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Representation of linear operators on spaces of vector valued functions

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This paper is concerned with an integral representation of some linear operators defined on an ordered space of vector valued functions. The terminology used is that of [1].

1. Preliminaries

Let \( X \) be a locally convex vector lattice with the topology \( \tau \) and \( P \) the set of all solid and \((\tau)\)-continuous semi-norms defined on \( X \). Let \( Y \) be a complete vector lattice. For any \( p \in P \) we denote by \( \mathcal{L}_p \) the set of all linear operators \( U: X \to Y \) for which the set \( \{U(x); p(x) \leq 1\} \) is order bounded. If \( U \in \mathcal{L}_p \), we put

\[
\|U\|_p = \sup \{|U(x)|; p(x) \leq 1\}.
\]

We set also

\[
\mathcal{L} = \bigcup_{p \in P} \mathcal{L}_p,
\]

If \( U \in \mathcal{L} \), then \( U \) is called a \((po)\)-bounded operator. If \( U \in \mathcal{L}_p \), we say that \( U \) is \((po)\)-bounded with respect to \( p \).

The set \( \mathcal{L} \) is a normal subspace of the space \( \mathcal{R}(X, Y) \) of all regular operators which map \( X \) into \( Y \).

2. The space \( M(T, X) \)

Let \( T \) be a locally compact space and \( \mathcal{K} \) the set of all compact subsets of \( T \). For any \( E \in \mathcal{K} \) we denote by \( \mathcal{B}_E \) the set of all borelian subsets of \( E \) and we put

\[
\mathcal{B} = \bigcup_{E \in \mathcal{K}} \mathcal{B}_E.
\]
A $\mathcal{B}$-simple function $f$: $T \to X$ is, by definition, of the form

$$f(t) = \sum_{i=1}^{k} \gamma_{A_i}(t) x_i, \quad (t \in T)$$  \hspace{1cm} (1)$$

where $A_i \in \mathcal{B}$, $x_i \in X$, and $\gamma_{A}$ being the characteristic function of $A$.

We denote by $M(T, X)$ the set of the functions $f$: $T \to X$ having the following properties: there exists $E \in \mathcal{K}$ and a generalized sequence $\{f_\delta\}_{\delta \in \Delta}$ of $\mathcal{B}$-simple functions (mapping $T$ into $X$) such that $\text{spt } f_\delta \subset E$ (where spt means "the support") and $\{f_\delta\}_{\delta \in \Delta}$ is uniformly convergent to $f$. We shall say that $\{f_\delta\}_{\delta \in \Delta}$ is an approximating sequence for $f$.

For any $p \in \mathcal{P}$ we define a semi-norm $\tilde{p}$ on the vector space $M(T, X)$ putting

$$\tilde{p}(f) = \sup \{p(f(t)); \, t \in T\}$$

if $f \in M(T, X)$.

The set $M(T, X)$ is a locally convex vector lattice with respect to the pointwise order and the topology defined by the set $\{\tilde{p}: p \in \mathcal{P}\}$ of semi-norms.

The set $C_0(T, X)$ of continuous functions with compact support (mapping $T$ into $X$) is a vector sublattice of the space $M(T, X)$.

For any $E \in \mathcal{K}$ we denote

$$M_E(T, X) = \{f \in M(T, X); \, \text{spt } f \subset E\}.$$ 

The set $M_E(T, X)$ is a vector sublattice of the vector lattice $M(T, X)$.

We shall consider on the vector subspaces $C_0(T, X)$ and $M_E(T, X)$ of $M(T, X)$ the induced topology.

3. The integral

Let $m$: $\mathcal{B} \to \mathcal{L}$ be an additive function which satisfies the condition: for any $E \in \mathcal{K}$ there exists $p \in \mathcal{P}$ such that $m(\mathcal{B}_E) \subset \mathcal{L}_p$ and the set

$$G(E; p) = \left\{ \sum_{i=1}^{k} ||m(A_i)||_p; \, (A_1, \ldots, A_k) \text{ a } \mathcal{B}-\text{partition of } E \right\}$$

is $(\sigma)$-bounded. We shall say that $m$ is of the $(bv)$-type and we shall denote

$$v_m(E, p) = \sup G(E, p).$$

If $f \in M(T, X)$ is a $\mathcal{B}$-simple function (see formula (1)), we define

$$\int_T f \, dm = \sum_{i=1}^{k} m(A_i)(x_i).$$

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It is easily verified that the operator \( f \to \int_T f \, dm \) (defined on the set of \( \mathcal{B} \)-simple functions) is linear and for any \( E \in \mathcal{H} \) there exists \( p \in \mathcal{P} \) such that
\[
\left| \int_T f \, dm \right| \leq \bar{p}(f) v_m(E, p) \tag{2}
\]
if \( \text{spt } f \subset E \).

Let now \( f \) be arbitrary in \( M(T, X) \) and let \( \{f_\delta\}_{\delta \in \Delta} \) be an approximating sequence for \( f \), with \( \text{spt } f_\delta \subset \text{spt } f = E \).

There exists (see also (2)) \( p \in \mathcal{P} \) such that
\[
\left| \int_T f_\delta \cdot dm - \int_T f_\delta' \cdot dm \right| \leq \bar{p}(f_\delta' - f_\delta) v_m(E, p).
\]

Since \( Y \) is a complete vector lattice, the generalized sequence \( \left\{ \int_T f_\delta \, dm \right\}_{\delta \in \Delta} \) is \((p)\)-convergent (convergent with regulator \([1]\)). We shall define
\[
\int_T f \, dm = (p) - \lim_{\delta \to \Delta} \int_T f_\delta \, dm
\]
the limit being independent of the approximating sequence.

