \geq d_i > 0 \) for \( t \geq t_2 \). From the \( n \)-th inequality of \( S \), we have, with the help of (b) and (6),

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\[ y_i'(t) \leq -f_n(t, y_i(t), y_i(h_i(t))) \leq -f_n(t, d_1, d_1) \text{ for } t \geq t_1 = \gamma(t_2). \]

Integrating the last inequality from \( t_1 \) to \( t \), we obtain

\[ \int_{t_1}^{t} f_n(s, d_1, d_1) \, ds \leq y_n(t) - y_n(t_1) \leq y_n(t_1), \]

which contradicts (7) for \( k = n \), as \( t \to \infty \).

II. Let \( l = 1 \) (\( n \) is odd). According to (8) and (b) we have \( y_1(t) < 0, y_2(h_1(t)) < 0 \) for \( t \geq t_1 = \gamma(t_0) \). Then the 1st equation of (S) gives that \( y_1(t) \) is nonincreasing and therefore \( \lim_{t \to \infty} y_1(t) = \delta \geq 0 \). We suppose that \( \delta > 0 \). Proceeding analogously as in the proof of I, we obtain a contradiction to (7). Therefore \( \delta = 0 \). Then applying Lemma 3 we get \( \lim y_i(t) = 0 \) for \( i = 1, 2, ..., n \).

The proof of Theorem 1 is complete.

Theorem 1 generalizes the results in [5, Theorem 1] and in [1, Remark 1].

**Theorem 2.** Suppose that (3) holds and in addition

\[ f_n (t, x, y) = p_n(t)g_n(x, y), \quad (9) \]

where \( p_n : [a, \infty) \to [0, \infty) \), \( g_n : \mathbb{R}^2 \to \mathbb{R} \) are continuous functions with \( p_n \) not identically zero on any subinterval of \( [a, \infty) \), \( yg_n(x, y) > 0 \) for \( xy > 0 \) and \( \lim \inf_{y \to \infty} |g_n(x, y)| > 0 \) for all \( x \neq 0 \).

If

\[ \int_{a}^{\infty} p_n(t) \, dt = \infty, \quad (10) \]

then the system (S) has the property A.

**Proof.** Arguing as in the proof of Theorem 1 we can show that (8) holds. a) In case I \( (i \geq 2) \) we have proved that \( y_i(t) \) is a nondecreasing function for which \( y_i(t) \geq d > 0 \) for \( t \geq t_2 \) and \( \lim_{t \to \infty} y_i(t) = d_2 \geq 0 \), where either \( d_2 < \infty \) or \( d_2 = \infty \). Then in view of (9) there exists a \( K > 0 \) such that

\[ g_n(y_i(t), y_i(h_i(t))) \geq K \text{ for } t \geq t_3 = \gamma(t_2). \]

From the \( n \)-th inequality of (S) with the help of the last inequality we have

\[ y_i'(t) \leq -f_n(t, y_i(t), y_i(h_i(t))) = -p_n(t)g_n(y_i(t), y_i(h_i(t))) \leq -Kp_n(t), \quad \text{for } t \geq t_3. \]

Integrating the last inequality from \( t_3 \) to \( t \), we obtain
which gives a contradiction to (10) as $t \to \infty$.

b) Let $l = 1$. Analogously as in case II of the proof of Theorem 1 we can show that $\lim_{t \to \infty} y_i(t) = 0$. Then by Lemma 3 we get $\lim_{t \to \infty} y_i(t) = 0$ for $i = 1, 2, \ldots, n$.

The proof of Theorem 2 is complete. This Theorem generalizes Theorem 2 [6].

We turn now to the system (S), where

$$f_i(t, x, y) = p_i(t)x, \quad i = 1, 2, \ldots, n-2$$

(11)

$$f_k(t, x, y) \text{sgn } y = p_k(t)|y|^\alpha, \quad \alpha_k > 0, \quad k = n-1, n,$$

where

$$p_i : [a, \infty) \to [0, \infty), \quad i = 1, 2, \ldots, n$$

(12)

are continuous functions and not identically zero on any subinterval of $[a, \infty)$,$$
\int_{p_i}^\infty(t) \, dt = \infty, \quad i = 1, 2, \ldots, n-1.$$

