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ON ONE–POINT $\mathcal{I}$–COMPACTIFICATION AND LOCAL $\mathcal{I}$–COMPACTNESS[1)]

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ABSTRACT. An ideal $\mathcal{I}$ on a set X is a nonempty subset of the power set $\mathcal{P}(X)$ which has heredity and is finitely additive. (Local) $\mathcal{I}$-compactness is the natural generalization of (local) compactness, where an $\mathcal{I}$-cover of $A \subseteq X$ covers all but an ideal member of A. If τ is a topology on X, $\mathcal{I}$ is τ-codense if each member of $\mathcal{I}$ is codense in (X,τ) and $\mathcal{I}$ is τ-local if each subset $A \subseteq X$ locally in $\mathcal{I}$ belongs to $\mathcal{I}$. If $\mathcal{I}$ is τ-local, then $\beta = \{U - I \mid U \in \tau,\ I \in \mathcal{I}\}$ is a topology. In any case, β is a basis for a topology $\tau^*(\mathcal{I})$ finer than τ on X. It is seen that a Hausdorff space (X,τ) has a one-point Hausdorff $\mathcal{I}$-compactification if and only if each point of X has a τ-closed $\mathcal{I}$-compact neighbourhood. This condition which is equivalent to $(X,\tau^*(\mathcal{I}))$ being locally $\mathcal{I}$-compact, properly implies that (X,τ) is locally $\mathcal{I}$-compact. However, the converse is implied by the τ-codenseness of $\mathcal{I}$. Further, when $\mathcal{I}$ is τ-codense, (X,τ) having a one-point Hausdorff $\mathcal{I}$-compactification implies that (X,τ) is locally H-closed, i.e. locally $\mathcal{N}(\tau)$-compact, where $\mathcal{N}(\tau)$ is the ideal of nowhere dense subsets of (X,τ).

§1. Introduction

Given a nonempty set X, an *ideal* $\mathcal{I}$ is defined to be a nonempty collection of subsets of X such that

(1) $B \in \mathcal{I}$ and $A \subseteq B \to A \in \mathcal{I}$ (heredity), and
(2) $A \in \mathcal{I}$ and $B \in \mathcal{I} \to A \cup B \in \mathcal{I}$ (finitely additive).

If, in addition, $\mathcal{I}$ satisfies the following condition:

(3) $\{A_n\colon n = 1,2,3,\dots\} \subseteq \mathcal{I} \to \cup A_n \in \mathcal{I}$ (countably additive),

then $\mathcal{I}$ is said to be a *σ-ideal.* If $X \notin \mathcal{I}$, then $\mathcal{I}$ is called a *proper ideal* and $\{X - I\colon I \in \mathcal{I}\}$ is a filter. For any family $\mathcal{S}$ of subsets of X, there is a smallest ideal (σ-ideal) containing $\mathcal{S}$, denoted $\langle\mathcal{S}\rangle$ ($\langle\mathcal{S}\rangle_\sigma$), since intersections of ideals (σ-ideals) are ideals (σ-ideals). The smallest ideal containing $\mathcal{I} \cup \mathcal{J}$ for ideals

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$\mathcal{I}$ and $\mathcal{J}$ is the *join* of $\mathcal{I}$ and $\mathcal{J}$, denoted by $\mathcal{I} \vee \mathcal{J} = \{I \cup J\colon I \in \mathcal{I}$ and $J \in \mathcal{J}\}$.

By a *space* we mean a topological space with no separation properties assumed unless explicitly stated. We denote by $(X,\tau,\mathcal{I})$ a topological space (X,τ) together with an ideal $\mathcal{I}$ on X. Of particular importance are the ideals of nowhere dense subsets, denoted $\mathcal{N}(\tau)$, and of meager or first category subsets, denoted $\mathcal{M}(\tau)$, of a space. Clearly, $\mathcal{M}(\tau)$ is the smallest σ-ideal containing $\mathcal{N}(\tau)$.

If A, $B \subseteq X$, we say $A = B\,[\mathrm{Mod}\,\mathcal{I}]$ if the symmetric difference of A and B is in $\mathcal{I}$; i.e., if $A \triangle B = (A - B) \cup (B - A) \in \mathcal{I}$. For a space $(X,\tau,\mathcal{I})$ an $\mathcal{I}$-*cover* of $A \subseteq X$ is a family $\mathcal{U}$ of subsets of X such that $\mathcal{U}$ covers $B \subseteq A$ with $B = A\,[\mathrm{Mod}\,\mathcal{I}]$. A subset $A \subseteq X$ is said to be $\mathcal{I}$-*compact* ([1], [2], [3]) iff (if and only if) each τ-open cover of A has a finite $\mathcal{I}$-subcover. It is shown in [3] that $A \subseteq X$ is $\mathcal{I}$-compact iff the subspace $(A, \tau|A,\ \mathcal{I}|A)$ is $\mathcal{I}|A$-compact, where $\mathcal{I}|A = \{A \cap I\colon I \in \mathcal{I}\}$. $\mathcal{I}$-compact spaces have been studiea in [1], [2], and [3]. It was shown in [1] that a Hausdorff space (X,τ) is $\mathcal{N}(\tau)$-compact iff (X,τ) is H-closed.

