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DOMINATION NUMBERS OF CARDINAL PRODUCTS

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ABSTRACT. For a graph G a subset D of the vertex-set of G is called dominating set if every vertex x not in D, is adjacent to at least one vertex of D. The domination number $\gamma(G)$ is the cardinality of the smallest dominating set.

Here we determine the domination numbers of $P_2 \times P_n$, $P_3 \times P_n$, $P_4 \times P_n$, and $P_5 \times P_n$ where $\times$ denotes the cardinal product.

1. Introduction

For any graph G we denote by $V(G)$ and $E(G)$ the vertex-set and edge-set of G, respectively. The cardinal product $G \times H$ of two graphs G and H is a graph with $V(G \times H) = V(G) \times V(H)$ and $\{(g_1, h_1), (g_2, h_2)\} \in E(G \times H)$ if and only if $\{g_1, g_2\} \in E(G)$ and $\{h_1, h_2\} \in E(H)$. $\gamma(G)$ is the cardinality of the smallest dominating set in G. In this paper we determine the domination numbers of certain classes of graphs. Such investigations were initiated by Vizing [12], who conjectured that

$$\gamma(G \square H) \geq \gamma(G)\gamma(H)$$

holds for the cartesian product of graphs G and H. While dominating numbers of the cartesian product of graphs were considered in many papers (see e.g. [2], [3], [4], [6], [7], [10]), only a few results about the domination numbers of cardinal products of graphs are known so far ([5], [8], [9], [11]).

The following observation will be frequently used in the sequel.

OBSERVATION 1. *Let C_n and P_n denote the cycle and path with n vertices, respectively. Then*

$$\gamma(C_n) = \gamma(P_n) = \left\lceil \frac{n}{3} \right\rceil.$$

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Following the investigations of the cartesian product we consider those cardinal products where one of the factors is a path.

PROPOSITION 1. *For any tree T and any graph G without cycles of odd length we have*

$$\gamma(P_2 \times T) = 2\gamma(T) > \gamma(P_2)\gamma(T)$$

and

$$\gamma(P_2 \times G) = 2\gamma(G) > \gamma(P_2)\gamma(G).$$

P r o o f. Obvious, since $P_2 \times T$ and $P_2 \times G$ consist of two disjoint copies of T and G, respectively. □

PROPOSITION 2. *For the path P_2 and any odd cycle C_{2n+1}, $n \geq 1$,*

$$\gamma(P_2 \times C_{2n+1}) = \left\lceil \frac{4n+2}{3} \right\rceil > \gamma(P_2)\gamma(C_{2n+1}).$$

P r o o f. Note that the cardinal product of P_2 and C_{2n+1} is isomorphic to C_{4n+2}. Then Observation 1 implies that

$$\gamma(C_{4n+2}) = \left\lceil \frac{4n+2}{3} \right\rceil > \left\lceil \frac{2n+1}{3} \right\rceil = \gamma(P_2)\gamma(C_{2n+1}).$$

□

2. Domination numbers of $P_k \times P_n$

In the sequel we consider the graphs $P_k \times P_n$ for $3 \leq k \leq 5$.

OBSERVATION 2. *The cardinal product $P_k \times P_n$, $k, n \geq 3$, consists of two components. If both, k and n are odd, these components are not isomorphic. If at least one of these two numbers is even, the components are isomorphic.*

DEFINITION 1. By C_1 we denote the component which contains the vertex $(1,1)$, by C_2 the other component.

DEFINITION 2. For a fixed m, $1 \leq m \leq n$, the set $(P_k)_m = \{(i,m) \mid i = 1,\dots,k\}$ is called a *column* of $P_k \times P_n$. The set $(P_n)_m = \{(m,j) \mid j = 1,\dots,n\}$ is called a *row* of $P_k \times P_n$.

A set $B = \{(P_k)_m, (P_k)_{m+1}, \dots, (P_k)_{m+l} \mid l \geq 0,\ m \geq 1,\ m+l \leq n\}$ of columns is called a *block* of size $k \times (l+1)$ of $P_k \times P_n$.

If another block B_1 contains the column $(P_k)_{m-1}$ or the column $(P_k)_{m+l+1}$, then we say that B_1 is *adjacent* to B. A block B is called *internal*, if it is adjacent to two other blocks, it is called *external* if it is only adjacent to one block.

THEOREM 1. *For every path P_n, $n \geq 2$,*

$$\gamma(P_3 \times P_n) = n.$$

Proof. The set $S = \{(2,j) \mid 1 \leq j \leq n\}$ dominates $P_3 \times P_n$. Thus $\gamma(P_3 \times P_n) \leq n$.

We now prove that $\gamma(P_3 \times P_n) \geq n$.

Case 1. n is even.

We only consider the component C_1.

LEMMA 1. *There is a minimum dominating set D, such that D only contains vertices of the row $(2,i)$, $i \in \{2,4,\dots,n\}$.*

Proof. Let D be a minimal dominating set which does not satisfy our assertion. Without loss of generality we assume that D contains a vertex of the row $(P_n)_1$. Let $(1,j)$ be this vertex for some fixed $j \in \{1,3,\dots,n-1\}$. Then the vertex $(3,j)$ is either contained in D or dominated by a vertex of D.

We first assume that $(3,j) \in D$. Let $j \notin \{1,n-1\}$. Then the set $D' = D \setminus \{(1,j),(3,j)\} \cup \{(2,j-1),(2,j+1)\}$ also dominates C_1 and $|D'| \leq |D|$. If $j = 1$ then D is not minimal since $D' = D \setminus \{(1,1),(3,1)\} \cup \{(2,2)\}$ also dominates C_1. If $j = n-1$ then $D' = D \setminus \{(1,n-1),(3,n-1)\} \cup \{(2,n-1)\}$ dominates C_1 and $|D'| = |D| - 1$.