The system (S), in the particular case where (11), (12) hold and $p_i(t) > 0$, $i = 1, 2, \ldots, n-1$, $\alpha_{n-1} = 1$, $h_n(t) = t$ on $[a, \infty)$, is equivalent to the $n$-th order scalar differential inequality

$$\{ \left( \frac{1}{p_{n-1}(t)} \left( \frac{1}{p_2(t)} \left( \frac{1}{p_1(t)} y'(t) \right)' \right)' \right)' \}'' + p_n(t)|y(h_1(t))|^\alpha \cdot \text{sgn } y(h_1(t)) \leq 0.$$

We introduce the notation. $\alpha_{n-1} = \alpha$, $\alpha_n = \beta$;

$$\tilde{p}_i(t) = \min \{ p_i(s); \, t/4 \leq s \leq t \}, \quad t \geq a, \quad i = 1, \ldots, n-1$$

$$P_i(t) = \tilde{p}_i(t)\tilde{p}_{i-1}(t) \ldots \tilde{p}_1(t) \quad \text{for} \quad i \leq j,$$

$$P_i(t) = 1 \quad \text{for} \quad i > j, \quad P_1(t) = P_1(t).$$

Let $i_k \in \{ 1, 2, \ldots, n \} 1 \leq k \leq n-1$ and $t, s \in [a, \infty)$. We define $I_0 = 1 = J_0$, and

$$I_k(t, s; p_{i_k}, \ldots, p_i) = \int_s^t p_{i_k}(x)I_{k-1}(x, s; p_{i_{k-1}}, \ldots, p_i) \, dx,$$

$$J_k(t, s; p_{i_k}, \ldots, p_i) = \int_s^t p_{i}(x)J_{k-1}(t, x; p_{i_{k-1}}, \ldots, p_i) \, dx.$$

**Lemma 4.** Suppose that (11), (12) hold. Let $y$ be a solution of (S) on the interval $[a, \infty)$. Then the following relations hold:
\[ y_i(s) = \sum_{i=0}^{n-i-1} (-1)^i y_i(t) I_i(t, s; p_{i+p-1}, \ldots, p_i) + \]
\[ + (-1)^n \int^{t}_{s} p_n(x) |y_n(h_n(x))|^{\alpha} \text{sgn} y_n(h_n(x)) I_n(x, s; p_{n-2}, \ldots, p_i) \, dx, \]

\[ \text{for } a \leq s \leq t, \quad i = 1, 2, \ldots, n-1; \]

\[ y_i(t) = \sum_{j=0}^{m} y_{i+j}(s) J_i(r, s; p_i, \ldots, p_{i+j-1}) + \]
\[ + \int^{t}_{s} y_{i+m+1}(x) p_{i+m}(x) J_m(r, x; p_i, \ldots, p_{i+m-1}) \, dx, \]

\[ \text{for } r \geq s \geq a, \quad i < n-1, \quad 0 \leq m < n-i-1. \]

**Proof.** a) Let \( a \leq s \leq t \). It is evident that

\[ y_i(s) = y_i(t) - \int^{t}_{s} y_i'(x) \, dx = y_i(t) - \int^{t}_{s} p_i(x) y_{i+1}(x) \, dx, \]

\[ \text{for } i \leq n-2, \]

\[ y_n(t) = y_{n-1}(t) - \int^{t}_{s} p_{n-1}(x) |y_n(h_n(x))|^{\alpha} \text{sgn} y_n(h_n(x)) \, dx. \]

We calculate the second integral in (15) by parts. Denote:

\[ v(x) = \int^{t}_{s} p_i(\tau) \, d\tau = I_i(x, s; p_i), \quad u(x) = y_{i+1}(x). \]

Then we obtain

\[ y_i(s) = y_i(t) - y_{i+1}(t) I_i(t, s; p_i) + \int^{t}_{s} y_{i+1}'(x) I_i(x, s; p_i) \, dx = \]
\[ = y_i(t) - y_{i+1}(t) I_i(t, s; p_i) + \int^{t}_{s} y_{i+2}(x) p_{i+1}(x) I_i(x, s; p_i) \, dx \]

\[ \text{for } i < n-2. \]

If \( i = n-2 \), we get (13).