Given a space (X,τ), $x \in X$, we denote by $\tau(x) = \{U \in \tau\colon x \in U\}$. A subset $A \subseteq X$ is called a *neighbourhood*, abbreviated nbd, of x if there exists $U \in \tau(x)$ such that $x \in U \subseteq A$. We will say that a space $(X,\tau,\mathcal{I})$ is (strongly) *locally* $\mathcal{I}$-*compact* if each point in X has an $\mathcal{I}$-compact (τ-closed) nbd. A direct proof is given in [4] of the surprising result that a Hausdorff space (X,τ) is locally $\mathcal{N}(\tau)$-compact iff (X,τ) is locally H-closed (in the sense of Porter [5]). In this paper we obtain this result indirectly, as well as several other new results, via the concept of a one-point $\mathcal{I}$-compactification. In addition, we find a sufficient condition on $\mathcal{I}$ for a Hausdorff locally $\mathcal{I}$-compact space to have a Hausdorff one-point $\mathcal{I}$-compactification and show by example that not every Hausdorff locally $\mathcal{I}$-compact space has a Hausdorff one-point $\mathcal{I}$-compactification.

Recall that a space (X,τ) is said to be *quasi* H-*closed*, abbreviated QHC, iff every open cover of X has a finite subcollection which covers a dense subset of X. A space is said to be H-*closed* iff it is Hausdorff and QHC. Porter [5] defines a Hausdorff space (X,τ) to be *locally* H-*closed* if each point in X has a nbd which is H-closed as a subspace of (X,τ). It was shown in [3] that a space (X,τ) is QHC iff (X,τ) is $\mathcal{N}(\tau)$-compact.

In the following, for $A \subseteq (X,\tau)$, we denote by $\mathrm{Cl}_\tau(A)$ and $\mathrm{Int}_\tau(A)$ the closure and interior of A, respectively, with respect to τ. We will simply write $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$ when no ambiguity is present.

Given a space $(X,\tau,\mathcal{I})$, we denote by $\tau^*(\mathcal{I})$ the topology generated by the basis $\beta(\mathcal{I},\tau) = \{U - I\colon U \in \tau,\ I \in \mathcal{I}\}$ [6]. We will simply write τ^* for $\tau^*(\mathcal{I})$ and β for $\beta(\mathcal{I},\tau)$ when no ambiguity is present. It is shown in [2] ([4]) that $(X,\tau,\mathcal{I})$ is $\mathcal{I}$-compact (locally $\mathcal{I}$-compact) iff $(X,\tau^*,\mathcal{I})$ is $\mathcal{I}$-compact (locally

$\mathcal{I}$-compact). It is also shown in [3] that τ^*-closed subsets of $\mathcal{I}$-compact spaces are $\mathcal{I}$-compact and $\mathcal{I}$-compact subsets of Hausdorff spaces are τ^*-closed.

Two important properties that an ideal may have in relation to the topology on a space are defined as follows. Given a space $(X,\tau,\mathcal{I})$ we say that $\mathcal{I}$ is *codense with respect to* τ, or τ*-codense*, iff $\mathcal{I}\cap\tau=\{\emptyset\}$. This property is called "τ-boundary" elsewhere in the literature [2]. We say that a subset A of X is *locally in* $\mathcal{I}$ *with respect to* τ [7], or τ*-locally in* $\mathcal{I}$, iff for each point $x\in A$ there exists $U\in\tau(x)$ such that $U\cap A\in\mathcal{I}$. The ideal $\mathcal{I}$ is called *local with respect to* τ, or τ*-local*, if $\mathcal{I}$ contains all subsets of X which are locally in $\mathcal{I}$; i.e., if A being locally in $\mathcal{I}$ implies $A\in\mathcal{I}$. Elsewhere in the literature, local ideals are called "compatible" ([8], [3], [10]), "adherent" [6], "supercompact" [11], and having "strong Banach's localization property" [12]. For all spaces (X,τ), it is known that $\mathcal{N}(\tau)$ [11] and $\mathcal{M}(\tau)$ ([13], Banach Category Theorem) are τ-local ideals, and $\mathcal{N}(\tau)$ is τ-codense. Also it is well known that $\mathcal{M}(\tau)$ is τ-codense iff (X,τ) is a Baire space. It is noted in [9] that in a hereditarily Lindelöf space, every σ-ideal is local. We conclude this section by noting that in a space $(X,\tau,\mathcal{I})$, $\mathcal{I}$ is τ-local iff $\mathcal{I}$ is τ^*-local [8], and $\mathcal{I}$ is τ-codense iff $\mathcal{I}$ is τ^*-codense. If (X,τ) is Hausdorff, then certainly (X,τ^*) is Hausdorff since τ^* is finer than τ; the converse is true provided $\mathcal{I}$ is τ-codense [14].