Let $(3,j) \notin D$. Since $(3,j)$ is dominated by a vertex of D, either $(2,j-1)$ or $(2,j+1)$ is contained in D. If $(2,j-1) \in D$ then $D' = D \setminus \{(1,j)\} \cup \{(2,j+1)\}$ also dominates C_1. If $(2,j+1) \in D$, $j > 1$, then $D' = D \setminus \{(1,j)\} \cup \{(2,j-1)\}$ dominates C_1. If $j = 1$, and $(2,j+1) \in D$ then D is not minimal. □

If D only contains vertices of the row $(2,i)$, $1 \leq i \leq n$, then obviously $|D| = n$ holds.

Case 2. n is odd.

For both components the assertion of Lemma 1 can be proved analogously which again implies that $\gamma(P_3 \times P_n) = n$ holds. □

THEOREM 2. *Let $n \geq 2$. Then*

$$\gamma(P_4 \times P_n) = \begin{cases} n & n \equiv 0 \pmod 4, \\ n+1 & n \equiv 1 \pmod 4;\ n \equiv 3 \pmod 4, \\ n+2 & n \equiv 2 \pmod 4. \end{cases}$$

Proof. We consider the set

$$D = \{(2,4m+2),(2,4m+3),(3,4m+2),(3,4m+3) \mid m = 0,1,\dots,\lfloor \tfrac{n}{4} \rfloor - 1\}.$$

D dominates all vertices if n is divisible by 4. If $n = 4k+1$ then we add $(2,4k),(3,4k)$ to D, if $n = 4k+2$ we add $(2,4k),(3,4k),(2,4k+1),(3,4k+1)$

and if $n = 4k+3$ we add $(2,4k+2),(3,4k+2),(2,4k+3),(3,4k+3)$. The set D is dominating and hence

$$\gamma(P_4 \times P_n) \le |D| = \begin{cases} n & n \equiv 0 \pmod 4, \\ n+1 & n \equiv 1 \pmod 4;\ n \equiv 3 \pmod 4, \\ n+2 & n \equiv 2 \pmod 4. \end{cases}$$

In the sequel we prove that $\gamma(P_4 \times P_n) \ge |D|$. Since $P_4 \times P_n$ consists of two isomorphic components, all the considerations are done for only one component, namely C_1.

We partition the graph $P_4 \times P_n$ into $\lfloor \frac{n}{4} \rfloor$ 4×4-blocks. If $n \equiv k \pmod 4$, where $k \neq 0$, then we also have one $4 \times k$ block E'.

Without loss of generality we assume $E' = \{(P_4)_n, \dots, (P_4)_{n-k+1}\}$.

Case 1. $n \equiv 0 \pmod 4$.

Lemma 2. *There is no dominating set D such that, for some 4×4 block B,*

$$|D \cap B| \le 1.$$

Proof. First, let B be external block. Without loss of generality we assume that $B = \{(P_4)_1, \dots, (P_4)_4\}$. Even if the column $(P_4)_4$ is dominated by vertices from the adjacent block we still need at least two vertices contained in B to dominate all vertices of the first three columns.

Let B be now any internal block. At most the first and the last column of B can be dominated by vertices not in B. To dominate the remaining vertices we need at least two vertices which are contained in B. □

It follows from Lemma 2 that the domination number of one component of $P_4 \times P_n$ is equal to $n/2$ hence $\gamma(P_4 \times P_n) = n$.

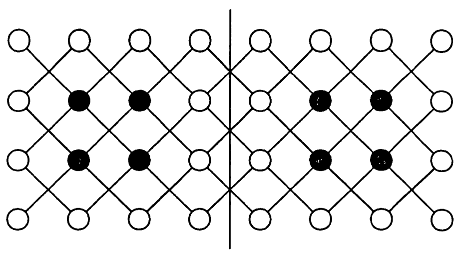

Figure 1. Dominating set of $P_4 \times P_8$.

Case 2. $n \equiv 1 \pmod 4$.

LEMMA 3. *If* $|D \cap E'| = 0$, *then there exists at least one block* B_i *of size* 4×4 *such that* $|D \cap B_i| \geq 3$, *for every dominating set* D.

Proof. If $|D \cap E'| = 0$, then the column $(P_4)_{n-1}$ (of the adjacent block B_1) contains at least one vertex of D. If $(4, n-1) \in D$ then D must also contain the vertex $(2, n-1)$. But then it is clear that B_1 must contain at least a third vertex of D.

We now assume that $(4, n-1)$ is not in D. Then $(2, n-1) \in D$ must hold. To dominate the remaining vertices of B_1 we need at least two more vertices. If both of these vertices are contained in B_1, then we are done.

If $|B_1 \cap D| = 2$, then $(3, n-2) \in D$ must hold since the vertices $(2, n-3)$, $(4, n-3)$ and $(4, n-1)$ can only be dominated by vertices which are contained in B_1. But then both vertices of the first column of B_1, namely $(1, n-4)$ and $(3, n-4)$ are dominated by vertices of the last column of the 4×4 block adjacent to B_1. Then we have the same situation as above: either both vertices, $(2, n-5)$ and $(4, n-5)$, are contained in D or only $(2, n-2) \in D$ holds.