Using further the method by parts \((n-2-i)\) times \((i < n-2)\) on the last integral, we obtain (13).

b) Let \( a \leq s \leq t \) and let \( i < n-1 \). It is clear that

\[ y_i(t) = y_i(s) + \int^{t}_{s} y_i'(x) \, dx = y_i(s) + \int^{t}_{s} y_{i+1}(x) p_i(x) \, dx. \]
For \( i = n - 2 \) (14) is true. Let \( i < n - 2 \). We calculate the last integral in (16) by parts. Denote \( v(x) = -\int_{x}^{t} p_i(\tau) \, d\tau \), \( u(x) = y_{i+1}(x) \). Then we have

\[
y_i(t) = y_i(s) + y_{i+1}(s) \int_{x}^{t} p_i(x) \, dx + \int_{x}^{t} y'_{i+1}(x) J_i(t, x; p_i) \, dx =
\]

\[
y_i(s) + y_{i+1}(s) J_i(t, s; p_i) + \int_{x}^{t} y'_{i+2}(x) p_{i+1}(x) J_i(t, x; p_i) \, dx.
\]

Using further the method by parts \( m - 1 \) times on the last integral, we get (14).

**Lemma 5.** Suppose that (11), (12) and the assumption (i) of Lemma 2 hold. Then there exist \( l \in \{1, 2, \ldots, n\} \), \( n + l \) is even and a \( T \geq a \) such that (4), (5) hold and

\[
|y_i(t/2)| \geq C_l t^{n-i} P_{n-i-1}(t) |y_{n}(t)|^a \quad \text{for} \quad t \geq T,
\]

where

\[
C_l = \frac{2^{-2(n-i)}}{(n-1)! (n-i)!}, \quad i = 1, 2, \ldots, n - 1.
\]

**Proof.** The inequality (4), (5) follows from Lemma 2. Without loss of generality we suppose that \( y_i(t) > 0 \) for \( t \geq t_0 \). Then from (13) we obtain for \( s = t/2, \) in view of (5) and the monotonicity of \( y_n(t) \)

\[
(-1)^i y_i(t/2) \geq \int_{t/2}^{t} (y_n(x))^a p_{n-i-1}(x) I_{n-i-1}(x, t/2; p_{n-2}, \ldots, p_i) \, dx
\]

\[
\geq (y_n(t))^a \tilde{p}_{n-i-1}(t) \int_{t/2}^{t} (t - x) p_{n-2}(x) I_{n-i-2}(x, t/2; p_{n-3}, \ldots, p_i) \, dx \geq \ldots \geq
\]

\[
\geq (y_n(t))^a \tilde{p}_{n-i-1}(t) \ldots \tilde{p}_i(t) \int_{t/2}^{t} \frac{(t-x)^{n-i-1}}{(n-i-1)!} \, dx.
\]

Calculating the last integral we get

\[
(-1)^i y_i(t/2) \geq \left( \frac{t}{2} \right)^{n-i} \frac{P_i(t)}{(n-i)!} (y_n(t))^a \quad \text{for} \quad t \geq 2t_0,
\]

and \( i = l, l + 1, \ldots, n - 1 \).

According to (4) and the monotonicity of \( y_n(t) \) we have from (14) for \( m = l - i - 1, r = 1/2, s = t/4 \)

\[
y_i(t/2) \geq y_i(t/2) \int_{t/4}^{t/2} p_{l-i}(x) J_{l-i-1}(t/2, x; p_i, \ldots, p_{i-2}) \, dx \geq
\]

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\( \geq y_i(t/2) \hat{p}_i, i(t) \int_{t/4}^{t^2} (x - t/4) p_{i+1} z(x) J_{\alpha} J_{\beta}(x) p_{i+1} \, dx \geq \cdots \geq \)

\( \geq y_i(t/2) \hat{p}_i, i(t) \cdots \hat{p}_i(t) \int_{t/4}^{t^2} (x - t/4)^{i-1} (l-i-1)! \, dx . \)

If we calculate the last integral we obtain

\[
y_i(t/2) \geq \left( \frac{t}{4} \right)^i \frac{p_i, i(t)}{(l-i)!} y_i(t/2) \quad \text{for} \quad t \geq 4t_0 = T_i, i = 1, 2, \ldots, l - 1.
\]

Combining (18) for \( i = l \) and (19) we get (17).

