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ON CARATHÉODORY VECTOR LATTICES

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ABSTRACT. To each generalized Boolean algebra B there corresponds a vector lattice V; this correspondence goes back to Gofman. In general, B cannot be uniquely reconstructed from V. In this paper we investigate pairs of generalized Boolean algebras B and B' which generate the same vector lattice V. Further, we deal with the relations between the internal direct product decompositions of V and B.

1. Introduction

Gofman [4] investigated the elementary Carathéodory functions corresponding to a Boolean algebra B; the author [5] applied this notion for dealing with cardinal properties of lattice ordered groups.

The elementary Carathéodory functions corresponding to a Boolean algebra B are defined in [4] to be forms

$$a_1 b_1 + \cdots + a_n b_n ,$$

where a_i are reals and b_i are elements of B $(i = 1, 2, \dots, n)$ with appropriately defined operations and relations.

In the same way we can define the elementary Carathéodory functions corresponding to a generalized Boolean algebra B (for the sake of completeness the definition is recalled in Section 2 below). These were applied by the author [7] for studying sequential convergences on generalized Boolean algebras. We denote by $C(B)$ the vector lattice of all elementary Carathéodory functions corresponding to B. The relations between higher degrees of distributivity of B and of $C(B)$ were considered by the author [10].

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Let $\mathcal{C}$ be the class of all vector lattices V such that there exists a generalized Boolean algebra B with $V \simeq C(B)$. The elements of $\mathcal{C}$ will be called Carathéodory vector lattices.

In Section 2 we show that a Carathéodory vector lattice V can be characterized by the following condition:

(α) There exists a generalized Boolean algebra B such that

(i) B is a sublattice of the underlying lattice $\ell(V)$ of V;

(ii) the least element of B coincides with the neutral element 0 of V;

(iii) each nonzero element x of V can be represented as

$$x = a_1 b_1 + \cdots + a_n b_n,$$

where a_i are nonzero reals and b_i are nonzero elements of B.

If the above conditions are satisfied, then we say that (V, B) is a correct pair. There can exist a generalized Boolean algebra $B' \neq B$ such that (V, B') is a correct pair as well. The description of all such B' is given in Section 3.

If the condition (iii) is modified in such a way that all a_i are assumed to be integers, then we obtain the notion of the Specker lattice ordered group corresponding to B; let us denote it by $S(B)$. Specker lattice ordered groups were investigated by Conrad and Darnel [2]; cf. also Conrad and Martinez [3], and the author [8].

In Section 4 we prove that each direct product decomposition of a Carathéodory vector lattice has only a finite number of nonzero direct factors. The same result holds for Specker lattice ordered groups.

The relations between internal direct product decompositions of B, $C(B)$ and $S(B)$ are dealt with in Section 5.

2. The class $\mathcal{C}_1$

For the notation and the terminology concerning lattices, lattice ordered groups and vector lattices, cf. Birkhoff [1], and Luxemburg and Zaanen [11].

We start by recalling the definition of elementary Carathéodory functions corresponding to a generalized Boolean algebra B (cf. [1], [3], [4]).

We denote by $C(B)$ the system consisting of all forms

$$f = a_1 b_1 + \cdots + a_n b_n$$

(where a_i are nonzero reals, $b_i \in B$, $b_i > 0$, $b_{i(1)} \wedge b_{i(2)} = 0$ for any $i(1), i(2) \in \{1, 2, \ldots, n\}$, $i(1) \neq i(2)$, and of the "empty form". If g is another such form,

$$g = a'_1 b'_1 + \cdots + a'_m b'_m,$$

then f and g are considered as equal if $\bigvee_{i=1}^{n} b_i = \bigvee_{j=1}^{m} b'_j$ and if $a_i = a'_j$ whenever $b_i \wedge b'_j > 0$.