Repeating the above considerations we either obtain a block B_m with $|D \cap B_m| = 3$, for some I, $2 \leq m < \lfloor \frac{n}{4} \rfloor$, or $|D \cap B_i| = 2$ holds for all i, $2 \leq i \leq \lfloor \frac{n}{4} \rfloor$. But then the block $B_{\lfloor \frac{n}{4} \rfloor}$ contains at least three vertices of D since no vertex of $B_{\lfloor \frac{n}{4} \rfloor}$ is dominated by vertices outside $B_{\lfloor \frac{n}{4} \rfloor}$ if $|D \cap B_i| = 2$ holds for all i, $2 \leq i < \lfloor \frac{n}{4} \rfloor$. □

Of course Lemma 2 also holds if $n \equiv 1 \pmod 4$. Hence, together with Lemma 3 we obtain

$$|D| \geq n + 1.$$

If $|D \cap E'| \geq 1$, then it again follows from Lemma 2 that $|D| \geq n + 1$.

Case 3. $n \equiv 2 \pmod 4$.

LEMMA 4.

1) $|D \cap E'| \geq 1$ *for every dominating set* D.
2) *If* $|D \cap E'| = 1$, *then there exists at least one block* B_i *of size* 4×4 *such that* $|D \cap B_i| \geq 3$ *for every dominating set* D.

Proof.

1) With vertices from the adjacent block, we can only dominate vertices of $(P_4)_{n-1}$.

2) Similar to the proof of Lemma 3. □

Again, Lemma 2 also holds. These fact, together with Lemma 4, imply that $|D \cap C_1| \geq \frac{n}{2} + 1$, and therefore

$$|D| \geq n + 2.$$

Case 4. $n \equiv 3 \pmod 4$.
It is easy to see that $|D \cap E'| \geq 2$ holds for every dominating set D. From this and Lemma 2 we obtain

$$|D| \geq 2 \cdot \left(\frac{n-3}{4} \cdot 2 + 2\right) = n + 1 .$$

□

THEOREM 3. *We have*

$$\gamma(P_5 \times P_n) = \begin{cases} n+2 & \text{if } n = 2,3,4, \\ 11 & \text{if } n = 7, \\ \frac{4n+6}{3} & \text{if } n \equiv 0 \pmod 6;\ n \equiv 3 \pmod 6, \\ \frac{4n+4}{3} & \text{if } n \equiv 2 \pmod 6;\ n \equiv 5 \pmod 6, \\ \frac{4n+8}{3} & \text{if } n \equiv 4 \pmod 6;\ n \equiv 1 \pmod 6,\ n > 7. \end{cases}$$

Proof. For $n \in \{2,3,4\}$ it was already shown. For $n = 7$ it is easy to check.

If n is odd, we have to consider both components separately, since they are not isomorphic. For even n, the components are isomorphic, hence we consider only one component, namely C_1.

Case 1. n is even.
A dominating set S of C_1 is given as follows: It contains the vertices $(2,2)$, $(4,2)$, $(4,4)$ and $(1,5)$. If $n \geq 12$ it also contains all vertices $(5, 7+6m)$, $(2, 8+6m)$, $(4, 10+6m)$, $(1, 11+6m)$, $m = 0, 1, \ldots, \lfloor \frac{n}{6} \rfloor - 2$. In addition it contains the vertices

$$\begin{array}{rl} (4,n) & \text{if } n \equiv 0 \pmod 6, \\ (5,n-1), (2,n) & \text{if } n \equiv 2 \pmod 6, \\ (2,n-2), (2,n), (4,n), (5,n-3) & \text{if } n \equiv 4 \pmod 6. \end{array}$$

Then

$$|S| = \begin{cases} \frac{2n+3}{3} & \text{if } n \equiv 0 \pmod 6, \\ \frac{2n+2}{3} & \text{if } n \equiv 2 \pmod 6, \\ \frac{2n+4}{3} & \text{if } n \equiv 4 \pmod 6. \end{cases}$$

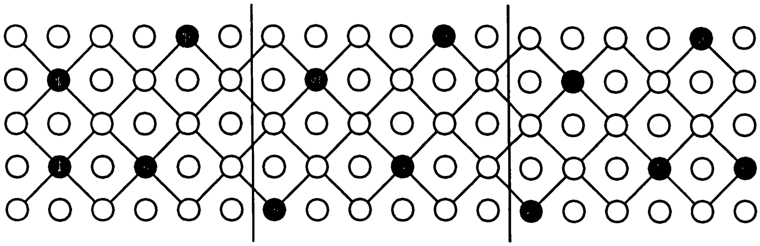

FIGURE 2. Dominating set of $P_5 \times P_{18}$ (component C_1).

Case 2. n is odd.
We first consider the component C_2. A dominating set S_2 of C_2 is given as follows: $S_2 = \{(1, 4+6m), (2, 1+6m), (4, 5+6m), (5, 2+6m) \mid m = 0, 1, \ldots, \lfloor \frac{n}{6} \rfloor - 1\}$. In addition S_2 contains the vertices

$$\begin{aligned} (2,n),(4,n) &\quad \text{if} \quad n \equiv 1 \pmod 6, \\ (2,n-2),(2,n),(5,n-1) &\quad \text{if} \quad n \equiv 3 \pmod 6, \\ (1,n-1),(2,n-4),(4,n),(5,n-3) &\quad \text{if} \quad n \equiv 5 \pmod 6. \end{aligned}$$

Then

$$|S_2| = \begin{cases} \frac{2n+4}{3} & \text{if } n \equiv 1 \pmod 6, \\ \frac{2n+3}{3} & \text{if } n \equiv 3 \pmod 6, \\ \frac{2n+2}{3} & \text{if } n \equiv 5 \pmod 6. \end{cases}$$