§2. One-point $\mathcal{I}$-compactification

We begin with the following definition.

DEFINITION. *A space $(Y,\sigma,\mathcal{J})$ is said to be a $\mathcal{J}$-compactification of $(X,\tau,\mathcal{I})$ iff*

(1) $X\subseteq Y$,
(2) $\tau=\sigma|X=\{V\cap X:\ V\in\sigma\}$,
(3) $\mathcal{J}|X=\{J\cap X:\ J\in\mathcal{J}\}=\mathcal{I}$, *and*
(4) $(Y,\sigma,\mathcal{J})$ *is* $\mathcal{J}$*-compact.*

If, in addition, we have

(5) $Cl_\sigma(X)=Y$,

then $(Y,\sigma,\mathcal{J})$ is said to be a $\mathcal{J}$-compact extension of $(X,\tau,\mathcal{I})$. Furthermore, if $Y-X=\{r\}$, then $(Y,\sigma,\mathcal{J})$ is said to be a one-point $\mathcal{J}$-compactification (or $\mathcal{J}$-compact extension) of $(X,\tau,\mathcal{I})$.

Note that if $(Y,\sigma,\mathcal{J})$ is a $\mathcal{J}$-compact extension of $(X,\tau,\mathcal{I})$, then $\mathcal{J}\cap\sigma=\{\emptyset\}$ iff $\mathcal{I}\cap\tau=\{\emptyset\}$. Also, since $\mathcal{J}|X=\mathcal{I}$, $\mathcal{J}=\langle\mathcal{J}|(Y-X)\cup\mathcal{I}\rangle$ and since $\mathcal{I}\subseteq\mathcal{J}$, (Y,σ) is $\mathcal{J}$-compact if (Y,σ) is $\mathcal{I}$-compact. Furthermore, the converse is true if the remainder $Y-X$ is finite. Thus in the following discussion of one-point compactifications and extensions, we consider only $\mathcal{I}$-compactness.

THEOREM 2.1. *If (Y, σ) is a Hausdorff one-point $\mathcal{I}$-compactification of $(X, \tau, \mathcal{I})$, then we have the following:*

(1) $\tau \subseteq \sigma$,

(2) *$(X, \tau, \mathcal{I})$ is Hausdorff and strongly locally $\mathcal{I}$-compact, and*

(3) *if $Y - X = \{r\} \in \sigma$, then $(X, \tau, \mathcal{I})$ is $\mathcal{I}$-compact.*

Furthermore, the converse of (3) holds if $\mathcal{I}$ is τ-codense.

Proof.

(1) Since points are closed in (Y, σ), $X \in \sigma$ and hence $\sigma|X = \tau \subseteq \sigma$.

(2) Clearly (X, τ) is Hausdorff. If $x \in X$ and $Y - X = \{r\}$, then $x \neq r$ and there are disjoint σ-open sets U and V with $x \in U$, $r \in V$. Then $U \subseteq \mathrm{Cl}_\sigma(U) = \mathrm{Cl}_\tau(U) \subseteq Y - V \subseteq X$, so that $(X, \tau, \mathcal{I})$ is strongly locally $\mathcal{I}$-compact since closed subsets of $\mathcal{I}$-compact spaces are $\mathcal{I}$-compact.

(3) If $Y - X = \{r\} \in \sigma$, then X is $\mathcal{I}$-compact since it is a closed subset of an $\mathcal{I}$-compact space (Y, σ). Thus, $(X, \sigma|X) = (X, \tau)$ is $\mathcal{I}$-compact.

For the converse, suppose that $(X, \tau, \mathcal{I})$ is $\mathcal{I}$-compact. Then, since (Y, σ) is Hausdorff, X is σ^*-closed. Thus $\{r\} = Y - X$ implies that for some $U \in \sigma$ and $I \in \mathcal{I}$, $r \in U - I$ and $U - I \subseteq Y - X$, but $U = (U - I) \cup (U \cap I)$ and $U \cap I = U \cap X$ since $I \subseteq X$ and $U - I \subseteq Y - X$. Since $U \cap X \in \tau \cap \mathcal{I}$, $\mathcal{I} \cap \tau = \{\emptyset\}$ implies that $U \cap I = \emptyset$ and $U = U - I$. Thus, $Y - X \in \sigma$. □

From Theorem 2.1, we see that only Hausdorff strongly locally $\mathcal{I}$-compact spaces $(X, \tau, \mathcal{I})$ need be considered for one-point Hausdorff $\mathcal{I}$-compactifications. Also, if (Y, σ) is a one-point Hausdorff $\mathcal{I}$-compactification of $(X, \tau, \mathcal{I})$ and if $(X, \tau, \mathcal{I})$ is not $\mathcal{I}$-compact, then (Y, σ) is a one-point $\mathcal{I}$-compact extension of $(X, \tau, \mathcal{I})$.