For $b, b' \in B$ let $b -_1 b'$ be the relative complement of $b \wedge b'$ in the interval $[0, b]$. If f and g are as above, then we put

$$f + g = \sum_{i=1}^{n} \sum_{j=1}^{m} (a_i + a'_j)(b_i \wedge b'_j) + \sum_{i=1}^{n} a_i \Big(b_i -_1 \bigvee_{j=1}^{m} b'_j \Big) + \sum_{j=1}^{m} a'_j \Big(b'_j -_1 \bigvee_{i=1}^{n} b_i \Big)$$

where in the summations only those terms are taken into account in which $a_i + a'_j \neq 0$ and the elements $b_i \wedge b'_j$, $b_i -_1 \bigvee_{j=1}^{m} b'_j$, $b'_j -_1 \bigvee_{i=1}^{n} b_i$ are nonzero. The empty form is considered to be the neutral element of $C(B)$ (with respect to the operation $+$) and it is identified with the element 0 of B. If b is the neutral element of $C(B)$ and $a \in \mathbb{R}$, then we put $ab = b$. If $0 \in \mathbb{R}$ and $b \in B$, we set $0b = 0 \in C(B)$. Each element $b \in B$ is identified with $1b \in C(B)$; hence $B \subseteq C(B)$. If f is as above and $a \in \mathbb{R}$, then we put $af = (aa_1)b_1 + \cdots + (aa_n)b_n$. Under this definition, $C(B)$ is a vector lattice; its elements are called elementary Carathéodory functions corresponding to B.

Let us remark that we have the same symbol for the zero element of $\mathbb{R}$, the least element of B and the neutral element of $C(B)$; the meaning of this symbol will be always clear from the context.

Now let us denote by $\mathcal{C}_1$ the class of all vector lattices V satisfying the condition (α) from Section 1. Further, let $\mathcal{C}$ be as in Section 1. The aim of the present section is to verify that $\mathcal{C}_1 = \mathcal{C}$.

For $V \in \mathcal{C}_1$ we apply the above formulated remark concerning the different meanings of the symbol 0.

An indexed system $\{x_i\}_{i \in I}$ of elements of a vector lattice V is called *orthogonal* (or *disjoint*) if $x_i \geqq 0$ for each $i \in I$ and $x_{i(1)} \wedge x_{i(2)} = 0$ whenever $i(1)$ and $i(2)$ are distinct elements of I.

LEMMA 2.1. *Assume that $V \in \mathcal{C}_1$ and let $0 \neq x \in V$. Then there are $n \in \mathbb{N}$, $0 \neq a_i \in \mathbb{R}$, $0 \neq b_i \in B$ $(i = 1, 2, \dots, n)$ such that*

$$x = a_1 b_1 + \cdots + a_n b_n \tag{1}$$

and the system $\{b_i\}_{i=1,2,\dots,n}$ is orthogonal.

P r o o f. In view of the assumption, the element x can be expressed in the form

$$x = a'_1 b'_1 + \cdots + a'_m b'_m$$

where a'_j are nonzero reals and b'_j are nonzero elements of B for $j = 1, 2, \dots, m$.

We proceed by induction on m. For $m = 1$, our assertion is valid. Suppose that $m > 1$ and that the assertion is valid for $m - 1$. Put

$$y = a'_1 b'_1 + \cdots + a'_{m-1} b'_{m-1} .$$

If $y = 0$, then the assertion under consideration holds. Suppose that $y \neq 0$. Thus y can be expressed as

$$y_1 = a''_1 b''_1 + \cdots + a''_t b''_t ,$$

where $0 \neq a''_k \in \mathbb{R}$, $0 \neq b''_k \in B$ for $k = 1, 2, \ldots, t$ and the system $(b''_k)_{1 \leqq k \leqq t}$ is orthogonal. Put

$$b = b''_1 \vee \cdots \vee b''_t , \qquad b \wedge b_n = b_{n1}$$

and let b_{n2} be the complement of b_{n1} in the interval $[0, b_n]$ of B. This complement exists, since b, b_n and b_{n1} belong to B. We have

$$b \wedge b_{n2} = b \wedge (b_{n2} \wedge b_n) = (b \wedge b_n) \wedge b_{n2} = b_{n1} \wedge b_{n2} = 0 .$$