A dominating set S_1 of C_1 is given as follows: It contains the vertices $(2,2)$, $(4,2)$, $(4,4)$ and $(1,5)$. If $n \geq 13$ it also contains all vertices $(5, 7+6m)$, $(2, 8+6m)$, $(4, 10+6m)$, $(1, 11+6m)$, $m = 0, 1, \ldots, \lfloor \frac{n}{6} \rfloor - 2$. In addition it contains the vertices

$$\begin{aligned} (1,n),(4,n-1) &\quad \text{if} \quad n \equiv 1 \pmod 6, \\ (2,n-1),(5,n-2),(5,n) &\quad \text{if} \quad n \equiv 3 \pmod 6, \\ (2,n-3),(2,n-1),(4,n-1),(5,n-4) &\quad \text{if} \quad n \equiv 5 \pmod 6. \end{aligned}$$

Then

$$|S_1| = \begin{cases} \frac{2n+4}{3} & \text{if } n \equiv 1 \pmod 6, \\ \frac{2n+3}{3} & \text{if } n \equiv 3 \pmod 6, \\ \frac{2n+2}{3} & \text{if } n \equiv 5 \pmod 6. \end{cases}$$

and

$$|S| = |S_1 \cup S_2| = \begin{cases} \frac{4n+8}{3} & \text{if } n \equiv 1 \pmod 6, \\ \frac{4n+6}{3} & \text{if } n \equiv 3 \pmod 6, \\ \frac{4n+4}{3} & \text{if } n \equiv 5 \pmod 6. \end{cases}$$

Obviously the set S is a dominating set of $P_5 \times P_n$ for every odd n.

We now prove that $\gamma(P_5 \times P_n) \geq |S|$.

We partition the graph $P_5 \times P_n$ into 5×6 blocks.

DEFINITION 3. If a block is external we denote it by E, if it is internal by I. If $n \equiv k \pmod 6$, where $k \neq 0$, then we also have a block E', which is $5 \times k$ block.

Without loss of generality we assume that $E' = \{(P_5)_n, \dots, (P_5)_{n-k+1}\}$.

The next three Lemmas are all proven for the component C_1, not depending on the parity of n. If it cannot be seen immediately, that the respective result also holds for C_2 if n is odd, then remarks following the respective Lemmas indicate why this is true.

LEMMA 5. *There is no dominating set D such that $|D \cap E| \leq 3$.*

Proof. W.l.o.g. we assume that E is the first block in the graph $P_5 \times P_n$ (it contains $(1,1)$). If the column $(P_5)_6$ is dominated with vertices from the adjacent block, there still is one undominated block of size 5×5. To dominate the vertices of this 5×5 block we need at least four vertices:

a) If the column $(P_5)_6$ of E contains no vertex of D, we need at least four vertices of this 5×5 block, to dominate it.

b) We now assume that the column $(P_5)_6$ contains at least one vertex of D. This vertex cannot dominate any vertices of $(P_5)_4$. To dominate the three vertices of the column $(P_5)_1$ we need at least two vertices. These vertices can dominate at most the first three columns of E. Then at least the column $(P_5)_4$ is not dominated. So, $D \cap E$ contains at least one more vertex, i.e. $|D \cap E| \geq 4$. □

Remark. For C_2 Lemma 5 can be shown analogously, since we also need at least four vertices contained in E to dominate the vertices of $(P_5)_1, \dots, (P_5)_5$.

LEMMA 6. *There is no dominating set D such that $|D \cap I| \leq 2$.*

Proof. Let $I = \{(P_5)_j, (P_5)_{j+1}, \dots, (P_5)_{j+5}\}$, $j \geq 7$, be some internal block. Only vertices of the columns $(P_5)_j$ and $(P_5)_{j+5}$ can be dominated by vertices of adjacent blocks. To dominate the vertices of the columns $(P_5)_{j+1}, \dots, (P_5)_{j+4}$ we always need at least three vertices, where it does not matter if $(P_5)_j$ or $(P_5)_{j+5}$ contain any vertex of D. Of course this fact also does neither depend on the parity of n nor on the component we consider. □

LEMMA 7. *If $|D \cap B_k| = 3$ for some internal 5×6 block B_k, $n \geq 18$, then $|D \cap B_{k-1}| \geq 5$, and $|D \cap B_{k+1}| \geq 5$. If B_{k+1} is external then $|D \cap B_{k+1}| \geq 6$.*

Proof. Let $B_k = \{(P_5)_j, (P_5)_{j+1}, \dots, (P_5)_{j+5}\}$, $j = 6(k-1)+1$, $k \in \{2, \dots, \lfloor \frac{n}{6} \rfloor - 1\}$. By vertices not in B_k we can dominate only the first and the last column of B_k. Hence, if $|D \cap B_k| \asymp 3$, we need these 3 vertices to dominate all vertices of the columns $(P_5)_{j+1}, \dots, (P_5)_{j+4}$.

It is easy to see that

Case 1. $|D \cap (P_5)_j| \geq 1$ and $|D \cap (P_5)_{j+5}| \geq 1$,
and

Case 2. $|D \cap (P_5)_{j+5}| \geq 1$ and $|D \cap (P_5)_j| = 0$
are not possible.

Case 3. $|D \cap (P_5)_j| = |D \cap (P_5)_{j+5}| = 0$.
There is exactly one possibility to dominate the vertices of the columns $(P_5)_{j+1}, \dots, (P_5)_{j+4}$ by three vertices, namely $(3, j+2), (2, j+3), (4, j+3) \in D$. But then we have to dominate all vertices of $(P_5)_j$ by vertices of the block B_{k-1}. Hence $(2, j-1), (4, j-1) \in D$. To dominate the vertices of $(P_5)_{j-3}$, $(P_5)_{j-4}$, $(P_5)_{j-5}$ we need at least three additional vertices which are contained in B_{k-1}. Hence $|D \cap B_{k-1}| \geq 5$.

