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THE k-TUPLE DOMATIC NUMBER OF A GRAPH

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ABSTRACT. A node of a graph $G = (V, E)$ dominates itself and all nodes adjacent to it. A subset $S \subset V$ is a dominating set for G if each node is dominated by some node of S. This concept can be extended to k-tuple domination by requiring that each node in V be dominated by at least k nodes in S. The domatic number of G has been defined as the largest number of sets in a partition of V into dominating sets. Similarly, we define the k-tuple domatic number of G as the largest number of sets in a partition of V into k-tuple dominating sets. We derive bounds for the k-tuple domatic number. Results involving the ordinary domination and domatic numbers are improved as a consequence of this generalized approach.

1. Introduction

In general, we follow the terminology and notation of [4]. A node in $G = (V, E)$ is said to *dominate* itself and all nodes adjacent to it, the nodes in its closed neighbourhood $N[v]$. A *dominating set* $S \subset V$ has each node of G dominated by some node in S. The *domination number* $\gamma(G)$ is the smallest cardinality of a dominating set. A *domatic partition* is a partition of V into dominating sets, and the *domatic number* $d(G)$ is the largest number of sets in a domatic partition.

In [5], we defined $S \subset V$ to be a *multiple dominating set* or a *k-tuple dominating set* by requiring that each node in V be dominated by at least k nodes in S. The order of a smallest k-tuple dominating set is the *k-tuple domination number*, written $\gamma_k(G)$. It is easy to see that not every connected nontrivial graph has a k-tuple domination number for $k \geq 2$. For example, no tree has a 3-tuple domination number, and no cycle C_n has a 4-tuple domination number.

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However, any graph G without isolates has a 2-tuple domination number, and in general, any graph G with $\delta(G) \geq k-1$ has a k-tuple dominating set ([5]). We extend the concept of a domatic partition to a *k-tuple domatic partition* by partitioning V into k-tuple dominating sets. Then the *k-tuple domatic number* $d_k(G)$ is defined as expected. Note that for $k=1$, $d_k(G)$ is simply the domatic number introduced by Cockayne and Hedetniemi [3]. Zelinka [10] and Kulli [8] studied a different type of multiple domination. For each generic invariant μ of a graph G, let $\mu = \mu(G)$ and $\overline{\mu} = \mu(\overline{G})$.

2. The k-tuple domatic number

We establish a bound on the k-tuple domatic number in terms of the minimum degree δ.

THEOREM 1. *Let k be a positive integer, and G have $\delta \geq k-1$. Then*

$$d_k \leq \left\lfloor \frac{(\delta+1)}{k} \right\rfloor .$$

Proof. Let G be a graph with $\delta \geq k-1$, and consider a partition of V into k-tuple dominating sets $V_1, V_2, \dots, V_{d_k}$. Without loss of generality, let $u \in V_{d_k}$. Then u must have at least k neighbours in each V_i, $1 \leq i \leq d_k - 1$, and at least $k-1$ neighbours in V_{d_k}. Hence, each node has degree at least $kd_k - 1$, so $d_k \leq \lfloor (\delta+1)/k \rfloor$. Complete graphs K_n with $n \geq k$ achieve the upper bound with $\delta = n-1$, $\gamma_k(K_n) = k$, and $d_k(K_n) = \lfloor n/k \rfloor$. □

A result due to Cockayne and Hedetniemi [3] follows.

COROLLARY 1.1. ([3]) *For any graph G, $d \leq \delta + 1$.*

Since the k-tuple domination number is defined only for graphs G with $\delta \geq k-1$, we also have the following corollary.

COROLLARY 1.2. *If G has $k-1 \leq \delta \leq k$, then $d_k = 1$.*

A natural question to ask is for which G is $d_k \geq 2$. Corollary 1.2 implies that if $d_2 \geq 2$, then $\delta \geq 3$. However, the converse is not true. Consider the Petersen graph P which has $\delta(P) = 3$ and $\gamma_2(P) = 6$, implying that $d_2(P) = 1$. But a cubic graph can have $d_2 \geq 2$ as can be seen with $d_2(K_4) = 2$. A characterization of the graphs with $d_k \geq 2$ remains an open question.

