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$(\mathcal{L},\mathcal{L}')$-PRODUCTS OF ALGEBRAS

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ABSTRACT. An $(\mathcal{L},\mathcal{L}')$-product of algebras A_i $(i \in I)$ is a subdirect product of A_i satisfying certain conditions involving $\mathcal{L}$ and $\mathcal{L}'$, where $\mathcal{L}$ and $\mathcal{L}'$ are ideals of the power set of I. Direct, full subdirect and weak direct representations of algebras are special cases of $(\mathcal{L},\mathcal{L}')$-representations. Theorem 1 of this paper characterizes such representations in terms of congruence relations.

1. Introduction

Let I be a nonvoid set. $\mathcal{P}(I)$ and $\mathcal{F}(I)$ denote the set of all subsets of I and the set of all finite subsets of I, respectively. We denote by $P(I)$ the Boolean algebra

$$\langle \mathcal{P}(I), \cap, \cup, ', \emptyset, I \rangle .$$

If $\langle A_i : i \in I \rangle$ is a system of similar algebras, then $\prod \langle A_i : i \in I \rangle$, or $\prod A_i$, denotes the direct product of algebras A_i, $i \in I$. If $A = A_i$ for all $i \in I$, we write A^I for the direct product and call it a *direct power* of A.

For two elements $x, y \in \prod \langle A_i : i \in I \rangle$ we define

$$I(x,y) = \{ i \in I : x(i) \neq y(i) \} .$$

A *full subdirect product* of the A_i, $i \in I$, is a subalgebra A of $\prod A_i$ satisfying the following condition:

(A1) If $x \in A$, $y \in \prod A_i$ and if $I(x,y)$ is finite, then $y \in A$.

It is easy to verify that a subalgebra A of $\prod A_i$ is a full subdirect product if condition (iii) on p. 45 of [7] holds.

Let $A \subseteq \prod \langle A_i : i \in I \rangle$ be a subdirect product and let $\mathcal{L}$ be an ideal of $P(I)$. A is called an $\mathcal{L}$*-restricted subdirect product* (see [4; p. 92]) if it satisfies the following condition:

(A2) For every $x, y \in A$, $I(x,y) \in \mathcal{L}$.

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Let a subdirect product $A \subseteq \prod A_i$ satisfy (A2). If A has the property that for every $x \in A$ and for every $y \in \prod A_i$, $I(x,y) \in \mathcal{L}$ implies $y \in A$, then we say that A is an *L-restricted direct product* (see [3; p. 140] or [6; p. 219]). A subalgebra A of $\prod A_i$ is an *L-restricted full subdirect product* of algebras A_i, $i \in I$, (see [7; p. 45]) if conditions (A1) and (A2) are satisfied.

Now we generalize these notions in the following way:

Definition 1. Let A be a subdirect product of algebras A_i, $i \in I$, and let $\mathcal{L}$, $\mathcal{L}'$ be ideals of $P(I)$. We say that A is an *$(\mathcal{L},\mathcal{L}')$-product* of A_i, and we write

$$A = \prod_{\mathcal{L}}^{\mathcal{L}'} \langle A_i : \ i \in I \rangle, \qquad \text{or} \qquad A = \prod_{\mathcal{L}}^{\mathcal{L}'} A_i$$

if A satisfies (A2) and the following condition:

(A3) $x \in A$, $y \in \prod A_i$ and $I(x,y) \in \mathcal{L}'$ imply that $y \in A$.

If $C = A_i$ for all $i \in I$, we call $A = \prod_{\mathcal{L}}^{\mathcal{L}'} \langle A_i : \ i \in I \rangle$ an *$\langle \mathcal{L},\mathcal{L}' \rangle$-power of C with exponent I.*

If $\mathcal{L} = \mathcal{L}'$, we write $A = \prod^{\mathcal{L}} \langle A_i : \ i \in I \rangle$ for the $\langle \mathcal{L},\mathcal{L} \rangle$-product.