THEOREM 2.2. *If (Y, σ) is a one-point Hausdorff $\mathcal{I}$-compactification of $(X, \tau, \mathcal{I})$ and $\mathcal{I}$ is τ-codense, then (X, τ) is locally* H*-closed. If, in addition, X is dense in Y, then (X, τ) is not* H*-closed and hence not $\mathcal{I}$-compact.*

Proof. Since $\mathcal{I} \cap \sigma = \mathcal{I} \cap \tau = \{\emptyset\}$, and since (Y, σ) is $\mathcal{I}$-compact, (Y, σ) is H-closed. It follows from Theorem 2.1, (a), of [5], that (X, τ) is locally H-closed. If X is not closed in the Hausdorff space (Y, σ), then $(X, \tau) = (X, \sigma|X)$ is not H-closed and hence not $\mathcal{I}$-compact. □

A natural question is whether every Hausdorff (strongly) locally $\mathcal{I}$-compact space $(X, \tau, \mathcal{I})$ has a one-point Hausdorff $\mathcal{I}$-compactification. We answer this question in the affirmative by considering the following one-point $\mathcal{I}$-compactification for a space $(X, \tau, \mathcal{I})$. In what follows, if $(X, \tau, \mathcal{I})$ is a space, let $X^\Lambda = X \cup \{r\}$, where $r \notin X$, and let $\tau^\Lambda = \tau \cup \big\{\{r\} \cup V \colon V \in \tau$ and $X - V$ is $\mathcal{I}$-compact$\big\}$.

THEOREM 2.3. *For any space $(X,\tau,\mathcal{I})$, τ^Λ is a topology on X^Λ and (X^Λ,τ^Λ) is a one-point $\mathcal{I}$-compactification of $(X,\tau,\mathcal{I})$.*

Proof. Clearly $\{W\cap X \mid W\in\tau^\Lambda\}=\tau$ so that if τ^Λ is a topology, $\tau^\Lambda|X=\tau$. Since finite unions of $\mathcal{I}$-compact sets are $\mathcal{I}$-compact and τ is closed under finite intersection, τ^Λ is closed under finite intersection. Now, if $\emptyset\neq\{V_\alpha \mid \alpha\in A\}\subseteq\tau$ with each $X-V_\alpha$ $\mathcal{I}$-compact, then $\bigcup\limits_\alpha(\{r\}\cup V_\alpha)=\{r\}\cup\Big(\bigcup\limits_\alpha V_\alpha\Big)\in\tau^\Lambda$ since $\bigcup\limits_\alpha V_\alpha\in\tau$ and $X-\Big(\bigcup\limits_\alpha V_\alpha\Big)=\bigcap\limits_\alpha(X-V_\alpha)$ is $\mathcal{I}$-compact being a closed subset of an $\mathcal{I}$-compact set. Similarly, $U\cup\big(\{r\}\cup V\big)\in\tau^\Lambda$ if $U,V\in\tau$ and $X-V$ is $\mathcal{I}$-compact. Therefore, τ^Λ is closed under arbitrary union and is a topology. To see that (X^Λ,τ^Λ) is $\mathcal{I}$-compact, let $\mathcal{W}$ be a τ^Λ-open cover of X^Λ. If $r\in W_0\in\mathcal{W}$, $W_0=\{r\}\cup V$ for some V with $V\in\tau$ and $X-V$ $\mathcal{I}$-compact. Since $\tau^\Lambda|X=\tau$, $\{W\cap X \mid W\in\mathcal{W}$ and $W\neq W_0\}$ is a τ-open cover of $X-V$. Hence there is a finite subset $\{W_1,W_2,\dots,W_n\}\subseteq\mathcal{W}$ such that $\{W_1\cap X,\dots,W_n\cap X\}$ is a finite $\mathcal{I}$-cover of $X-V$. Thus, $\{W_0,W_1,\dots,W_n\}$ is a finite $\mathcal{I}$-subcover of $\mathcal{W}$ for X^Λ. □

We note that (X^Λ,τ^Λ) is an $\mathcal{I}$-compact extension of $(X,\tau,\mathcal{I})$ if and only if $(X,\tau,\mathcal{I})$ is not $\mathcal{I}$-compact. In any case, (X^Λ,τ^Λ) is T_1 (i.e. points are closed) iff (X,τ) is T_1 since finite and hence singleton subsets of X are always $\mathcal{I}$-compact for any ideal $\mathcal{I}$. The smallest T_1 topology possible for any one-point compactification of a T_1 space (X,τ) is locally cofinite at the remainder point r. The next example illustrates that this can and does happen with $\mathcal{I}=\{\emptyset\}$ for (X^Λ,τ^Λ) precisely when (X,τ) is T_1 and anticompact [15] in the sense that the only compact subsets of (X,τ) are finite.