Thus $b''_1 \wedge b_{n2} = 0, \ldots, b''_t \wedge b_{n2} = 0$ and hence the system $\{b''_1, \ldots, b''_t, b_{n2}\}$ is orthogonal. Further,

$$b_{n1} = b_{n1} \wedge b = b_{n1} \wedge (b''_1 \vee \cdots \vee b''_t) = (b_{n1} \wedge b''_1) \vee \cdots \vee (b_{n1} \wedge b''_t) .$$

For $k \in \{1, 2, \ldots, t\}$ put $b^*_k = b_{n1} \wedge b''_k$ and let b''_{k1} be the complement of b^*_k in the interval $[0, b''_k]$ of B. Hence

$$b''_k = b^*_k \vee b^*_{k1} = b^*_k + b^*_{k1} ,$$

$$y = \sum_{k=1}^{t} a''_k b^*_k + \sum_{k=1}^{t} a''_k b^*_{k1} ,$$

$$a_n b_n = a_n (b_{n1} + b_{n2}) = a_n b_{n1} + a_n b_{n2} = \sum_{k=1}^{t} a_n b^*_k + a_n b_{n2} ,$$

$$x = y + a_n b_n = \sum_{k=1}^{t} (a''_k + a_n) b^*_k + \sum_{k=1}^{t} a''_k b^*_{k1} + a_n b_{n2} . \tag{2}$$

The system $(b^*_1, \ldots, b^*_t, b^*_{11}, \ldots, b^*_{t1}, b_{n2})$ is orthogonal. Now it suffices to omit all members on the right side of (2) with $a''_k + a_n = 0$, $b^*_k = 0$, $b^*_{k1} = 0$ or $b_{n2} = 0$. We obtain the desired expression for x. □

For any vector lattice V and any element x of V we denote, as usual, $x^+ = x \vee 0$, $-x^- = x \wedge 0$. The following assertion can be verified by a simple calculation; we omit the proof.

LEMMA 2.2. *Let $I = \{1,2,\dots,n\}$ and let $(x_i)_{i\in I}$ be an orthogonal system of elements of a vector lattice V. For each $i \in I$ let $y_i \in V$ such that either $y_i = x_i$ or $y_i = -x_i$. Put $I_1 = \{i \in I:\ y_i = x_i\}$, $I_2 = \{i \in I:\ y_i = -x_i\}$, $y = y_1 + \dots + y_n$. Then*

$$y^+ = \sum_{i\in I_1} y_i\,, \qquad -y^- = \sum_{i\in I_2} y_i\,, \qquad |y| = \sum_{i\in I} x_i\,.$$

In the remaining part of this section we assume that V is a vector lattice belonging to $\mathcal{C}_1$.

Under the notation as above, we say that (1) is a regular representation of the element x. Let y be another nonzero element of V having a regular representation

$$y = a_1' b_1' + \dots + a_m' b_m'\,.$$

Put $I = \{1,2,\dots,n\}$, $J = \{1,2,\dots,m\}$.

LEMMA 2.3. *Assume that $x = y$. Then*

(i) $\bigvee_{i\in I} b_i = \bigvee_{j\in J} b_j'$;

(ii) *if $b_i \wedge b_j' > 0$ for some $i \in I$, $j \in J$, then $a_i = a_j'$.*

P r o o f. In view of the assumption we have $|x| = |y|$. Hence according to 2.2 we get