Obviously, $A = \prod^{\mathcal{L}} A_i$ if A is an $\mathcal{L}$-restricted direct product of algebras A_i, $i \in I$. In particular, $A = \prod^{\mathcal{F}(I)} \langle A_i : \ i \in I \rangle$ if and only if A is a weak direct product (see [3; p. 139]). If $\mathcal{L} = \mathcal{L}' = \mathcal{P}(I)$ we obtain the direct product.

If $\mathcal{L}' = \{\emptyset\}$ in Definition 1, we get the concept of an $\mathcal{L}$-restricted subdirect product. We note that if $\mathcal{L} = \mathcal{P}(I)$, then an $\mathcal{L}$-restricted subdirect product is a subdirect product.

It is easily seen that $\prod_{\mathcal{L}}^{\mathcal{F}(I)} A_i$ is an $\mathcal{L}$-restricted full subdirect product of the A_i, $i \in I$. Finally, a full subdirect product is a $\big(\mathcal{P}(I),\mathcal{F}(I)\big)$-product.

Example. Let I be an index set and let $G = Z_2^I$ where Z_2 is the two element group. For $x \in G$, we define the support of x, denoted $\operatorname{supp}(x)$, as

$$\operatorname{supp}(x) = \{ i \in I : \ x(i) \neq 0 \} .$$

Let I' be a subset of I, and set

$$\mathcal{L} = \{ X \cup Y : \ X \text{ is a finite subset of } I' \text{ and } Y \subseteq I - I' \} .$$

Define

$$\begin{aligned} H_1 &= \{x \in G : \ x(i) = x(j) \ \text{for all} \ i, j \in I - I'\}, \\ H_2 &= \{x \in G : \ I' \cap \operatorname{supp}(x) \ \text{is finite}\}, \\ H_3 &= \{x \in G : \ \operatorname{supp}(x) \ \text{is finite}\}, \\ H_4 &= \{x \in G : \ \operatorname{supp}(x) \ \text{is finite or} \ I - \operatorname{supp}(x) \ \text{is finite}\}. \end{aligned}$$

It is easy to see that H_1 is a $\langle \mathcal{P}(I), \mathcal{P}(I')\rangle$-power of Z_2 with exponent I, and H_2 is an $\mathcal{L}$-restricted direct power (and also an $\mathcal{L}$-restricted full subdirect power). $H_1 \cap H_2$ is an $\langle \mathcal{L}, \mathcal{F}(I')\rangle$-power of Z_2, and H_3 is a weak direct power. Finally, H_4 is a full subdirect power of Z_2, but it is not a weak direct power.

In the present paper we characterize $(\mathcal{L},\mathcal{L}')$-products in terms of congruence relations.

2. Preliminaries on congruence relations

Let A be an arbitrary algebra. We denote by $\operatorname{Con}(A)$ the set of all congruence relations on A. $\operatorname{Con}(A)$ forms a complete lattice with 0_A and 1_A, the smallest and the greatest congruence relations, respectively.

Let I be a nonvoid set and let $\mathcal{L}$, $\mathcal{L}'$ be ideals of the Boolean algebra $P(I)$. Let $\Theta = \langle \theta_i : \ i \in I\rangle$ be a system of congruences on A. For an arbitrary set $M \subseteq I$, we define a congruence relation $\theta(M)$ of A by

$$\theta(M) = \bigwedge \{\theta_j : \ j \in I - M\}.$$

We shall use the notion $\overline{\theta}_i$ for $\theta(\{i\})$, $i \in I$. We write

$$0_A = \prod_{\mathcal{L}}^{\mathcal{L}'} \langle \theta_i : \ i \in I\rangle$$

if the following conditions hold:

(i) $0_A = \bigwedge\{\theta_i : \ i \in I\}$,

(ii) $1_A = \bigvee\{\theta(M) : \ M \in \mathcal{L}\}$,

(iii) if $M \in \mathcal{L}'$ and if x, y_i $(i \in I)$ are elements of A such that $\langle x, y_i\rangle \in \theta_i$ for all $i \in I - M$, then there exists $z \in A$ satisfying $\langle z, y_i\rangle \in \theta_i$ for each $i \in I$.