Remark 1. a) The inequality (4) implies \( |y_i(t)| \geq |y_i(t/2)| \) for \( i = 1, 2, \ldots, l - 1 \). Then (17) can be written in the form

\[
|y_i(t)| \geq C_i t^a P_n, i(t) |y_n(t)|^a \quad \text{for} \quad t \geq T, i = 1, \ldots, l - 1.
\]

b) If \( 0 < \alpha \leq 1 \), then it is evident that (17) holds also for \( i = n \).

**Theorem 3.** Suppose that (11), (12) hold. If \( 0 < \alpha \beta < 1 \) and

\[
\int_T^\infty (h_i(t))^{\alpha} p_{i+1}(t)(P_n, i(h_i(t)))^\beta dt = \infty,
\]

then the system (S) has the property A.

**Proof.** Suppose that the system (S) has a nonoscillatory solution \( y = (y_1, \ldots, y_n) \in W \). Without loss of generality we may suppose that \( y_1(t) > 0 \) for \( t \geq t_0 \geq a \). According to (a) we have \( y_i(h_i(t)) > 0 \) for \( t \geq t_i = \gamma(t_0) \). Then the \( n \)-th inequality of (S) implies that \( y_n(t) \leq 0 \) for \( t \geq t_1 \) and it is not identically zero on any subinterval of \( [t_1, \infty) \). As \( y_1(t) > 0 \) and \( y_n(t) \leq 0 \) for \( t \geq t_1 \), then by Lemma 5 we get (4), (5) and (17), resp. (17').

I. Let \( l \geq 2 \). From (17') we have for \( i = 1 \)

\[
y_i(t) \geq C_i t^a P_n, i(t) |y_n(t)|^a, \quad t \geq t_2 > t_1.
\]

Then the \( n \)-th inequality of (S) implies

\[
y_n'(t) \leq -C_i^a p_n(t)(h_i(t))^{\alpha_1} (P_n, i(h_i(t)))^\beta (y_n(h_i(t)))^{\alpha_2} \leq \quad (21)
\]

\[
\leq -C_i^a p_n(t)(h_i(t))^{\alpha_1} (P_n, i(h_i(t)))^\beta (y_n(t))^{\alpha_2}
\]

for \( t \geq t_3 = \gamma(t_2) \).

In (21) we have used the fact that \( y_n(t) \) is nonincreasing.

Dividing (21) by \( (y_n(t))^{\alpha_2} \) and then integrating from \( t_3 \) to \( t \), we obtain

\[
\frac{(y_n(t))^{\alpha_2} - (y_n(t_3))^{\alpha_2}}{1 - \alpha_2 \beta} \leq -C_i^a \int_{t_3}^{t} p_n(s)(P_n, i(h_i(s)))^\beta (h_i(s))^{\alpha_1} h_i(s) ds.
\]

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From the last inequality we get
\[ C^\alpha \int_{t_0}^\infty p_n(s)(h_1(s))^{(\alpha-1)|\beta|}(P_{n-1}(h_1(s)))^\alpha \, ds \leq \frac{(y_n(t))^{1-\alpha|\beta|}}{1-\alpha\beta} < \infty, \]
which contradicts (20).

II. Let \( l = 1 \) (\( m \) is odd). Then by (5) the function \( y_1(t) \) is nonincreasing and with regard to \( y_1(t) > 0 \) it follows that \( \lim_{t \to \infty} y_1(t) = \delta \geq 0 \). We suppose that \( \delta > 0 \). Therefore there exists a \( K > 0 \) such that
\[ \inf_{t_2 \geq t_1} \frac{y_1(t)}{y_1(t/2)} = K. \] (22)

From (17) we get for \( i = 1 \) with the help of (22)
\[ y_1(t) = \frac{y_1(t)}{y_1(t/2)} y_1(t/2) \geq K C_i t^{n-1} P_{n-i}(t)(y_n(t))^{\alpha} \]
for \( t \geq t_1, t \geq 2t_1 \).

Proceeding further in the same way as in case I, we get a contradiction to (20).

Then \( \lim_{t \to \infty} y_1(t) = 0 \) and by Lemma 3 we have \( \lim_{t \to \infty} y_k(t) = 0 \) for \( k = 1, 2, \ldots, n \).