Also the two vertices of $(P_5)_{j+5}$ must be dominated by vertices of B_{k+1}. We first assume that $D \cap (P_5)_{j+6} = \{(3, j+6)\}$. Then all vertices of $(P_5)_{j+8}$, $(P_5)_{j+9}$, $(P_5)_{j+10}$ as well as $(1, j+6)$ and $(5, j+6)$ must be dominated by vertices of B_{k+1}. But then B_{k+1} contains four additional vertices and $|D \cap B_{k+1}| \geq 5$. If B_{k+1} is external also the vertices of $(P_5)_{j+11}$ are dominated by vertices of B_{k+1}. Therefore $|D \cap B_{k+1}| \geq 6$ in this case.

If $(3, j+6) \notin D$ then $(1, j+6), (5, j+6) \in D$. Both assertions about the cardinality of $D \cap B_{k+1}$ follow immediately since $(3, j+6)$ must be dominated by $(2, j+7)$ or $(4, j+7)$ in this case. If all three vertices of $(P_5)_{j+6}$ are contained in D our assertions obviously hold.

Case 4. $|D \cap (P_5)_j| \geq 1$ and $|D \cap (P_5)_{j+5}| = 0$.
To dominate the vertices of $(P_5)_{j+2}, \dots, (P_5)_{j+4}$ we need at least two vertices, namely $(2, j+3)$ and $(4, j+3)$. Hence, if $|D \cap B_k| = 3$, then D contains $(3, j)$, $(2, j+3)$ and $(4, j+3)$ in this case. The assertions about B_{k+1} can be shown as in Case 3.

Since the vertices $(1, j)$ and $(5, j)$ are dominated by vertices of $(P_5)_{j-1}$, the vertices $(2, j-1)$ and $(4, j-1)$ are both contained in D. To dominate the vertices of the columns $(P_5)_{j-3}$, $(P_5)_{j-4}$, $(P_5)_{j-5}$ at least three additional vertices of B_{k-1} must be contained in D. Therefore $|D \cap B_{k-1}| \geq 5$. □

Remark. For the component C_2 an analogous result holds with the roles of B_{k-1} and B_{k+1} interchanged.

Case 1. n is even.

Case 1.1. $n = 6m$.

We first assume that $n \geq 18$ and consider the component C_1.

Let D be any dominating set. $|D \cap B_k| \geq 3$ holds for each block B_k, $1 \leq k \leq \frac{n}{6}$, by Lemma 6. Assume that there are s 5×6 blocks which contain only three vertices of D. By Lemma 5 these blocks are internal. Then, by Lemma 7, there are at least $s+1$ 5×6 blocks which contain at least five vertices of D. Let B_{i_j}, $1 \leq j \leq 2s+1$, denote these blocks which either contain three or five vertices. Then $\mathcal{B} = \bigcup_{j=1}^{2s+1} B_{i_j}$ contains at least $8s+5$ vertices of D. By the above description of S, the set $\mathcal{B}$ contains at most $8s+5$ vertices of S. Hence $|D| \geq |S|$ holds for any dominating set D.

Let $n = 12$. $|D \cap B_k| \geq 4$ holds for each block B_k, $k = 1, 2$, by Lemma 5. If $|D \cap B_1| = 4$, at least one vertex of B_1 is dominated by vertices of B_2. Then it is obviously $|D \cap B_2| \geq 5$ and therefore $|D| \geq |S|$.

Case 1.2. $n = 6m + 2$.
We first assume that $n \geq 20$ and consider the component C_2 now.

Lemma 8. *There is no dominating set D such that $|D \cap E'| \leq 1$.*

Proof. To dominate the vertices of E' we clearly need at least two vertices which are contained in E' since the vertices of $(P_5)_n$ cannot be dominated by vertices not in E'. □

Let D be any dominating set. Again we assume that there are s blocks containing only three vertices of D. Since it may happen that $|B_m \cap D| = 3$ holds in this case, Lemma 7 now only implies that there are s blocks containing at least 5 vertices of D. But together with Lemma 8 this is again sufficient to show that $|D| \geq |S|$.

Let n=8. From $|D \cap B_1| \geq 4$ (Lemma 5) and from Lemma 8 we get $|D| \geq |S|$.

Let n=14. If $|D \cap B_1| = 4$, these 4 vertices cannot dominate any vertex of B_2. Vertices of E' can at most dominate the column $(P_5)_{12}$ of B_2. Then at least $(P_5)_7, \ldots, (P_5)_{11}$ and one vertex of $(P_5)_6$ are dominated by the vertices of B_2. This implies that $|D \cap B_2| \geq 4$. Together with Lemma 8 it follows that $|D| \geq |S|$.

Case 1.3. $n = 6m + 4$.

We again consider the component C_1.

Lemma 9. $|D \cap (B_m \cup E')| > 6$ *for any dominating set D.*

Proof. $|D \cap (B_m \cup E')| \leq 5$ cannot hold by the fact that for every D, we have $|D \cap E'| \geq 3$ and Lemma 6.

Assume that $|D \cap (B_m \cup E')| = 6$. Then E' and B_m both must contain exactly three vertices of D. As we have already seen in the proof of Lemma 10, $|D \cap (P_5)_{n-4}| = 0$ must hold if $|B_m \cap D| = 3$. Hence the three vertices of D in E' must dominate all vertices of E'. But this is only possible if $|(P_5)_{n-3} \cap D| = 0$. Hence the two vertices of $(P_5)_{n-4}$ must be dominated by vertices of $(P_5)_{n-5}$. But this immediately implies that B_m contains at least four vertices of D. □

LEMMA 10. *If* $|D \cap (E' \cup B_m)| = 7$ *then* $|D \cap (E' \cup B_m \cup B_{m-1})| \ge 12$.