Example. Let (X,τ) be a Hausdorff dense-in-itself space which is anticompact. For example, (X,τ) could be $\big(\mathbb{R},u^*(\mathcal{N}(u))\big)$, where u is the usual topology on the set $\mathbb{R}$ of real numbers [9]. Then for $\mathcal{I}=\{\emptyset\}$ or $\mathcal{I}=\{F\subseteq X \mid F \text{ is finite}\}$, $(X,\tau,\mathcal{I})$ is not locally $\mathcal{I}$-compact and hence (X^Λ,τ^Λ) is not Hausdorff. In fact, τ^Λ is locally cofinite at r and is thus as far from being Hausdorff at r as possible. (i.e. $U\cap V\neq\emptyset$ for every open U containing r and every open nonempty V.)

COROLLARY 2.3. *The space $(X,\tau,\mathcal{I})$ has a Hausdorff one-point $\mathcal{I}$-compactification if and only if $(X,\tau,\mathcal{I})$ is a strongly locally $\mathcal{I}$-compact Hausdorff space.*

Proof. The necessity is part (2) of Theorem 2.1. For the sufficiency it is enough to show that (X^Λ,τ^Λ) is Hausdorff. Since (X,τ) is Hausdorff, it remains only to see that each $x\in X$ can be separated from $r\in X^\Lambda-X$ by disjoint

τ^{Λ}-open sets. Let K be a τ-closed $\mathcal{I}$-compact neighbourhood of $x \in X$. Then $x \in \operatorname{Int}_{\tau} K \in \tau^{\Lambda}$ since $\tau \subseteq \tau^{\Lambda}$, and $r \in X^{\Lambda} - K \in \tau^{\Lambda}$. □

Since $\mathcal{I}$-compact subsets of a Hausdorff space are τ^*-closed, it follows that (X, τ^*) is Hausdorff and locally $\mathcal{I}$-compact if and only if it is Hausdorff and strongly locally $\mathcal{I}$-compact. Hence, if $\mathcal{I}$ is τ-codense, (X, τ) is Hausdorff and locally $\mathcal{I}$-compact if and only if (X, τ^*) is Hausdorff and strongly locally $\mathcal{I}$-compact. Thus, we have the following corollaries also.

COROLLARY 2.4. *The space $(X, \tau^*(\mathcal{I}), \mathcal{I})$ is Hausdorff and locally $\mathcal{I}$-compact iff $(X^{\Lambda}, \tau^*(\mathcal{I})^{\Lambda})$ is Hausdorff.*

COROLLARY 2.5. *If $(X, \tau, \mathcal{I})$ is Hausdorff and locally $\mathcal{I}$-compact, then $(X^{\Lambda}, \tau^{*\Lambda})$ is Hausdorff. The converse holds if $\mathcal{I}$ is τ-codense.*

THEOREM 2.6. *If $\mathcal{I}$ is τ-codense, then $(X^{\Lambda}, \tau^{\Lambda})$ is Hausdorff iff $(X, \tau, \mathcal{I})$ is Hausdorff and locally $\mathcal{I}$-compact.*

Proof. It is enough to show that when $\mathcal{I}$ is τ-codense and (X, τ) is Hausdorff and locally $\mathcal{I}$-compact, then (X, τ) is strongly locally $\mathcal{I}$-compact. To this end, let $x \in U \in \tau$ with $U \subseteq K$ and K an $\mathcal{I}$-compact subset of X. Since (X, τ) is Hausdorff, K is τ^*-closed so that $\operatorname{Cl}_{\tau^*} U \subseteq K$. But $\mathcal{I}$ is τ-codense, implies $\operatorname{Cl}_{\tau^*} U = \operatorname{Cl}_{\tau} U$ for $U \in \tau$ so that (X, τ) is strongly locally $\mathcal{I}$-compact. □

The following example shows $\mathcal{I}$ being τ-codense cannot be dropped for Theorem 2.6. In particular, locally $\mathcal{I}$-compact spaces exist which are not strongly locally $\mathcal{I}$-compact.