$$|a_1 b_1| + \dots + |a_n b_n| = |a_1' b_1'| + \dots + |a_m' b_m'|\,.$$

Since $|a_i b_i| = |a_i| b_i$, $|a_j' b_j'| = |a_j'| b_j'$, we obtain

$$|a_1| b_1 + \dots + |a_n| b_n = |a_1'| b_1' + \dots + |a_m'| b_m'\,. \tag{3}$$

(i) Put $\bigvee_{j=1}^{m} b_j' = b'$. Let $i \in I$. Denote $b' \wedge b_i = c_1$ and let c_2 be the complement of c_1 in the interval $[0, b_i]$ of B. Then $0 = c_1 \wedge c_2 = b' \wedge c_2$, whence $b_j' \wedge c_2 = 0$ for each $j \in J$. This yields $|a_i| c_2 \wedge |a_j'| b_j' = 0$ for each $j \in J$. Hence $|a_i| c_2 \wedge |y| = 0$. On the other hand, $0 \leqq |a_i| c_2 \leqq |a_i| b_i$ and thus $|a_i| c_2 \leqq |x|$. Since $|y| = |x|$, we get $c_2 = 0$, whence $b_i \leqq b'$. Thus $\bigvee_{i\in I} b_i \leqq \bigvee_{j\in J} b_j'$. Similarly we obtain the dual relation.

(ii) We denote

$$I_1 = \{i \in I:\ a_i > 0\}\,, \qquad I_2 = \{i \in I:\ a_i < 0\}\,,$$
$$J_1 = \{j \in J:\ a_j' > 0\}\,, \qquad J_2 = \{j \in J:\ a_j' < 0\}\,.$$

In view of 2.2 we have

$$x^+ = \sum_{i\in I_1} a_i b_i\,, \quad -x^- = \sum_{i\in I_2} a_i b_i\,, \qquad y^+ = \sum_{j\in J_1} a_j' b_j'\,, \quad -y^- = \sum_{j\in J_2} a_j' b_j'\,.$$

Since $x = y$, we get $x^+ = y^+$, $x^- = y^-$. It is well known that $x^+ \wedge (x^-) = 0$. This yields that whenever $i \in I_1$ and $j \in J_2$, then $b_i \wedge b'_j = 0$. Similarly, if $i \in I_2$ and $j \in J_1$, then $b_i \wedge b'_j = 0$.

Let $i(1) \in I_1$ and $j(1) \in J_1$; assume that $b_{i(1)} \wedge b'_{j(1)} > 0$. Denote $c_1 = b_{i(1)} \wedge b'_{j(1)}$. There exists $c_2 \in B$ such that c_2 is the complement of c_1 in the interval $[0, b'_{j(1)}]$ of B. We get

$$a_{i(1)}c_1 \leqq a_{i(1)}b_{i(1)} \leqq x^+ = y^+ ,$$

$$a_{i(1)}c_1 = a_{i(1)}c_1 \wedge y^+ = a_{i(1)}c_1 \wedge \Big(\sum_{j \in J_1} a'_j b'_j \Big)$$

$$= a_{i(1)}c_1 \wedge \Big(\bigvee_{j \in J_1} a'_j b'_j \Big) = \bigvee_{j \in J_1} \big(a_{i(1)}c_1 \wedge a'_j b'_j \big) .$$

Since $c_1 \leqq b'_{j(1)}$, we obtain $c_1 \wedge b'_j = 0$ whenever $j \in J_1$, $j \neq j(1)$; for such j we have also

$$a_{i(1)}c_1 \wedge a'_j b'_j = 0 .$$

Hence $a_{i(1)}c_1 = a_{i(1)}c_1 \wedge a'_{j(1)}b'_{j(1)}$. From the definition of c_2 we get

$$b'_{j(1)} = c_1 \vee c_2 , \qquad c_1 \wedge c_2 = 0 ,$$

therefore

$$a_{i(1)}c_1 = a_{i(1)}c_1 \wedge \big(a'_{j(1)}c_1 \vee a'_{j(1)}c_2 \big) = a_{i(1)}c_1 \wedge a'_{j(1)}c_1 ,$$

since $a_{i(1)}c_1 \wedge a'_{j(1)}c_2 = 0$. Thus $a_{i(1)}c_1 \leqq a'_{j(1)}c_1$ and so $a_{i(1)} \leqq a'_{j(1)}$. Analogously we obtain the dual relation, whence $a_{i(1)} = a'_{j(1)}$.