Proof. By Lemma 9, $D \cap (E' \cup B_m)$ contains at least seven vertices. If $D \cap B_m$ now contains only three vertices of D, then B_{m-1} contains at least five vertices of D by Lemma 7.

Let $|B_m \cap D| = 4$. Then $|E' \cap D| = 3$. If all vertices of $(P_5)_{n-3}$ are dominated by vertices of B_m, then $|(P_5)_{n-4} \cap D| = 2$ and $|B_m \cap D| > 4$, a contradiction.

Let $|(P_5)_{n-4} \cap D| = 1$. Without loss of generality we can assume that $(2, n-4) \in D$. Then $(4, n-4)$ cannot be dominated by a vertex of E' since $|E' \cap D| = 3$ cannot hold if a vertex of $(P_5)_{n-3}$ is contained in D. Hence $|(P_5)_{n-5} \cap D| \ge 1$ must hold. But in this case we immediately get a contradiction to $|D \cap B_m| = 4$.

Hence $|(P_5)_{n-4} \cap D| = 0$. Then, since $|E' \cap D| = 3$, also $|(P_5)_{n-3} \cap D| = 0$. So all vertices of B_m, except those of the column $(P_5)_{n-9}$ must be dominated by vertices of B_m. Since $|B_m \cap D| = 4$, this implies that either $\{(3, n-9), (2, n-6), (4, n-6), (3, n-5)\} \subset D$ or $\{(3, n-7), (2, n-6), (4, n-6), (3, n-5)\} \subset D$. In both cases the vertices $(1, n-9)$ and $(5, n-9)$ must be dominated by vertices of $(P_5)_{n-10}$. Hence $(2, n-10) \in D$ and $(4, n-10) \in D$. But the vertices of the columns $(P_5)_{n-12}$, $(P_5)_{n-13}$ and $(P_5)_{n-14}$ are also dominated by vertices of B_{m-1} which immediately implies that $|D \cap B_{m-1}| \ge 5$. □

We now assume that there exist s blocks B_{j_i}, $1 \le s$, $j_i < m-1$, with $|B_{j_i} \cap D| = 3$. Of course $j_i > 1$ holds for all j_i, $1 \le i \le s$, by Lemma 5. Then by Lemma 7 there are also s blocks B_{k_i}, $k_i \notin \{m-1, m\}$, $1 \le i \le s$, with $|B_{k_i} \cap D| \ge 5$. This again implies that $|D| \ge |S|$ for every dominating set D.

Finally, let $|D \cap (B_m \cup E')| \ge 8$. Again we assume that there are s blocks B_{j_i}, $j_i \le m-1$, which contain only three vertices of D. As above Lemma 7 now immediately implies that $|D| \ge |S|$.

Let $n = 10$. By Lemma 5, $|D \cap B_1| \ge 4$ holds. If $|D \cap B_1| = 4$, the vertices of E' must dominate E' and at least one vertex of B_1. Then $|D \cap E'| \ge 4$, and $|D| \ge |S|$. If $|D \cap B_1| = 5$, the statement follows from Lemma 9.

Let $n = 16$. Same as in Lemma 9, $|D \cap (B_m \cup E')| > 6$. If $|D \cap B_2| = 3$, then as in Lemma 7 it follows $|D \cap B_1| \ge 5$, and hence $|D| \ge |S|$.

Let $|D \cap B_2| = 4$ and $|D \cap E'| = 3$. Three vertices in E' cannot dominate any vertex from $(P_5)_{12}$. As we have already seen, four vertices cannot dominate all vertices of 5×6 block. Some vertices of $(P_5)_7$ are dominated by vertices from B_1. By the same arguments as in Lemma 10 it follows $|D \cap B_1| \ge 5$, and $|D| \ge |S|$.

Case 2. n is odd.

Case 2.1. $n = 6m + 1$.

We first consider the component C_2.

LEMMA 11. *If* $|D \cap E'| = 0$, *then there exists at least* 1 *block* B *such that* $|D \cap B| \ge 6$, *or at least* 2 *blocks* B_i, B_j, *such that* $|D \cap B_i| = |D \cap B_j| = 5$.

P r o o f . E' contains two vertices. We first consider the following two characteristic possibilities to dominate them:

a) $(3, n-1) \in D$, $(1, n-1), (5, n-1) \notin D$
b) $(1, n-1), (5, n-1) \in D$, $(3, n-1) \notin D$.

Case a) $(3, n-1) \in D$, $(1, n-1), (5, n-1) \notin D$.
Since $(1, n-1)$ and $(5, n-1)$ are not in D, they must be dominated by the vertices $(2, n-2)$ and $(4, n-2)$. But then the vertices of the columns $(P_5)_{n-4}$, $(P_5)_{n-5}$ and $(P_5)_{n-6}$ are still not dominated. If all those vertices are dominated by vertices of B_m, then $|B_m \cap D| \geq 6$ holds. If B_m is external this is clearly satisfied.

If $|B_m \cap D| = 5$, then at least one vertex of the first column ($(P_5)_{n-6}$) of B_m is dominated by a vertex of B_{m-1}. Hence the last column of B_{m-1} contains at least one vertex of D. This immediately implies that B_{m-1} contains at least four vertices of D (cf. proof of Lemma 7). If B_{m-1} contains exactly four vertices of D, then again at least one vertex of the first column of B_{m-1} is dominated by a vertex of the adjacent block. Continuing this way we obtain that there must be a second 5×6 block besides B_m which contains at least five vertices of D. At least the external block B_1 must have this property.