Example. Let $X = \mathbb{R}$, the set of real numbers, let $\mathbb{N}$ be the set of positive integers, and let $S = \bigcup_{n \in N} (n, n+1)$. Let $\mathcal{I} = \mathcal{P}(S)$ be the ideal of all subsets of S. Let τ be the topology on X having for its neighbourhood base at each $x \neq 0$, the usual one but for which the open neighbourhoods of $x = 0$ are of the form $\{0\} \cup \bigcup_{n \geq k} (n, n+1)$, where $k \in \mathbb{N}$. Then $(X, \tau, \mathcal{I})$ is Hausdorff and each $x \neq 0$ has a compact and hence $\mathcal{I}$-compact neighbourhood. Also, each neighbourhood of $x = 0$ is the union of the singleton set $\{0\}$ with a member of $\mathcal{I}$ and is therefore $\mathcal{I}$-compact. So $(X, \tau, \mathcal{I})$ is locally $\mathcal{I}$-compact. We claim that $(X^{\Lambda}, \tau^{\Lambda})$ is not Hausdorff and that in particular r and $x = 0$ cannot be separated with disjoint τ^{Λ}-open sets. For if $r \in U \in \tau^{\Lambda}$ and $0 \in V \in \tau^{\Lambda}$ with $U \cap V = \emptyset$, then $U = \{r\} \cup W$ with $W \in \tau$ and $X - W$ $\mathcal{I}$-compact and $V \in \tau$ so that $\operatorname{Cl}_{\tau}(V) \subseteq X - W$ and $\operatorname{Cl}_{\tau}(V)$ is $\mathcal{I}$-compact. We may assume that V is a basic open neighbourhood and that $V = \{0\} \cup \bigcup_{n \geq k} (n, n+1)$ for some $k \in \mathbb{N}$. Then $\mathrm{Cl}_\tau(V) = \{0\} \cup [n, +\infty)$ which is not $\mathcal{I}$-compact. For if $\mathcal{U} = \{V\} \cup \{(n - .25, n + .25) \mid n \geq k \text{ and } n \in \mathbb{N}\}$, $\mathcal{U}$ is a τ-open cover of $\mathrm{Cl}_\tau(V)$ having no finite $\mathcal{I}$-subcover of $\mathrm{Cl}_\tau(V)$. Apparently (X, τ) is not strongly locally $\mathcal{I}$-compact.

One can let $\mathcal{I} = \mathcal{P}(X)$ for any Hausdorff space (X, τ) and note that $(X^\Lambda, \tau^\Lambda)$ is Hausdorff so that the condition $\mathcal{I} \cap \tau = \{\emptyset\}$ is not necessary for Theorem 2.6. Also for any locally compact Hausdorff space (X, τ) and for any ideal $\mathcal{I}$ of subsets of X, the topology τ^Λ on X^Λ is finer than the topology for the standard Alexandroff one-point compactification of (X, τ) [16] and hence $(X^\Lambda, \tau^\Lambda)$ is Hausdorff.

Since many important ideals are codense, we summarize the results of this section in this case.

THEOREM 2.7. *The following are equivalent for any space $(X, \tau, \mathcal{I})$ when $\mathcal{I}$ is τ-codense.*

(1) *(X, τ) is Hausdorff and locally $\mathcal{I}$-compact.*
(2) *(X, τ) is Hausdorff and strongly locally $\mathcal{I}$-compact.*
(3) *(X, τ^*) is Hausdorff and (strongly) locally $\mathcal{I}$-compact.*
(4) *(X, τ) has a Hausdorff one-point $\mathcal{I}$-compactification.*
(5) *(X, τ^*) has a Hausdorff one-point $\mathcal{I}$-compactification.*
(6) *$(X^\Lambda, \tau^\Lambda)$ is Hausdorff.*
(7) *$(X^\Lambda, \tau^{*\Lambda})$ is Hausdorff.*

We conclude this section with a question.

Question 1. For any space $(X, \tau, \mathcal{I})$, does $\tau^{*\Lambda} = \tau^{\Lambda *}$ if $\mathcal{I}$ is τ-codense?

We know the answer is yes if $\tau^* = \beta$ and this condition is satisfied by τ-local ideals such as $\mathcal{N}(\tau)$, $\mathcal{M}(\tau)$, and principal ideals ($\mathcal{P}(A)$ for any subset A), as well as some non-local ideals. An affirmative answer in general would give an alternate proof for the converse of Corollary 2.5.

§3. Applications and locally H-closed spaces

From Theorem 2.7 we have the following observation.

COROLLARY 3.2. *Whenever $(X, \tau, \mathcal{I})$ is a Hausdorff space with $\mathcal{I}$ codense with respect to τ, local $\mathcal{I}$-compactness of (X, τ) implies that (X, τ) is locally H-closed.*

It is known [3] that (X, τ) is QHC if and only if (X, τ) is $\mathcal{N}(\tau)$-compact and hence for a Hausdorff space (X, τ), $\mathcal{N}(\tau)$-compactness is equivalent to (X, τ) being H-closed. The following parallel result was obtained in [4] in another way.