Theorem 3 extends the results of Ševelo and Varech [5, Theorem 2].

**Theorem 4.** Suppose that (11) and (12) hold. In addition there exists a differentiable function \( g: [a, \infty) \to R \) such that
\[ g'(t) \geq 0, \quad 0 \leq g(t) \leq h_i(t) \quad \text{for} \quad t \geq T \geq a. \] (23)

If \( \alpha = 1, \beta > 1 \) and
\[ \int_T^\infty p_n(t) \int_T^t (g(s))^{n-2} P_{n-i}(g(s))g'(s) \, ds \, dt = \infty, \] (24)
then the system (S) has the property A.

**Proof.** Suppose that the system (S) has a nonoscillatory solution \( y = (y_1, \ldots, y_n) \in W \). We suppose that \( y_1(t) > 0 \) for \( t \geq t_0 \). Proceeding in the same way as in the proof of Theorem 2 we get (4), (5) and (17). With regard to \( y_1(t) > 0 \), (4) and (5) we have either

\[ y_2(t) > 0 \quad \text{or} \quad y_2(t) < 0 \quad \text{for} \quad t \geq t_1 > t_0. \]

I. Let \( y_2(t) > 0 \) for \( t \geq t_1 \). Then the 1st equation of (S) implies that \( y_1'(t) \geq 0 \) for \( t \geq \tilde{\gamma}(t) \), where \( \tilde{\gamma}(t) = \max (y_1(t), \sup \\{s ; g(t) < t\}) \) for \( t \geq a \).
We define the function $z$ as follows

$$z(t) = -v(t)\int_{t_2}^t (g(s))^{n-2} g'(s) P_{n-1}(g(s)) \, ds$$

for $t \geq t_2 = \max \{ T, \bar{y}(t) \}$.

It is evident that

$$z(t) < 0 \quad \text{for} \quad t > t_2. \quad (26)$$

In view of the $n$-th inequality of (S), (23) and the monotonicity of $y_i$ we get from (25) the following

$$z'(t) \geq p_n(t)(y_i(h_i(t)))^\beta \int_{t_2}^t (g(s))^{n-2} g'(s) P_{n-1}(g(s)) \, ds -$$

$$- y_n(t) (g(t))^{n-2} g'(t) P_{n-1}(g(t)) \geq$$

$$\geq p_n(t) \int_{t_2}^t (g(s))^{n-2} g'(s) P_{n-1}(g(s)) \, ds -$$

$$- \frac{y_n(g(t))}{(y_i(g(t)/2))^{n-2}} (g(t))^{n-2} g'(t) P_{n-1}(g(t)) p_i(g(t)/2).$$

If we use (17) for $i = 2, \alpha = 1$ and we substitute $g(t)$ for $t$, then from the last inequality we obtain

$$z'(t) \geq p_n(t) \int_{t_2}^t (g(s))^{n-2} g'(s) P_{n-1}(g(s)) \, ds -$$

$$- \frac{y_2(g(t)/2)g'(t)p_i(g(t)/2)}{C_2(y_i(g(t)/2))^{n-1}}. \quad (27)$$

Using the 1st equation of (S) and then integrating (27) from $t_2$ to $t$, we obtain

$$z(t) \geq z(t_2) + \int_{t_2}^t p_n(x) \int_{t_2}^x (g(s))^{n-2} g'(s) P_{n-1}(g(s)) \, ds \, dx -$$

$$- \frac{2y_1(g(t_2/2))^{n-1}}{C_2(\beta - 1)}. \quad (28)$$

In view of (24) the last inequality implies $\lim_{t \to \infty} z(t) = \infty$, which contradicts (26).

II. Let $y_2(t) < 0$ for $t \geq t_1$. The first equation of (S) implies that $y_1(t)$ is a nonincreasing function. Then in view of $y_1(t) > 0$ it follows that $\lim_{t \to \infty} y_1(t) = \delta \geq 0$. We suppose that $\delta > 0$. 83
We now define the function $w$ as follows:

$$w(t) = -y_n(t) \int_{t_2}^{t} (g(s))^{n-2} g'(s) P_{n-1}(g(s)) \, ds, \quad t \geq t_2. \tag{28}$$

It is clear that $w(t) < 0$ for $t \geq t_2$.