By Corollary 1.2, any graph with no isolates and $\delta \leq 2$ has $d_2 = 1$ and $d_2 = \lfloor (\delta+1)/2 \rfloor$. For example, cycles and trees fall into this category.

We now give a Nordhaus-Gaddum inequality involving the k-tuple domatic numbers of G and $\overline{G}$.

THEOREM 2. *For a graph G with δ, $\overline{\delta} \geq k-1$,*

$$d_k + \overline{d_k} \leq \frac{(n - \Delta + \delta + 1)}{k}.$$

Proof. Let G be a graph with δ, $\overline{\delta} \geq k-1$. By Theorem 1,

$$d_k \leq \frac{(\delta + 1)}{k}.$$

Hence

$$d_k + \overline{d_k} \leq \frac{(\delta + 1)}{k} + \frac{(\overline{\delta} + 1)}{k} = \frac{(\delta + 1)}{k} + \frac{(n - \Delta)}{k}.$$

And the theorem holds. □

Again a result concerning the domatic number follows as a corollary.

COROLLARY 2.1. (Cockayne and Hedetniemi [3]) *For any graph G, $d + \overline{d} \leq n + 1$ with equality if and only if $G = K_n$ or $\overline{K}_n$.*

A *full node* has degree $n-1$. In [5], we showed that a graph G has $\gamma_2 = 2$ if and only if G has two full nodes. From the definition of the k-tuple domatic number, we have $\gamma_k \times d_k \leq n$. We use these facts and a proof technique similar to one used by Joseph and Arumugam in [7] to establish a upper bound on $\gamma_k + d_k$. A consequence of this result improves the known upper bound on $\gamma + d$.

THEOREM 3. *If G is a graph with $\delta \geq k - 1 \geq 1$ and $d_k \geq 2$, then*

$$\gamma_k + d_k \leq \lfloor n/2 \rfloor + 2$$

with equality if and only if one of the statements (1) *to* (4) *holds.*

(1) $d_k = 2$ *and* $\gamma_k = \lfloor n/2 \rfloor$.
(2) $k = 2$, $n = 9$, *and* $d_2 = \gamma_2 = 3$.
(3) $k = 2$ *and* $G \cong K_n$.
(4) $k = 3$ *and* $G \cong K_9$.

Proof. Let G be a graph with $\delta \geq k-1 \geq 1$ and $d_k \geq 2$. Then $\gamma_k \times d_k \leq n$, $\gamma_k \geq k$ imply $\gamma_k + d_k \leq n/d_k + d_k$ and $2 \leq d_k \leq n/k$. Note that $f(x) = n/x + x$ is decreasing for $1 \leq x \leq \sqrt{n}$ and increasing for $\sqrt{n} \leq x$. Thus

$$\frac{n}{d_k} + d_k \leq \max\left\{\frac{n}{2} + 2, \frac{n}{n/k} + \frac{n}{k}\right\} \leq \frac{n}{2} + 2.$$

Obviously, if any one of statements (1) to (4) holds, then $\gamma_k + d_k = \lfloor n/2 \rfloor + 2$. Conversely, let G be a graph with $\delta \geq k-1 \geq 2$ and $\gamma_k + d_k = \lfloor n/2 \rfloor + 2$.

Assume that $\gamma_k = 2$ or $d_k = 2$. If $d_k = 2$ and $\gamma_k = \lfloor n/2 \rfloor$, then statement (1) of the theorem holds. If $\gamma_k = 2$ and $d_k = \lfloor n/2 \rfloor$, then $\gamma_k \geq k$ implies $k = 2$. As mentioned above, $\gamma_2 = 2$ if and only if G has two full nodes ([5]). Further, since $d_k = \lfloor n/2 \rfloor$, $G \cong K_n$ and statement (3) of the theorem holds.