By the same method we can deal with the situation when $i(1) \in I_2$, $j(1) \in J_2$, $b_{i(1)} \wedge b'_{j(1)} > 0$. □

LEMMA 2.4. *Let $x, y \in V$ be expressed as above. Assume that the conditions* (i) *and* (ii) *from* 2.3 *are satisfied. Then $x = y$.*

P r o o f . Let $i \in I$. In view of the condition (i) we have

$$b_i = b_i \wedge \Big(\bigvee_{j \in J} b'_j \Big) = \bigvee_{j \in J} (b_i \wedge b'_j) = \sum_{j \in J} (b_i \wedge b'_j) .$$

Analogously, for each $j \in J$

$$b'_j = b'_j \wedge \Big(\bigvee_{i \in I} b_i \Big) = \bigvee_{i \in I} (b'_j \wedge b_i) = \sum_{i \in I} (b_i \wedge b'_j) .$$

Hence we obtain

$$x = \sum_{i \in I} \sum_{j \in J} a_i (b_i \wedge b'_j) , \qquad y = \sum_{j \in J} \sum_{i \in I} a'_j (b'_j \wedge b_i) .$$

Thus in view of the condition (ii) we get $x = y$. □

LEMMA 2.5. *Let x and y be as above. For $i \in I$ let c_i be the complement of the element $c_{i1} = b_i \wedge \Big(\bigvee_{j\in J} b'_j \Big)$ in the interval $[0, b_i]$ of B. Similarly, for $j \in J$ let c'_j be the complement of the element $c'_{j1} = b'_j \wedge \Big(\bigvee_{i\in I} b_i \Big)$ in the interval $[0, b'_j]$ of B. Then*

$$x + y = \sum_{i\in I} \sum_{j\in J} (a_i + a'_j)(b_i \wedge b'_j) + \sum_{i\in I} a_i c_i + \sum_{j\in J} a'_j c'_j . \tag{4}$$

Proof. For each $i \in I$ we have $b_i = c_{i1} \vee c_i = c_{i1} + c_i$. Further

$$c_{i1} = \bigvee_{j\in J} (b_i \wedge b'_j) = \sum_{j\in J} (b_i \wedge b'_j) ,$$

$$a_i b_i = a_i (c_{i1} + c_i) = \sum_{j\in J} a_i (b_i \wedge b'_j) + a_i c_i ,$$

$$x = \sum_{i\in I} \sum_{j\in J} a_i (b_i \wedge b'_j) + \sum_{i\in I} a_i c_i .$$

Similarly we obtain

$$y = \sum_{j\in J} \sum_{i\in I} a'_j (b'_j \wedge b_i) + \sum_{j\in J} a'_j c'_j .$$

Therefore the formula (4) is valid. □

LEMMA 2.6. *Let x be as in (1). Then $x > 0$ if and only if $a_i > 0$ for $i = 1, 2, \dots, n$.*

Proof. If $a_i > 0$ for $i = 1, 2, \dots, n$, then clearly $x > 0$. The converse assertion is a consequence of 2.2. □

PROPOSITION 2.7. *We have $\mathcal{C} = \mathcal{C}_1$.*

Proof. Assume that $V_1 \in \mathcal{C}$. Then from the definition of $\mathcal{C}$ we immediately obtain that the conditions (i), (ii) and (iii) from Section 1 are satisfied.

Conversely, assume that V belongs to $\mathcal{C}_1$. From 2.1, 2.3–2.6 and from the definition of elementary Carathéodory functions corresponding to the generalized Boolean algebra B we obtain that V belongs to $\mathcal{C}$. □

3. Correct pairs

In this section we assume that V is a vector lattice belonging to the class $\mathcal{C}_1$ and that B is a generalized Boolean algebra such that (V,B) is a correct pair.

Our aim is to characterize all generalized Boolean algebras B' such that (V,B') is a correct pair as well.

The case $B=\{0\}$ being trivial, we will suppose that B is not a one-element set. Hence also $V\neq\{0\}$.