We write $\prod^{\mathcal{L}} \langle \theta_i : \ i \in I\rangle$ for $\prod_{\mathcal{L}}^{\mathcal{L}} \langle \theta_i : \ i \in I\rangle$.

We begin with the following three lemmas.

LEMMA 1. (see [6; Lemma 4]) *If* $\mathcal{L} = \mathcal{P}(I)$, *then*

$$1_A = \bigvee \{\theta(M) : \ M \in \mathcal{L}\} .$$

LEMMA 2. *Let* $\mathcal{L}'$ *be an ideal of* $P(I)$ *containing all finite subsets of* I. *Then* (iii) *implies the following condition:*

(iv) *For every* $i \in I$, $1_A = \theta_i \circ \overline{\theta}_i$, *where* $\circ$ *denotes the relational product of two binary relations on* A.

Proof. Let i_0 be an arbitrary element of I and let $x, y \in A$. We define

$$y_i = \begin{cases} x & \text{if } i = i_0 , \\ y & \text{if } i \neq i_0 . \end{cases}$$

Obviously, $\langle y, y_i \rangle \in \theta_i$ for each $i \in I - M$, where $M = \{i_0\}$. Since $M \in \mathcal{L}'$, by (iii) we conclude that there is an element $z \in A$ such that $\langle z, y_i \rangle \in \theta_i$ for all $i \in I$. Then $\langle x, z \rangle \in \theta_{i_0}$ and $\langle z, y \rangle \in \overline{\theta}_{i_0}$. Hence (iv) holds. □

LEMMA 3. *If* $\mathcal{L}' = \mathcal{F}(I)$, *then* (iii) *is equivalent to* (iv).

Proof. Let Θ satisfy (iv). To prove (iii), we apply induction on the cardinality of M. Let $M = \{i_0\}$, x and y_i $(i \in I)$ be elements of A with $\langle x, y_i \rangle \in \theta_i$ for $i \neq i_0$.

By (iv), there is an element $z \in A$ satisfying $\langle y_{i_0}, z \rangle \in \theta_{i_0}$ and $\langle z, x \rangle \in \overline{\theta}_{i_0}$. Then $\langle z, y_i \rangle \in \theta_i$ for each $i \in I$.

Now suppose that the assertion is true for all $M \subseteq I$ with $|M| < n$. Let $M = \{i_1, \dots, i_n\}$ and let $x, y_i \in A$ $(i \in I)$ such that $\langle x, y_i \rangle \in \theta_i$ for $i \in I - M$. Again by (iv), there exists an element $y \in A$ satisfying $\langle y_{i_n}, y \rangle \in \theta_{i_n}$ and $\langle x, y \rangle \in \overline{\theta}_{i_n}$. Then $\langle y, y_i \rangle \in \theta_i$ for each $i \in I - \{i_1, \dots, i_{n-1}\}$. By the induction hypothesis, there is a $z \in A$ with $\langle z, y_i \rangle \in \theta_i$ for all $i \in I$. This ends the proof of (iii). The implication (iii) $\Longrightarrow$ (iv) follows from Lemma 2. □

From Lemmas 1 and 3 we have

PROPOSITION 1.

(a) $0_A = \prod_{\mathcal{P}(I)}^{\{\emptyset\}} \langle \theta_i : \ i \in I \rangle$ *if and only if* $0_A = \bigwedge \{\theta_i : \ i \in I\}$.

(b) $0_A = \prod_{\mathcal{L}}^{\{\emptyset\}} \langle \theta_i : \ i \in I \rangle$ *if and only if* Θ *satisfies* (i) *and* (ii).

(c) $0_A = \prod_{\mathcal{L}}^{\mathcal{F}(I)} \langle \theta_i : \ i \in I \rangle$ *if and only if* Θ *has properties* (i), (ii) *and* (iv).