THEOREM 3.3. *A Hausdorff space (X,τ) is locally* H*-closed if and only if it is locally $\mathcal{N}(\tau)$-compact.*

Proof. As noted earlier, $\mathcal{N}(\tau)$ is a codense ideal, so that the sufficiency follows from Theorem 3.1. For the necessity, note that an H-closed subspace K is always $\mathcal{N}(\tau|K)$-compact, and hence $\mathcal{N}(\tau)$-compact, since $\mathcal{N}(\tau|K) \subseteq \mathcal{N}(\tau)$. □

It is clear from the proof above that the necessity part of Theorem 3.3 holds even without the Hausdorff assumption. Note that non-Hausdorff spaces exist which are (locally Hausdorff and even) locally H-closed. Thus, the class of locally $\mathcal{N}(\tau)$-compact spaces properly contains the class of Hausdorff locally H-closed spaces. (In fact, the partition topology τ on a set X shows that even a space which is not locally Hausdorff can be locally $\mathcal{N}(\tau)$-compact.)

Question 2. For a space (X,τ), is local $\mathcal{N}(\tau)$-compactness equivalent to (X,τ) being locally QHC?

THEOREM 3.4. *If (X,τ) is a Hausdorff Baire space, then the following are equivalent.*

(1) (X,τ) *is locally $\mathcal{M}(\tau)$-compact.*
(2) (X,τ) *is locally $\mathcal{N}(\tau)$-compact.*
(3) (X,τ) *is locally* H*-closed.*

If also (X,τ) is regular, each of the above is equivalent to the following.

(4) (X,τ) *is locally compact.*

Proof. It was noted earlier that (X,τ) is a Baire space if and only if $\mathcal{M}(\tau) \cap \tau = \{\emptyset\}$. Thus, by Corollary 3.2, (1) implies (3). By Theorem 3.3, (2) and (3) are equivalent, and since $\mathcal{N}(\tau) \subseteq \mathcal{M}(\tau)$, (2) implies (1). Now if also (X,τ) is regular, each H-closed subspace is compact so that (3) and (4) are equivalent. □

Question 3. In Theorem 3.4 above, does the equivalence of (1) and (2) hold true without the Hausdorff assumption? It is known that for any Baire space (X,τ), $\mathcal{M}(\tau)$-compactness is equivalent to $\mathcal{N}(\tau)$-compactness.

§4. Ideal expansions

In [10], for any space $(X,\tau,\mathcal{I})$, an expansion of $\mathcal{I}$ by an ideal $\mathcal{J}$ is defined by $\mathcal{I} * \mathcal{J} = \{A \subseteq X \mid A^*(\mathcal{I}) \in \mathcal{J}\}$ where $A^*(\mathcal{I}) = \{x \in X \mid x \in U \in \tau \to U \cap A \notin \mathcal{I}\}$ is the set of all points in X where A is not locally in $\mathcal{I}$. In particular, $A^*(\mathcal{I}) = \emptyset \in \mathcal{J}$ if $A \in \mathcal{I}$ so that $\mathcal{I} \subseteq \mathcal{I} * \mathcal{J}$ for any $\mathcal{J}$. Further, it is not difficult to show that $\mathcal{I} * \mathcal{J}$ is an ideal. When $\mathcal{J} = \mathcal{N}(\tau)$, $\mathcal{I} * \mathcal{J}$ is denoted

$\tilde{\mathcal{I}}$, and it may be noted that for any $\mathcal{I}$, $\mathcal{N}(\tau) \subseteq \tilde{\mathcal{I}}$ so that $\mathcal{I} \vee \mathcal{N}(\tau) \subseteq \tilde{\mathcal{I}}$. Most importantly, for any $\mathcal{I}$, $\tilde{\mathcal{I}}$ is τ-local. It is also observed in [10] that when $\mathcal{I}$ is τ-local, then $\tilde{\mathcal{I}} = \mathcal{I} \vee \mathcal{N}(\tau)$. We observe the following.

THEOREM 4.1. *For any space* $(X,\tau,\mathcal{I})$, $\tilde{\tilde{\mathcal{I}}} = \tilde{\mathcal{I}}$.

Proof. Since $\tilde{\mathcal{I}} \sim \tau$, $\tilde{\tilde{\mathcal{I}}} = \tilde{\mathcal{I}} \vee \mathcal{N}(\tau) = \tilde{\mathcal{I}}$ since $\mathcal{N}(\tau) \subseteq \tilde{\mathcal{I}}$. □

Observe that for any space (X,τ), $\{\mathcal{J} \mid \mathcal{J}$ is an ideal with $\mathcal{N}(\tau) \subseteq \mathcal{J}$ and $\mathcal{J} \sim \tau\} = \{\tilde{\mathcal{I}} \mid \mathcal{I}$ is an ideal of subsets of $X\}$. For if $\mathcal{J}$ is an ideal with $\mathcal{N}(\tau) \subseteq \mathcal{J}$ and $\mathcal{J} \sim \tau$, then $\tilde{\mathcal{J}} = \mathcal{J}$.