Using the $n$-th inequality of (S), the monotonicity of $y_1$ and (17) for $i = 2$, we obtain from (28):

$$w'(t) \geq p_n(t)(y_1(h_1(t)))^n \int_{t_2}^{t} (g(s))^{n-2} g'(s) P_{n-1}(g(s)) \, ds -$$

$$- y_n(t)(g(t))^{n-2} g'(t) P_{n-1}(g(t)) \geq$$

$$\geq \delta^n p_n(t) \int_{t_2}^{t} (g(s))^{n-2} g'(s) P_{n-1}(g(s)) \, ds +$$

$$+ \frac{1}{C_2} y_2(g(t)/2) g'(t) p_1(g(t)/2). \tag{29}$$

Integrating (29) from $t_2$ to $t$, we get

$$w(t) \geq w(t_2) + \delta \int_{t_2}^{t} p_n(x) \int_{t_2}^{x} (g(s))^{n-2} g'(s) P_{n-1}(g(s)) \, ds \, dx -$$

$$- \frac{2}{C_2} y_1(g(t_2)/2).$$

In view of (24) the last inequality implies $\lim_{t \to \infty} w(t) = \infty$, which contradicts $w(t) < 0$ for $t \geq t_2$. Therefore $\delta = 0$, i.e. $\lim_{t \to \infty} y_1(t) = 0$. Then by Lemma 3 we have

$$\lim_{t \to \infty} y_k(t) = 0 \quad \text{for} \quad k = 1, 2, \ldots, n.$$

Remark 2. Consider now the scalar equation

$$y^{(n)}(t) + p_n(t) |y(h_1(t))|^\beta \text{ sgn } y(h_1(t)) = 0, \quad n \geq 2, \beta > 1, \tag{E}$$

which is a special case of the system (S).

It is easy to prove that

$$\int_{t}^{\infty} p_n(t) \int_{t}^{t} (g(s))^{n-2} g'(s) \, ds \, dt = \infty$$

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$$\int_{t}^{\infty} p_n(t) (g(t))^{n-1} \, dt = \infty.$$

Then from Theorem 3 we get the following very well-known
**Corollary.** Suppose that (12), (23) hold. If

\[
\int_{1}^{\infty} p_n(t) g(t)^{n-1} \, dt = \infty,
\]

then every solution of (E) is oscillatory if \( n \) is even while for \( n \) odd it is either oscillatory or tends monotonically to zero as \( t \to \infty \).

**Theorem 5.** Suppose that (11), (12) and (23) hold. In addition we assume that \( \alpha\beta > 1 \). If

\[
\int_{1}^{\infty} p_n(t) \, dt < \infty
\]

and

\[
\int_{1}^{\infty} (g(t))^{n-2} g'(t) P_{n-1}(g(t)) \left( \int_{t}^{\infty} p_n(s) \, ds \right)^{\alpha} \, dt = \infty,
\]

(30)

then the system (S) has the property A.

**Proof.** Let \( y = (y_1, \ldots, y_n) \in \mathbb{W} \) be a nonoscillatory solution of (S). Proceeding in the same way as in the proof of Theorem 4 we get (4), (5) and (17). We may suppose that \( y_i(t) > 0 \) for \( t \geq t_1 \). Integrating the \( n \)-th inequality of (S) from \( t(\equiv t_2 = \gamma(t_1)) \) to \( \tau \), we get

\[
y_n(\tau) - y_n(t) \leq - \int_{t}^{\tau} p_n(s)(y_i(h_i(s)))^{\alpha} \, ds,
\]

and then we have for \( \tau \to \infty \)

\[
y_n(t) \geq \int_{t}^{\infty} p_n(s)(y_i(h_i(s)))^{\alpha} \, ds, \quad t \geq t_2. \tag{31}
\]

1. Let \( t \geq 2 \). Since \( y_1 \) is nondecreasing and \( y_n \) is nonincreasing, (31) implies

\[
y_n(g(t))^{\alpha} \geq (y_1(g(t)))^{\alpha} \left( \int_{t}^{\infty} p_n(s) \, ds \right)^{\alpha}, \quad t \geq t_1 = \tilde{\gamma}(t_2).
\]