(d) $0_A = \prod_{\mathcal{P}(I)}^{\mathcal{F}(I)} \langle \theta_i : \ i \in I \rangle$ *if and only if conditions* (i) *and* (iv) *are satisfied.*

Now we prove the following proposition.

PROPOSITION 2. $0_A = \prod^{\mathcal{L}}\langle\theta_i : i \in I\rangle$ *if and only if* Θ *satisfies* (i), (ii) *and the following condition* (given in [6; p. 222])*:*

(v) *For every* $\emptyset \neq M \in \mathcal{L}$ *and for every* $\langle x_i : i \in M\rangle \in A^M$*, if* $\langle x_i, x_j\rangle \in \theta(M)$ *for all* $i,j \in M$*, then there is a* $z \in A$ *such that* $\langle z, x_i\rangle \in \theta(M - \{i\})$ *for all* $i \in M$.

Proof. Assume that $0_A = \prod^{\mathcal{L}}\langle\theta_i : i \in M\rangle$. Clearly, Θ satisfies (i) and (ii). To prove (v), let $\emptyset \neq M \in \mathcal{L}$, x_i $(i \in M)$ be elements of A, and suppose that $\langle x_i, x_j\rangle \in \theta(M)$ for all $i,j \in M$. Let i_0 be an arbitrary element of M.

We set $x = x_{i_0}$ and define

$$y_i = \begin{cases} x_i & \text{if } i \in M, \\ x & \text{if } i \notin M. \end{cases}$$

Obviously, $\langle x, y_i\rangle \in \theta_i$ for all $i \in I - M$. By (iii), there exists an element $z \in A$ such that $\langle z, y_i\rangle \in \theta_i$ for each $i \in I$.

Let $i \in M$. Then $\langle z, y_i\rangle \in \theta_i$, and since $y_i = x_i$ we also have $\langle z, x_i\rangle \in \theta_i$. Observe that

$$\langle z, x_i\rangle \in \theta(M).$$

Indeed, if $j \notin M$, then $\langle z, x\rangle = \langle z, y_i\rangle \in \theta_j$. Hence $\langle z, x_{i_0}\rangle = \langle z, x\rangle \in \theta(M)$, and by the assumption, $\langle x_{i_0}, x_i\rangle \in \theta(M)$. Therefore, $\langle z, x_i\rangle \in \theta(M)$. Consequently, $\langle z, x_i\rangle \in \theta(M - \{i\})$ for each $i \in M$. Thus (v) is true.

Suppose now that conditions (i), (ii) and (v) are satisfied.

We conclude that (iv) holds by using the proof of Lemma 1 in [6]. To prove (iii), let $\emptyset \neq M \in \mathcal{L}$ (if $M = \emptyset$, then it is obvious), and let $x, y_i \in A$ $(i \in I)$ such that $\langle x, y_i\rangle \in \theta_i$ for $i \in I - M$. From (iv) we deduce that for every $i \in I$, there exists an $x_i \in A$ satisfying

$$\langle x_i, y_i\rangle \in \theta_i \qquad \text{and} \qquad \langle x_i, x\rangle \in \overline{\theta}_i. \tag{1}$$

Hence $\langle x_i, x_j\rangle \in \overline{\theta}_i \vee \overline{\theta}_j$ for any $i,j \in I$. Therefore, $\langle x_i, x_j\rangle \in \theta(M)$ for all $i,j \in M$. By (v), there is an element $z \in A$ such that $\langle z, x_i\rangle \in \theta(M - \{i\})$ for each $i \in M$. If $i \in M$, then $\langle z, x_i\rangle \in \theta_i$ and, since $\langle x_i, y_i\rangle \in \theta_i$ (by (1)), we obtain that $\langle z, y_i\rangle \in \theta_i$. Let $i \in I - M$. Then $\langle z, x_j\rangle \in \theta_i$ for some $j \in M$. From (1) it follows that $\langle x_j, x\rangle \in \overline{\theta}_j \leq \theta_i$, and by assumption we have $\langle x, y_i\rangle \in \theta_i$. Consequently, $\langle z, y_i\rangle \in \theta_i$ for each $i \in I$, and therefore, (iii) holds for $\mathcal{L}' = \mathcal{L}$. Thus $0_A = \prod^{\mathcal{L}}\langle\theta_i : i \in I\rangle$. □