THEOREM 4.2. *For any spaces* $(X,\tau,\mathcal{I})$ *and* $(X,\tau,\mathcal{J})$, $\mathcal{I} * \mathcal{J}$ *is codense if both* $\mathcal{I}$ *and* $\mathcal{J}$ *are codense. If either* $\mathcal{N}(\tau) \subseteq \mathcal{J}$ *or* (X,τ) *is regular, the converse is true.*

Proof. For the converse assume that $\mathcal{I} * \mathcal{J}$ is codense and note that since $\mathcal{I} \subseteq \mathcal{I} * \mathcal{J}$, $\mathcal{I}$ is codense. Also, if $U \in \mathcal{J} \cap \tau$, then $U^*(\mathcal{I}) = \mathrm{Cl}(U) = (\mathrm{Cl}(U) - U) \cup U \in \mathcal{J}$ if $\mathcal{N}(\tau) \subseteq \mathcal{J}$ and so $U \in (\mathcal{I} * \mathcal{J}) \cap \tau = \{\emptyset\}$. Hence, $U = \emptyset$ and $\mathcal{J}$ is codense. If (X,τ) is regular, and $U \neq \emptyset$, there exists $V \in \tau$ and $\emptyset \neq \mathrm{Cl}(V) \subseteq U$. So $\mathrm{Cl}(V) = V^*(\mathcal{I}) \in \mathcal{J}$ and hence $V \in (\mathcal{I} * \mathcal{J}) \cap \tau$. This contradiction shows that $\mathcal{J}$ is codense.

Now if $\mathcal{I}$ and $\mathcal{J}$ are codense and $U \in (\mathcal{I} * \mathcal{J}) \cap \tau$, then $U^*(\mathcal{I}) = \mathrm{Cl}(U) \in \mathcal{J}$ implies that $U \in \mathcal{J} \cap \tau$ so that $U = \emptyset$. □

Part of Theorem 3.5 of [10] follows as a corollary. That is for any space $(X,\tau,\mathcal{I})$, $\mathcal{I}$ is codense if and only if $\tilde{\mathcal{I}}$ is codense.

In any resolvable space X with D and $E = X - D$ disjoint dense subsets, the ideals $\mathcal{I} = \mathcal{P}(D)$ and $\mathcal{J} = \mathcal{P}(E)$ of all subsets of D and E respectively are codense and yet $\mathcal{I} \vee \mathcal{J} = \mathcal{P}(X)$ is not codense.

THEOREM 4.3. *For any space* $(X,\tau,\mathcal{I})$, $\mathcal{I}$ *is codense if and only if* $\mathcal{I} \vee \mathcal{N}(\tau)$ *is codense.*

Proof. For the necessity let $\mathcal{I}$ be codense. Then $\mathcal{I} \vee \mathcal{N}(\tau) \subseteq \tilde{\mathcal{I}}$ and $\tilde{\mathcal{I}}$ is codense implies that $\mathcal{I} \vee \mathcal{N}(\tau)$ is codense.

The sufficiency is clear since $\mathcal{I} \subseteq \mathcal{I} \vee \mathcal{N}(\tau)$. □

THEOREM 4.4. *If* $(X,\tau,\mathcal{I})$ *is Hausdorff and* $\mathcal{I}$ *is codense, the following are equivalent.*

(1) (X,τ) *is locally* H*-closed.*
(2) (X,τ) *is locally* $\mathcal{N}(\tau)$*-compact.*
(3) (X,τ) *is locally* $\tilde{\mathcal{I}}$*-compact.*
(4) (X,τ) *is locally* $(\mathcal{I} \vee \mathcal{N}(\tau))$*-compact.*

Proof. Since $\mathcal{N}(\tau) \subseteq \mathcal{I} \vee \mathcal{N}(\tau) \subseteq \tilde{\mathcal{I}}$, the equivalence of (2) and (3) by Theorems 3.2 and 3.3 implies the equivalence of (2) and (4). □

Note that in Theorem 4.4, $\mathcal{I}$ may be non-local, in which case $\mathcal{I} \vee \mathcal{N}(\tau)$ is non-local. Also, the codenseness of $\mathcal{I} \vee \mathcal{N}(\tau)$ is not needed in the proof.

COROLLARY 4.5. *For any Hausdorff space $(X, \tau, \mathcal{I})$ with $\mathcal{I}$ codense and $\mathcal{N}(\tau) \subseteq \mathcal{I}$, the following are equivalent.*

(1) *(X, τ) is locally $\mathcal{I}$-compact.*
(2) *(X, τ) is locally $\mathcal{N}(\tau)$-compact.*
(3) *(X, τ) is locally* H*-closed.*

Question 4. Are (1) and (2) equivalent in Corollary 4.5 above without the Hausdorff assumption?

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