Case b) $(1, n-1), (5, n-1) \in D$, $(3, n-1) \notin D$.
In this case we have to dominate the vertex $(3, n-1)$ by a vertex of the column $(P_5)_{n-2}$. Without loss of generality we assume that $(4, n-2) \in D$. Then the vertex $(1, n-3)$ and the vertices of the columns $(P_5)_{n-4}$, $(P_5)_{n-5}$, $(P_5)_{n-6}$ are still not dominated. To dominate these vertices we need at least three vertices. If these three vertices are all contained in B_m, then our first assertion holds. Hence $|B_m \cap D| \geq 6$ is always satisfied if B_m is external.

If B_m is internal then B_m may only contain five vertices of D. But in this case at least one vertex of the first column of B_m must be dominated by a vertex of B_{m-1}. As in the above case we can now conclude that there exist at least one more 5×6 block which contains at least five vertices of D.

All other possibilities (e.g. if all vertices of $(P_5)_{n-1}$ are contained in D) lead to the same results using quite similar arguments. □

Lemma 12. *If $|D \cap E'| = 1$, then exists at least 1 block B such that $|D \cap B| \geq 5$.*

P r o o f . E' consists of 2 vertices: $(2, n)$ and $(4, n)$. W.l.o.g. we only consider the case $(2, n) \in D$.

Let n=7. Then we have only one 5×6 block B_1. If $(2, 7) \in D$, then $(4, 7)$ is undominated. To dominate it we need at least one vertex from the $(P_5)_6$. Only the vertices $(3, 6)$ and $(5, 6)$ dominate vertex $(4, 7)$.

If $(3,6) \in D$ then the vertices of $(P_5)_5$ are dominated, but the vertex $(5,6)$ and the columns $(P_5)_1$, $(P_5)_2$, $(P_5)_3$, $(P_5)_4$ are undominated. To dominate these vertices we need at least four more vertices of B_1. So $|D \cap B_1| \geq 5$.

The same holds if $(5,6)$ is in D.

Let $n > 7$. Then we can dominate all or some vertices in the first column of B_m (column $(P_5)_{n-6}$) by vertices of the column $(P_5)_{n-7}$. Then we have $|D \cap B_m| \geq 4$, and in the column $(P_5)_{n-7}$ we have at least one dominating vertex. Using the same arguments as in the proof of Lemma 11, Case a), we obtain that there exists at least one (maybe B_1) block B such that $|D \cap B| \geq 5$. □

Let D now be any dominating set of C_2, and $n \geq 19$. We assume that there are s 5×6 blocks which contain only three vertices of D. By Lemma 7 we then have $s+1$ blocks containing at least five vertices of D. If E' contains no vertex of D, then Lemma 11 implies that there are two blocks with at least 5 vertices. At most one of these blocks coincides with one of the former $s+1$ blocks. Hence we have at least $s+2$ blocks with five vertices of D if $|E' \cap D| = 0$ and $|B_i \cap D| = 3$ for s 5×6 blocks B_i. Therefore $|D| \geq |S|$ in this case.

If E' contains one vertex of D, then analogously Lemma 12 implies that there are at least $s+1$ 5×6 blocks which contain at least five vertices of D if there are s blocks which contain only three vertices of D. Again $|D| \geq |S|$.

If $|E' \cap D| = 2$, then $|D| \geq |S|$ immediately follows from Lemma 7.

For $n = 7$, and $n = 13$, $|D| \geq |S|$ follows from Lemma 12.

In the sequel we consider the component C_1:
The following two results can be shown analogously to the above.

Lemma 13. *If $|D \cap E'| = 0$, then there either exist at least 2 blocks B_i, B_j such that $|D \cap B_i| \geq 5$ and $|D \cap B_j| \geq 5$, or there exists at least 1 block B such that $|D \cap B| \geq 6$ for $n \geq 13$.*

Lemma 14. *If $|D \cap E'| = 1$, then there exists at least 1 block B such that $|D \cap B| \geq 5$.*

Lemma 15. *If $|D \cap E'| = 2$, then $|D| \geq |S|$.*

Proof. Also in this case at least one vertex of E' must be dominated by a vertex of the last column of B_k. Therefore $|B_k \cap D| \geq 4$ and the result follows immediately. □

Finally we can again argue as above to show that $|D| \geq |S|$ if $|E' \cap D| = 0$ or $|E' \cap D| = 1$ for any dominating set D. If E' contains two vertices of D then our result holds by Lemma 15. If E' contains three vertices of D, then $|B_k \cap D| \geq 3$ still holds. Together with Lemma 7 this again implies that $|D| \geq |S|$.

For $n = 13$ the result holds by Lemma 13.

Case 2.2. $n = 6m + 3$.
We first consider the component C_2.

Lemma 16.

1) *There is no dominating set* D *such that* $|D \cap E'| \leq 1$.
2) *If* $|D \cap E'| = 2$*, then there exists at least* 1 *block* B*, such that* $|D \cap B| \geq 5$.

Proof.

1) At most the first column of E' can be dominated by vertices not in E'. Then 1 block of size 5×2 remains undominated. To dominate it we need at least 2 vertices of E'.

2) If E' contains only two vertices of D, then it does not matter which two vertices of E' are contained in D, at least one vertex of the column $(P_5)_{n-3}$ must be dominated by a vertex of the adjacent 5×6 block B_m. Then we have the same situation as in the proof of Lemma 11 above, and our result follows by using similar arguments.