PROPOSITION 3. *The following three statements are equivalent.*

(a) $0_A = \prod^{\mathcal{P}(I)} \langle \theta_i : i \in I \rangle$.
(b) Θ *satisfies* (i), (iv) *and* (vi) *for all elements* x_i $(i \in I)$ *of* A *satisfying* $\langle x_i, x_j \rangle \in \overline{\theta}_i \vee \overline{\theta}_j$ *for all* $i, j \in I$, *there is an element* $y \in A$ *such that* $\langle y, x_i \rangle \in \theta_i$ *for every* $i \in I$ (*that is,* Θ *is consistent, see* [1; p. 92]).
(c) Θ *satisfies* (i) *and* (vii) *for every* $\langle x_i : i \in I \rangle \in A^I$, *there is an element* $y \in A$ *such that* $\langle y, x_i \rangle \in \theta_i$ *for every* $i \in I$.

Proof. Let $0_A = \langle \theta_i : i \in I \rangle$. It is obvious that Θ is consistent. By Lemma 2, condition (iv) is fulfilled. Thus statement (b) holds. Therefore, (a) $\Longrightarrow$ (b).

Now assume that conditions (i), (iv) and (vi) are satisfied. To prove that Θ also satisfies (vii), let x_i $(i \in I)$ be elements of A. We put $x = x_{i_0}$, where i_0 is an element of I. By (iv), for every $i \in I$, there exists an element $y_i \in A$ such that

$$\langle x_i, y_i \rangle \in \theta_i \qquad \text{and} \qquad \langle y_i, x \rangle \in \overline{\theta}_i . \tag{2}$$

Hence $\langle y_i, y_j \rangle \in \overline{\theta}_i \vee \overline{\theta}_j$ for arbitrary $i, j \in I$. From (vi) we conclude that there is an element $y \in A$ satisfying $\langle y, y_i \rangle \in \theta_i$ for each $i \in I$. Now, from (2) it follows that $\langle y, x_i \rangle \in \theta_i$ for all $i \in I$, and therefore (vii) is satisfied. This finishes the proof that (b) $\Longrightarrow$ (c).

Finally, suppose that Θ satisfies (i) and (vii). Clearly, (iii) holds for $\mathcal{L}' = \mathcal{P}(I)$. By Lemma 1, $1_A = \bigvee(\theta(M) : M \in \mathcal{P}(I))$. Thus (c) $\Longrightarrow$ (a). □

3. $(\mathcal{L}, \mathcal{L}')$-representations of algebras

Let I be a nonvoid set and let $\mathcal{L}$, $\mathcal{L}'$ be ideals of $P(I)$. Let A be arbitrary algebra. We say that a system $\langle \theta_i : i \in I \rangle \in (\mathrm{Con}(A))^I$ is an $(\mathcal{L}, \mathcal{L}')$*-representation* of A if the mapping $f \colon A \to \prod \langle A/\theta_i : i \in I \rangle$ defined by the rule $f(x)(i) = x/\theta_i$ (x/θ_i is the congruence class containing x) is one-to-one and $f(A) = \prod_{\mathcal{L}}^{\mathcal{L}'} \langle A/\theta_i : i \in I \rangle$.

For every $i \in I$, we set $A_i = A/\theta_i$ and denote by p_i the ith projection function from $\prod \langle A_i : i \in I \rangle$ onto A_i.