The integral is a linear operator (mapping \( M(T, X) \) into \( Y \)) and the inequality (2) holds for any \( f \in M_E(T, X) \).

### 4. Representation of some operators

If \( U: M(T, X) \to Y \) is a linear operator and \( E \in \mathcal{H} \), we shall denote by \( U_E \) the restriction of \( U \) to the subspace \( M_E(T, X) \). If \( U_E \) is \((po)\)-bounded with respect to \( \bar{p} \), we shall denote
\[
\|U\|_{E, p} = \|U_E\| \bar{p}.
\]

**Theorem.** A linear operator \( U: M(T, X) \to Y \) satisfies the condition

(i) \( U_E \) is \((po)\)-bounded, \( \forall E \in \mathcal{H} \),

if and only if

(ii) \( U(f) = \int_T f \, dm, \quad (f \in M(T, X)) \)

where \( m: \mathcal{B} \to \mathcal{I} \) is an additive function of the \((bv)\)-type. If (i) holds, then \( m \) can be chosen in (ii) such that the equality

(iii) \( \|U\|_{E, p} = v_m(E, p) \)

holds, as soon as the left-hand member exists.
Proof. As we saw in §3, the operator defined by (ii) satisfies the condition (i). From (2), which holds for any \( f \in M_E(T, X) \), it follows that
\[
\| U_E \| \hat{p} \leq v_m(E, p). \tag{3}
\]

Conversely, let \( U: M(T, X) \to Y \) be a linear operator satisfying (i). Hence, for any \( E \in \mathcal{K} \) there exist \( p \in \mathcal{P} \) and \( y_0 \in Y \) such that
\[
|U(f)| \leq \hat{p}(f) y_0, \quad (\forall f \in M_E(T, X)) \tag{4}
\]
Define \( m: \mathcal{B} \to \mathcal{L} \) by setting
\[
(m(A))(x) = U(\gamma_A \cdot x), \quad (\forall x \in X)
\]
(where \( (\gamma_A \cdot x)(t) = \gamma_A(t) \cdot x; \quad \forall t \in T \)).

The operator \( m(A): X \to Y \) is obviously linear. With (4), there exists \( p \in \mathcal{P} \), such that \( m(A) \in \mathcal{L}_p \) (and the function \( m: \mathcal{B} \to \mathcal{L} \) is obviously additive). By considering a \( \mathcal{B} \)-partition \( (A_1, \ldots, A_k) \) of a set \( E \in \mathcal{K} \), one has
\[
\sum_{i=1}^{k} \| m(A_i) \|_p = \sum_{i=1}^{k} \sup \{ |m(A_i)(x_i)|; \quad p(x_i) \leq 1 \} =
\]
\[
= \sup \left\{ \sum_{i=1}^{k} |m(A_i)(x_i)|; \quad p(x_i) \leq 1; \quad i = 1, \ldots, k \right\} \leq
\]
\[
\leq \sup \left\{ \sum_{i=1}^{k} |U| (|\gamma_A| x_i)|; \quad p(x_i) \leq 1; \quad i = 1, \ldots, k \right\} \leq
\]
\[
\leq \sup \{ |U|(|f|); \quad f \in M_E(T, X); \quad \hat{p}(f) \leq 1 \}
\]
by taking into account that (4) implies
\[
|U|(|f|) \leq \hat{p}(f) y_0, \quad (\forall f \in M_E(T, X)).
\]

It follows that
\[
\sum_{i=1}^{k} \| m(A_i) \|_p \leq y_0 \tag{5}
\]

The equality in (ii) obviously holds if \( f \) is a \( \mathcal{B} \)-simple function. Let now \( f \) be arbitrary in \( M(T, X) \) and \( \{f_\delta\}_{\delta \in \Delta} \) an approximating sequence for \( f \) such that \( \text{spt } f_\delta \subset \text{spt } f = E \). With (4) it follows that
\[
|U(f_\delta) - U(f)| \leq \hat{p}(f_\delta - f) y_0
\]
(where \( p \) and \( y_0 \) are suitably taken for \( E \)). Hence \( U(f) = (\mathcal{Q})_{\delta \in \Delta} U(f_\delta) \) and consequently (ii) hold.
If $U_E$ is $(po)$-bounded with respect to $\rho$, then we can take $y_0 = \|U_E\|_\rho$ in (4) and from (5) we get

$$w_m(E, p) \leq \|U_E\|_\rho;$$

with (3) it follows that (iii) holds.

**Corollary.** Any $(po)$-bounded linear operator $U: C_0(T, X) \rightarrow Y$ can be expressed in the form

$$U(f) = \int_T f \, dm$$

where $m: B \rightarrow X$ is an additive function of the $(bv)$-type.

Indeed, $U$ can be extended as a $(po)$-bounded linear operator on the space $M(T, X)$.

**REFERENCES**


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ПРЕДСТАВЛЕНИЕ ЛИНЕЙНЫХ ОПЕРАТОРОВ НА ПРОСТРАНСТВАХ ВЕКТОРНЫХ ФУНКЦИЙ

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Резюме

В данной работе устанавливается интегральное представление некоторых линейных операторов, заданных на упорядоченных пространствах, состоящих из векторных функций.