Let T be a nonempty set of indices and for each $t\in T$ let X_t be an ideal of B such that the following conditions are satisfied:

(a) Whenever $t(1),t(2)\in T$, $t(1)\neq t(2)$, then $X_{t(1)}\cap X_{t(2)}=\{0\}$.

(b) If $0<b\in B$, then there are distinct elements $t_1,\dots,t_n\in T$ and nonzero elements $b'_1,\dots,b'_n$ with $b'_i\in X_{t_i}$ $(i=1,2,\dots,n)$ such that $b=b'_1\vee\dots\vee b'_n$.

Lemma 3.1. *If $0<b\in B$, then the representation of b in the form described in the condition* (b) *is uniquely determined.*

Proof. Assume that T_1 and T_2 are nonempty finite subsets of T such that

$$b=\bigvee_{t\in T_1} b_t\,,\qquad 0<b_t\in X_t\quad\text{for each } t\in T_1\,,$$

$$b=\bigvee_{s\in T_2} b'_s\,,\qquad 0<b'_s\in X_s\quad\text{for each } s\in T_2\,.$$

Let $t(1)\in T_1$. Then

$$b_{t(1)}=b_{t(1)}\wedge b=\bigvee_{s\in T_2}\left(b_{t(1)}\wedge b'_s\right).$$

Hence there exists $s\in T_2$ with $b_{t(1)}\wedge b'_s>0$. Then we must have $s=t(1)$, $s'\neq t(1)$ for $s'\in T_2$, $s'\neq s$. Thus $b_{t(1)}\wedge b'_{s'}=0$ and

$$b_{t(1)}=b_{t(1)}\wedge b'_s\,,$$

whence $b_{t(1)}\leqq b'_s$. Further, we proved that $T_1\subseteq T_2$. By analogous argument we obtain $T_2\subseteq T_1$ and $b'_s\leqq b_{t(1)}$. □

For each $t\in T$ let c_t be a positive real such that $c_{t(1)}\neq c_{t(2)}$ if $t(1)$ and $t(2)$ are distinct elements of T. Put

$$Y_t=\{c_t x_t:\ x_t\in X_t\}\,.$$

Hence we have

$$Y_{t_1}\cap Y_{t_2}=\{0\}\qquad\text{whenever}\quad t_1,t_2\in T\,,\ t_1\neq t_2\,.$$

We denote by B' the set of all elements $x \in V$ such that either $x = 0$ or x can be expressed in the form

$$x = y_1 \vee \cdots \vee y_n\,, \tag{$*$}$$

where $y_1 \in Y_{t(1)}, \ldots, y_n \in Y_{t(n)}$, and $t(1), \ldots, t(n)$ are elements of T.

Lemma 3.2. *Let $t \in T$. Then Y_t is a sublattice of $\ell(V)$ isomorphic to X_t.*

Proof. Let $y_1, y_2 \in Y_t$. There exist $x_1, x_2 \in X_t$ with $y_i = c_t x_i$ $(i = 1, 2)$. Then $y_1 \vee y_2 = c_t(x_1 \vee x_2) \in Y_t$ and $y_1 \wedge y_2 = c_t(x_1 \wedge x_2) \in Y_t$. The mapping $x \mapsto c_t x$ is an isomorphism of X_t onto Y_t. □

In view of 3.2, Y_t is a generalized Boolean algebra.

Lemma 3.3. *B' is a sublattice of $\ell(V)$.*

Proof. Let $x, x' \in B'$. Hence there are $t(1), \ldots, t(n), s(1), \ldots, s(m) \in T$, and elements $y_1 \in Y_{t(1)}, \ldots, y_n \in Y_{t(n)}, y'_1 \in Y_{s(1)}, \ldots, y'_m \in Y_{s(m)}$ such that

$$x = y_1 \vee \cdots \vee y_n\,, \qquad x' = y'_1 \vee \cdots \vee y'_m\,.$$

Then $x \vee x' = y_1 \vee \cdots \vee y_n \vee y'_1 \vee \cdots \vee y'_m$ belongs to B'. Further,

$$x \wedge x' = \bigvee_{i \in I,\, j \in J} (y_i \wedge y'_j)\,,$$

where $I = \{1, 2, \ldots, n\}$, $J = \{1, 2, \ldots, m\}$. If $t(i) \neq s(j)$, then $y_i \wedge y'_j = 0$. If $t(i) = s(j)$, then in view of 3.2 we have $y_i \wedge y'_j \in Y_{t(i)}$. Therefore $x \wedge x'$ belongs to B'. □