The mapping $f_i = p_i \circ f$, which is a homomorphism of A onto A_i will be referred to as the ith f*-projection.*

If $\langle\theta_i : i \in I\rangle$ is an $(\mathcal{L},\mathcal{L}')$-representation of A, then this representation is called:

(a) *subdirect*, if $\mathcal{L} = \mathcal{P}(I)$ and $\mathcal{L}' = \{\emptyset\}$,
(b) *$\mathcal{L}$-restricted subdirect*, if $\mathcal{L}' = \{\emptyset\}$,
(c) *full subdirect*, if $\mathcal{L} = \mathcal{P}(I)$ and $\mathcal{L}' = \mathcal{F}(I)$,
(d) *direct*, if $\mathcal{L} = \mathcal{L}' = \mathcal{P}(I)$,
(e) *$\mathcal{L}$-restricted direct*, if $\mathcal{L} = \mathcal{L}'$,
(f) *$\mathcal{L}$-restricted full subdirect*, if $\mathcal{L}' = \mathcal{F}(I)$,
(g) *weak direct*, if $\mathcal{L} = \mathcal{L}' = \mathcal{F}(I)$.

The next result characterizes $(\mathcal{L},\mathcal{L}')$-representations internally.

THEOREM 1. *Let A be an algebra and let I be a nonvoid set. Let $\mathcal{L}$ and $\mathcal{L}'$ be ideals of the Boolean algebra $P(I)$.*

Then a system $\langle\theta_i : i \in I\rangle \in (\mathrm{Con}(A))^I$ is an $(\mathcal{L},\mathcal{L}')$-representation of A if and only if $0_A = \prod_{\mathcal{L}}^{\mathcal{L}'}\langle\theta_i : i \in I\rangle$.

Proof. We put $A_i = A/\theta_i$ for $i \in I$ and define the mapping $f\colon A \to \prod\langle A_i : i \in I\rangle$ by setting $f(x) = \langle x/\theta_i : i \in I\rangle$. Let $B = f(A)$, and denote by f_i the ith f-projection.

Suppose that f is one-to-one and that $B = \prod_{\mathcal{L}}^{\mathcal{L}'}\langle A_i : i \in I\rangle$. Obviously, $0_A = \bigwedge\{\theta_i : i \in I\}$, that is, the condition (i) holds. To prove (ii), let $x, y \in A$ and let $M = \{i \in I : f_i(x) \neq f_i(y)\}$. By the property (A2), $M \in \mathcal{L}$, and clearly $\langle x,y\rangle \in \theta(M)$. Then $\langle x,y\rangle \in \bigvee(\theta(M) : M \in \mathcal{L})$, and hence (ii) is satisfied.

Now we shall prove that (iii) holds. Let M be a set of $\mathcal{L}'$ and let x, y_i $(i \in I)$ be elements of A such that $\langle x, y_i\rangle \in \theta_i$ for every $i \in I - M$. Then $\{i \in I : x/\theta_i \neq y_i/\theta_i\} \subseteq M$. By the definition of ideal we conclude that $\{i : x/\theta_i \neq y_i/\theta_i\} \in \mathcal{L}'$, and hence $I(f(x), y) \in \mathcal{L}'$, where $y = \langle y_i/\theta_i : i \in I\rangle$. From (A3) it follows that $y \in B$.

Let $z \in A$ such that $f(z) = y$. It is obvious that $f_i(z) = f_i(y_i)$ for $i \in I$. Hence $\langle z, y_i\rangle \in \theta_i$ for every i, and consequently, (iii) holds. Thus $0_A = \prod_{\mathcal{L}}^{\mathcal{L}'}\langle\theta_i : i \in I\rangle$.

Conversely, assume that $\langle\theta_i : i \in I\rangle$ satisfies conditions (i), (ii) and (iii). The fact that f is an embedding is easy to check. Of course, B is a subdirect product of algebras A_i, $i \in I$. Let $x, y \in A$. Now we prove that

$$I(f(x), f(y)) \in \mathcal{L}. \tag{3}$$

By (ii), $\langle x,y\rangle \in \bigvee\{\theta(M) : M \in \mathcal{L}\}$. Then there exists a sequence of elements of A, $x = x_1, x_2, \dots, x_n = y$ and sets $M_1, M_2, \dots, M_{n-1} \in \mathcal{L}$ such that $\langle x_i, x_{i+1}\rangle \in \theta(M_i)$, for $i = 1, 2, \dots, n-1$.