From the last inequality we obtain in view of (17) for \( i = 2 \) and the monotonicity of \( y_1 \)

\[
y_2(g(t)/2) \geq C_2(g(t))^{n-2} P_{n-1}(g(t))(y_1(g(t)/2))^{\alpha} \left( \int_{t}^{\infty} p_n(s) \, ds \right)^{\alpha}. \tag{32}
\]

Multiplying (32) by \( g'(t)p_1(g(t)/2)(y_1(g(t)/2))^{-\alpha} \) and using the 1st equation of (S), we get

\[
\frac{y'_i(g(t)/2)g'(t)}{(y_1(g(t)/2))^{\alpha i}} \geq C_2(g(t))^{n-2} g'(t) P_{n-1}(g(t)) \left( \int_{t}^{\infty} p_n(s) \, ds \right)^{\alpha}.
\]
Integrating the last inequality from \( t_3 \) to \( u \), we obtain

\[
\frac{2}{\alpha \beta - 1} (y_1(g(t)/2))^{\alpha_{q_1} - 1} \geq 0
\]

\[
\geq C_2 \int_{t_3}^{u} (g(t))^{n-2} g'(t) P_{n-1}(g(t)) \left( \int_t^\infty p_n(s) \, ds \right)^\alpha \, dt,
\]

which contradicts (30) as \( u \to \infty \).

II. Let \( l = 1 \). According to Lemma 5, \( y_1(t) > 0 \) for \( t \geq t_1 \), we get from the 1st equation of (S) \( y_2(t) < 0, \ y'_1(t) \leq 0 \) for \( t \geq t_1 \). Therefore \( \lim_{t \to \infty} y_1(t) = \delta \geq 0 \). We suppose that \( \delta > 0 \). Then, in view of the monotonicity of \( y_n, y_1 \) we obtain from (31):

\[
(y_n(g(t)))' \geq \delta q(n) \left( \int_t^\infty p_n(s) \, ds \right)^\alpha, \quad t \geq t_4 = \max \{ T, t_3 \}.
\]

If we use (17) for \( i = 2 \), we get from the last inequality

\[
-y_2(g(t)/2) \geq C_2 \delta q(n) (g(t))^{n-2} P_{n-1}(g(t)) \left( \int_t^\infty p_n(s) \, ds \right)^\alpha,
\]

for \( t \geq t_4 \).

Multiplying (33) by \( p_1(g(t)/2) g'(t) \) and using the 1st equation of (S), we obtain

\[
-y_1'(g(t)/2) g'(t) \geq C_2 \delta q(n) (g(t))^{n-2} g'(t) P_{n-1}(g(t)) \left( \int_t^\infty p_n(s) \, ds \right)^\alpha. \tag{34}
\]

Integrating (34) from \( t_4 \) to \( u \), we obtain

\[
2 y_1(g(t)/2) \geq
\]

\[
\geq C_2 \delta q(n) \int_{t_4}^{u} (g(t))^{n-2} g'(t) P_{n-1}(g(t)) \left( \int_t^\infty p_n(s) \, ds \right)^\alpha \, dt,
\]

which contradicts (30) as \( u \to \infty \).

Therefore \( \delta = 0 \), i.e. \( \lim_{t \to \infty} y_1(t) = 0 \). Then in view of Lemma 3 we have \( \lim_{t \to \infty} y_k(t) = 0 \) for \( k = 1, 2, ..., n \).

The proof of Theorem 5 is complete.

This Theorem generalizes Theorem 5 [5].
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О КОЛЕБЛЕМОСТИ РЕШЕНИЙ НЕЛИНЕЙНЫХ СИСТЕМ С ЗАПАЗДЫВАНИЕМ

Pavol Marušiak

Резюме

В статье приведены достаточные условия колеблемости решений системы (S) и системы

\[ y_i'(t) = p_i(t)y_{i+1}(t), \quad i = 1, 2, ..., n-2, \]

\[ y_n'(t) = P_n(t)|y_n(h_n(t))|^\alpha \text{ sgn } y_n(h_n(t)). \]

\[ y_n'(t) \text{ sgn } y_n(h_n(t)) \leq -p_n(t)|y_n(h_n(t))|^{\beta}, \quad 0 < \alpha, 0 < \beta. \]