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EXAMPLES OF CLASSICAL AND FUZZY RIESZ PROXIMITIES

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ABSTRACT. Examples of proximities which are Riesz (respectively fuzzy Riesz) but not Lodato (respectively fuzzy Lodato) have been constructed.

1. Introduction

In the classical theory of proximities, the notion of $f$-proximities and, in particular, of Riesz (or RI) proximity is due to Thron [6], and that of a symmetric generalized proximity (now known as Lodato or LO-proximity) is due to Lodato [2]. A relationship between these two, that “every LO-proximity on a nonempty set is an RI-proximity”, is given by Thron [6]. In [5], we have continued the study of fuzzy $f$-proximities introduced in [3] and generalized the notion of classical RI-proximity to fuzzy Riesz (or RI) proximity. Fuzzy RI-proximity turns out to be a particular case of fuzzy $f$-proximities. In the fuzzy subset, setting also the result that “every fuzzy LO-proximity [4] on a set is a fuzzy RI-proximity” holds good [5].

In the present paper, we have constructed

(i) an example (Example 3.1) of an RI-proximity which is not an LO-proximity,

(ii) two examples of fuzzy RI-proximities both of which are not fuzzy LO-proximities.

Example 3.2 has been obtained with the help of Example 3.1, while Example 3.3 uses purely fuzzy behaviour in the sense that one cannot derive this example from a classical proximity using the technique of Example 3.2 (cf. Remark 3.4).

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2. Preliminaries

Let $X$ be a nonempty set, $P(X)$ be the power set of $X$, and $I = [0, 1]$ be the closed unit interval of the real line $\mathbb{R}$. A fuzzy set $\lambda$ in $X$ is an element of the family $I^X$ of all functions from $X$ to $I$. A fuzzy point $x_p$, $x \in X$, $0 < p \leq 1$, is a fuzzy set in $X$ defined by

$$x_p(y) = \begin{cases} p & \text{if } y = x, \\ 0 & \text{otherwise}. \end{cases}$$

For $A \in P(X)$, $\chi_A \in I^X$ is defined by

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{otherwise}; \end{cases}$$

and $|A|$ denotes the cardinality of $A$. For $\lambda \in I^X$, we write $\text{supp} \lambda = \{x \in X : \lambda(x) \neq 0\}$. A fuzzy set which assigns the value $t$, $t \in I$, to each $x$ in $X$ is denoted by $t$. For $\lambda \in I^X$ and a binary relation $\Pi$ on $I^X$ define $c_{\Pi}(\lambda) = \bigvee \{x_p : (x_p, \lambda) \in \Pi\}$.

A binary relation $\Pi$ on $I^X$ is called a fuzzy Lodato (or LO) proximity on $X$ if, for $\lambda, \mu, \nu \in I^X$, the following hold:

F1. $(\lambda, \mu) \in \Pi \implies (\mu, \lambda) \in \Pi$,
F2. $(0, 1) \notin \Pi$,
F3. $(\lambda \lor \mu, \nu) \in \Pi \iff (\lambda, \nu) \in \Pi$ or $(\mu, \nu) \in \Pi$,
F4. $\lambda \land \mu \neq 0 \implies (\lambda, \mu) \in \Pi$,
F5. $(\lambda, \mu) \in \Pi$ and $(x_p, \nu) \in \Pi$ for all $x_p \leq \mu \implies (\lambda, \nu) \in \Pi$ ([4]).

A binary relation $\Pi$ on $I^X$ is called a fuzzy Riesz (or RI) proximity on $X$ if it satisfies F1, F2, F3, F4, and

F5'. $c_{\Pi}(\lambda) \land c_{\Pi}(\mu) \neq 0 \implies (\lambda, \mu) \in \Pi$ ([5]).

3. Examples

Example 3.1. Let $X = \mathbb{R} \times \mathbb{R}$, $d$ be the Euclidean metric on $X$, and $d(A, B) = \inf\{d(\xi, \eta) : \xi \in A, \ \eta \in B\}$ for subsets $A, B$ of $X$. Denote by $\omega_0$ the first infinite cardinal. Define

$$\delta = \{(A, B) : d(A, B) = 0\}$$

$$\cup \{(A, B) : |A \cap \{(0, y) : -1 \leq y \leq 1\}| \geq \omega_0$$

$$\text{and } |B \cap \{(x, 0) : x < -1\}| \geq \omega_0\}$$

$$\cup \{(A, B) : |A \cap \{(x, 0) : x < -1\}| \geq \omega_0$$

$$\text{and } |B \cap \{(0, y) : -1 \leq y \leq 1\}| \geq \omega_0\}.$$
Then $\delta$ is a Čech proximity ([1]) on $X$ and $c_\delta(A) \equiv \{x : (x,A) \in \delta\} = \{x : d(x,A) = 0\}$. If $c_\delta(A) \cap c_\delta(B) \neq \emptyset$, then there exists $x$ in $X$ such that $d(x,A) = 0 = d(x,B)$. Consequently, $d(A,B) = 0$, and hence $(A,B) \in \delta$. Thus $\delta$ is an RI-proximity on $X$.