Consequently, $\langle x, y\rangle \in \theta(M)$, where $M = M_1 \cup M_2 \cup \cdots \cup M_{n-1} \in \mathcal{L}$. Therefore, $f_i(x) = f_i(y)$ for every $i \notin M$, and hence $\{i : f_i(x) \neq f_i(y)\} \subseteq M$. From this we obtain (3). It follows that B satisfies (A2).

Now let $\overline{x} \in B$ and $y \in \prod\langle A/\theta_i : i \in I\rangle$. Suppose that $M = I(\overline{x}, y) \in \mathcal{L}'$. From the fact that B is a subdirect product of the algebras A/θ_i, $i \in I$ we conclude that there is a system $\langle \overline{y}_i : i \in I\rangle \in B^I$ with $\overline{y}_i(i) = y(i)$ for $i \in I$.

Take $x, y_i \in A$, $i \in I$, such that $f(x) = \overline{x}$ and $f(y_i) = \overline{y}_i$ for $i \in I$. Let $i \in I - M$. Then $\overline{x}(i) = y(i)$, and therefore, $x/\theta_i = y_i/\theta_i$. Hence $\langle x, y_i\rangle \in \theta_i$ for $i \in I - M$. By (iii), there is an element $z \in A$ satisfying $\langle z, y_i\rangle \in \theta_i$ for every $i \in I$. Let $\overline{z} = f(z) \in B$. We have $\overline{z}(i) = f_i(z) = z/\theta_i = y_i/\theta_i = f_i(y_i) = \overline{y}_i(i) = y(i)$ for $i \in I$. Then $\overline{z} = y$, and since $\overline{z} \in B$ we also have that $y \in B$. Consequently, B satisfies (A3). Thus $\langle \theta_i : i \in I\rangle$ is an $(\mathcal{L}, \mathcal{L}')$-representation of A. □

Now we give some applications of Theorem 1.

Let $\Theta = \langle \theta_i : i \in I\rangle$ be a system of congruences of an algebra A. From Theorem 1 and Proposition 1(a) we obtain the following well-known fact:

Corollary 1. *Θ is a subdirect representation of A if and only if $0_A = \bigwedge\{\theta_i : i \in I\}$.*

An immediate consequence of Theorem 1 and Propositions 1(b) and 2 is:

Corollary 2. (cf. [6; Corollaries 3 and 4]) *Let $\mathcal{L}$ be an ideal of $P(I)$. Then:*

(a) *Θ is an $\mathcal{L}$-restricted subdirect representation of A if and only if conditions* (i) *and* (ii) *are fulfilled.*

(b) *Θ is an $\mathcal{L}$-restricted direct representation of A if and only if conditions* (i), (ii), *and* (v) *are satisfied.*

By Theorem 1 and Proposition 3 we obtain:

Corollary 3. (see [1; Theorem 11.7] and [5; Theorem 4.31]) *Θ is a direct representation of A if and only if Θ satisfies* (i), (iv) *and* (vi) (*or:* (i) *and* (vii)).

From Theorem 1 and Proposition 1(c) we get:

Corollary 4. (cf. [7; Theorem 1]) *If $\mathcal{L}$ is an ideal of $P(I)$, then Θ is an $\mathcal{L}$-restricted full subdirect representation of A if and only if conditions* (i), (ii) *and* (iv) *hold.*

Hence we have:

Corollary 5. *Θ is a weak direct representation of A if and only if Θ satisfies* (i), (iv) *and* (ii) *with $\mathcal{L} = \mathcal{F}(I)$.*

Finally, we obtain:

COROLLARY 6. (see [2; Lemma 1.1]) Θ *is a full subdirect representation of* A *if and only if conditions* (i) *and* (iv) *are satisfied.*

P r o o f . Follows from Theorem 1 and from Proposition 1(d). □

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