If E' contains only two vertices of D, then we can combine Lemma 16 and Lemma 7 as above, to obtain that $|D| \geq |S|$ holds. If E' contains at least three vertices of a dominating set D, then $|D \cap E'| \geq |S \cap E'|$ and Lemma 7 again implies that $|D| \geq |S|$. □

We now consider the component C_1.
The next two results can be shown in the same way as the corresponding Lemmas for the component C_2.

Lemma 17.

1) *There is no dominating set* D *such that* $|D \cap E'| \leq 1$.
2) *If* $|D \cap E'| = 2$*, then there exists at least* 1 *block* B *such that* $|D \cap B| \geq 5$.

The final conclusions that $|D| \geq |S|$ can now be done as for C_2 above.

Let $n = 15$. We will consider the component C_1. For C_2 the proof is similar. By Lemma 5 $|D \cap B_1| \geq 4$. If $|D \cap B_1| = 4$, By Lemma 17 it follows that $|D \cap B_2| \geq 5$ and $|D \cap E'| \geq 2$. For such D, we have $|D| \geq |S|$.

Let $|D \cap B_1| \geq 5$. Then by Lemma 6 $|D \cap B_2| \geq 3$. If $|D \cap B_2| = 3$, then by the same arguments as in Lemma 7, it follows that $|(P_5)_{11} \cap D| = 0$ and then $|D \cap E'| = 3$. So in this case it also holds that $|D| \geq |S|$.

Case 2.3. $n = 6m + 5$.
We first consider the component C_2.

Lemma 18.

1) *There is no dominating set* D *such that* $|D \cap E'| \leq 2$.
2) *If* $|D \cap E'| = 3$*, then* $|D \cap B_m| \geq 5$.

Proof.

1) Only $(P_5)_{n-4}$ of E' can be dominated by vertices not in E'. To dominate the other four columns of E' we need at least 3 vertices.

2) If E' contains only three vertices of D, then $E' \cap D = \{(3, n-1), (2, n-2), (4, n-2)\}$ must hold. Hence both vertices of the column $(P_5)_{n-4}$ are dominated by vertices of the column $(P_5)_{n-5}$. As in the analogous lemmas for $n = 6m + 1$ or $n = 6m + 3$ our assertion now follows. □

Also the fact that $|D| \geq |S|$ in this case now follows as above for $n = 6m + 1$ or $n = 6m + 3$.

We now consider the component C_1. Again the two auxiliary results follow with the same arguments as in former cases.

LEMMA 19.

1) *There is no dominating set D such that $|D \cap E'| \leq 2$.*
2) *If $|D \cap E'| = 3$, then there exists at least one block B such that $|D \cap B| \geq 5$.*

Proof.

1) It is easy to check.

2) In this case we have two possibilities for the set $D \cap E'$, namely $\{(2, n-1), (4, n-1), (3, n-2)\}$ or $\{(2, n-1), (4, n-1), (3, n-4)\}$. But in both cases the vertices $(2, n-5)$ and $(4, n-5)$ must be contained in D, which immediately implies that $|B_m \cap D| \geq 5$. □

The final conclusions that $|D| \geq |S|$ are now again done as above if $n \geq 23$.

Let $n = 17$. We will consider the component C_2. For C_1 the proof is similar. By Lemma 18 $|D \cap E'| \geq 3$ holds. If $|D \cap E'| = 3$, then $|D \cap B_2| \geq 5$. Let $|D \cap B_2| = 5$. Then at least one vertex of the column $(P_5)_7$ is dominated by vertices of B_1. Then $|D \cap B_1| \geq 5$ and $|D| > |S|$.

Let $|D \cap E'| = 4$ and $|D \cap B_1| = 4$. By Lemma 6 $|D \cap B_2| \geq 3$ holds. If $|D \cap B_2| = 3$, then $(2, 9)$ and $(4, 9)$ must be in D. Hence the vertices of $(P_5)_7$ are dominated by vertices of $(P_5)_6$. This is a contradiction to $|D \cap B_1| = 4$. Hence $|D \cap B_2| \geq 4$ and $|D| > |S|$. □

REFERENCES

[1] DE JAENISCH, C. F.: *Applications de l'Analyse Mathematique an Jenudes Echecs*, Petrograd, 1862.

[2] EL-ZAHAR, M. PAREEK, C. M.: *Domination number of products of graphs*, Ars Combin. **31** (1991), 223 227.

[3] FAUDREE, R. J.—SCHELP, R. H.: *The domination number for the product of graphs*, Congr. Numer. **79** (1990), 29–33.

[4] FISHER, D. C.: *The domination number of complete grid graphs*, J. Graph Theory (To appear).

[5] GRAVIER, S.—KHELLADI, A.: *On the dominating number of cross product of graphs*, Discrete Math. **145** (1995), 273–277.

[6] JACOBSON, M. S.—KINCH, L. F.: *On the domination number of products of graphs I*, Ars Combin. **18** (1983), 33–44.

[7] JACOBSON, M. S.—KINCH, L. F.: *On the domination number of the products of graphs II: Trees*, J. Graph Theory **10** (1986), 97–106.

[8] JHA, P. K.—KLAVŽAR, S.: *Independence and matching in direct-product graphs*. Preprint 1995.

[9] JHA, P. K.—KLAVŽAR, S.—ZMAZEK, B.: *Isomorphic components of Kronecker product of bipartite graphs*. Preprint 1994.

[10] KLAVŽAR, S.—SEIFTER, N.: *Dominating Cartesian products of cycles*, Discrete Appl. Math. **59** (1995), 129–136.

[11] KLAVŽAR, S.—ZMAZEK, B.: *On a Vizing-like conjecture for direct product graphs*. Preprint 1995.

[12] VIZING, V. G.: *The cartesian product of graphs*, Vychisl. Sistemy **9** (1963), 30–43.

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