Next, put $A = \{(x,0) : x < -1/2\}$, $B = \{(0,y) : -1 < y < 1\}$ and $C = \{(x,\sin 1/x) : x > 0\}$. Then $(A,B) \in \delta$, $(b,C) \in \delta$ for all $b \in B$. But $(A,C) \notin \delta$. This proves that $\delta$ is not an LO-proximity.

Example 3.2. Consider the metric space $(X,d)$ of Example 3.1. Define

$$\Pi = \{(\lambda,\mu) : (\text{supp}\lambda, \text{supp}\mu) \in \delta\}.$$ 

Then $\Pi$ satisfies F1 to F4, and, for $A$, $B$, $C$, as taken in Example 3.1, $(\chi_A,\chi_B) \in \Pi$, $(x_p,\chi_C) \in \Pi$ for all $x_p \leq \chi_B$; but $(\chi_A,\chi_C) \notin \Pi$. Thus $\Pi$ is not a fuzzy LO-proximity on $X$.

Since $c_\Pi(\lambda) = \chi_{c_\delta(\text{supp}\lambda)}$, if $c_\Pi(\lambda) \cap c_\Pi(\mu) \neq \emptyset$, then $c_\delta(\text{supp}\lambda) \cap c_\delta(\text{supp}\mu) \neq \emptyset$. Hence $(\text{supp}\lambda, \text{supp}\mu) \in \delta$, i.e., $(\lambda,\mu) \in \Pi$. Thus $\Pi$ is a fuzzy RI-proximity on $X$.

Example 3.3. Let $X$ be an infinite set. For $0 < t < 1$, define

$$\Pi = \{(\lambda,\mu) : \lambda \land \mu \neq 0\}$$

$$\cup \{(\lambda,\mu) : \lambda \neq 0, \mu \neq 0 \text{ and } (\lambda \lor \mu)(x) > t \text{ for infinitely many elements } x \text{ of } X\}.$$ 

The relation $\Pi$ satisfies F1 to F4. Let $c_\Pi(\lambda) \cap c_\Pi(\mu) \neq \emptyset$. Then $\lambda \neq 0$ and $\mu \neq 0$. If at least one of $\lambda$ and $\mu$ takes values greater than $t$ for infinitely many elements of $X$, then $(\lambda,\mu) \in \Pi$. Otherwise, $\text{supp}\lambda \cap \text{supp}\mu \neq \emptyset$, which implies that $\lambda \land \mu \neq 0$, and again $(\lambda,\mu) \in \Pi$. It may be noted that, for $\lambda \in I^X$,

$$c_\Pi(\lambda) = \begin{cases} 1 & \text{if } (\lambda(x) > t \text{ for infinitely many elements of } X,} \\ \chi_{\text{supp}\lambda} & \text{otherwise.} \end{cases}$$ 

That $\Pi$ is not a fuzzy LO-proximity, follows from the following arguments:

Let $\lambda(\neq 0)$, $\nu(\neq 0) \in I^X$ be such that $\lambda \land \nu \neq 0$ and $\lambda(x) \leq t$, $\nu(x) \leq t$, for all $x$ in $X$. Choose $\mu \in I^X$ such that supp $\mu = \text{supp}\nu$ and $\mu(x) \geq t$ for infinitely many points $x$ of $X$. Then $(\lambda,\mu) \in \Pi$. Also, for $x_p \leq \mu$, $\mu(x) \neq 0$, and, consequently, $\nu(x) \neq 0$. Hence $(x_p,\nu) \in \Pi$. But $(\lambda,\nu) \notin \Pi$. Thus $\Pi$ is not a fuzzy Lodato proximity on $X$.

Remark 3.4. Let $\delta$ be a relation on $P(X)$. Define $\hat{\delta} = \{(\lambda,\mu) : (\text{supp}\lambda, \text{supp}\mu) \in \delta\}$. It may be verified that $\delta$ is an LO-proximity if and only if $\hat{\delta}$ is a fuzzy LO-proximity. Suppose that the fuzzy proximity $\Pi$ of Example 3.3 can be derived from a classical proximity $\delta$ as in Example 3.2, i.e.,
Π = ʰ. Since Π is not a fuzzy LO-proximity, ʰ is not an LO-proximity. But

\[ \tilde{\Pi} \equiv \{(A, B) : (x_A, x_B) \in \Pi\} \]
\[ = \{(A, B) : (x_A, x_B) \in ʰ\} \]
\[ = ʰ. \]

and \(\tilde{\Pi}\) is an LO-proximity, i.e., ʰ is an LO-proximity. This provides a contradiction.

Thus Example 3.3 cannot be derived from a classical proximity using the technique of Example 3.2.

REFERENCES