If $\gamma_k \geq 4$ and $d_k \geq 4$, then

$$\gamma_k + d_k \leq n/4 + n/4 < \lfloor n/2 \rfloor ,$$

and hence, equality is impossible.

The remaining possibility is that $\gamma_k = 3$ or $d_k = 3$ implying

$$3 + \lfloor n/3 \rfloor = \lfloor n/2 \rfloor + 2 .$$

But this equation is true only for $n = 6, 7$, or 9. If $n = 6$ or $n = 7$, then $d_k = 2$ or $\gamma_k = 2$, and we have already considered this case. Hence let $n = 9$. Then $\gamma_k + d_k = \lfloor n/2 \rfloor + 2 = 6$. Since $\gamma_k = 3$ or $d_k = 3$, we have $\gamma_k = d_k = 3$ implying that $2 \leq k \leq 3$. If $k = 2$, then statement (2) holds. If $k = 3$, then each of the three disjoint 3-tuple dominating sets induces a triangle. Furthermore, any node not in a given 3-tuple dominating set much be adjacent to all 3 nodes in the set. Hence $G \cong K_9$ and statement (4) holds. □

The first corollary to this theorem is a known upper bound from [3].

COROLLARY 3.1. (Cockayne and Hedetniemi [3]) *For any graph G, $\gamma + d \leq n + 1$.*

Ore [9] showed that for any graph without isolates, $\gamma(G) \leq n/2$ implying that G has $d \geq 2$ if and only if G has no isolates. Hence the upper bound of Corollary 3.1 is improved for graphs with no isolates and $\gamma \geq 2$ by our next corollary.

COROLLARY 3.2. *If G has no isolates and $\gamma \geq 2$, then $\gamma + d \leq \lfloor n/2 \rfloor + 2$.*

Proof. Let G be a graph with no isolates and $\gamma \geq 2$. Then $d \geq 2$. Thus $d \leq \lfloor n/2 \rfloor$ and $\gamma \leq \lfloor n/2 \rfloor$. The corollary follows when $k = 1$. □

The composition $P_3[P_3]$ of two P_3 paths is an example of a graph with $\gamma_2 = d_2 = 3$. Note that the condition $d_k = 2$ alone is not sufficient for sharpness as can be seen by $G = K_4 + \overline{K}_p$ for $p \geq 3$ and $k = 2$. Any two nodes in the K_4 form a 2-tuple dominating set for G, and no other pair of nodes is a 2-tuple dominating set. Hence

$$\gamma_2 + d_2 = 2 + 2 < \lfloor (4 + p)/2 \rfloor + 2 .$$

Jaeger and Payan [6] showed that $\overline{\gamma} \leq d$, and Cockayne and Hedetniemi [3] combined this fact with Corollary 3.1 to establish a Nordhaus-Gaddum inequality involving domination numbers of complementary graphs.

Considering this approach, we tried to find a similar relationship between $\overline{\gamma}_2$ and d_2. However, the 2-tuple domination number of $\overline{G}$ is not a lower bound for d_2 as can be seen in the following examples:

- For the corona $G \circ K_1$, $\gamma_2 = n$, $d_2 = 1$, and $\overline{\gamma}_2 \geq 2$.
- For the complete bipartite graph $K_{r,s}$, $3 \leq r \leq s$, it is a simple exercise to show

$$\gamma_2 = 4 \quad \text{and} \quad d_2 = \min(\lfloor r/2 \rfloor,\ \lfloor s/2 \rfloor).$$

Hence $\gamma_2(K_{6,8}) = 4$, $\gamma_2(\overline{K}_{6,8}) = 4$, and $d_2(K_{6,8}) = 3$.

We conclude with two interesting open problems:

- Characterize the graphs for which $d_k = 2$ and $\gamma_k = \lfloor n/2 \rfloor$.
- Characterize the graphs for which $d_k = \lfloor (\delta + 1)/2 \rfloor$.

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