Lemma 3.4. *Let $t \in T$. Then Y_t is an ideal of B'.*

Proof. The relation $Y_t \subseteq B'$ is obvious. Further, in view of 3.2 and 3.3, Y_t is a sublattice of B'. Let $y \in Y_t$ and $z \in B'$, $z \leqq y$. Hence there are $t(1), \ldots, t(n) \in T$ and $z_1 \in Y_{t(1)}, \ldots, z_n \in Y_{t(n)}$ with $z = z_1 \vee \cdots \vee z_n$. We obtain

$$z = z \wedge y = (z_1 \wedge y) \vee \cdots \vee (z_n \wedge y)\,.$$

If $i \in \{1, 2, \ldots, n\}$ and $t(i) \neq t$, then $z_i \wedge y = 0$; if $t(i) = t$, then in view of 3.2 we have $z_i \wedge y \in Y_t$. Therefore $z \in Y_t$. □

Lemma 3.5. *The lattice B' is a generalized Boolean algebra.*

Proof. In view of 3.3 and of the relation $0 \in B'$ it suffices to verify that whenever $0 < x \in B'$ and $x_1 \in B'$, $x_1 \leqq x$, then x_1 has a complement in the interval $[0, x]$ of B'.

Let x and x_1 satisfy the mentioned assumptions. Let x be as in $(*)$. Hence

$$x_1 = x_1 \wedge x = (x_1 \wedge y_1) \vee \cdots \vee (x_1 \wedge y_n).$$

According to 3.4 we have $x_1 \wedge y_1 \in Y_{t(1)}, \ldots, x_1 \wedge y_n \in Y_{t(n)}$.

Without loss of generality we can suppose that the elements $t(1), \ldots, t(n)$ are mutually distinct. Let $i \in \{1, 2, \ldots, n\} = I$. Since $Y_{t(i)}$ is a generalized Boolean algebra, there exists $z_i \in Y_{t(i)}$ such that z_i is the complement of $x_1 \wedge y_i$ in the interval $[0, y_i]$ of the lattice $Y_{t(i)}$. Put

$$x_1' = z_1 \vee \cdots \vee z_n.$$

An easy calculation shows that x_1' is a complement of x_1 in the interval $[0, x]$ of the lattice B'. □

If $v_1, \ldots, v_n \in V$, $a_1, \ldots, a_n \in \mathbb{R}$, then we say, as usual, that $a_1 v_1 + \cdots + a_n v_n$ is a linear combination of elements $v_1, \ldots, v_n$.

Let $v \in V$, $v \neq 0$. Since the pair (V, B) is correct, the element v can be expressed as a linear combination of some elements $b_1, \ldots, b_m$ of B. Let $j \in \{1, 2, \ldots, m\}$. In view of the above conditions (a) and (b), there are $t(1), \ldots, t(n) \in T$ and $b_1' \in X_{t(1)}, \ldots, b_n' \in X_{t(n)}$ such that $b_j = b'_{t(1)} \vee \cdots \vee b'_{t(n)}$. Since all X_t are ideals in B, we can assume without loss of generality that the elements $t(1), \ldots, t(n)$ are mutually distinct. Then the system $\{b'_{t(1)}, \ldots, b'_{t(n)}\}$ is orthogonal, whence $b_j = b'_{t(1)} + \cdots + b'_{t(n)}$. In view of the definition of Y_t for $t \in T$ we get

$$c^{-1}_{t(1)} b'_{t(1)} \in Y_{t(1)}, \ \ldots, \ c^{-1}_{t(n)} b'_{t(n)} \in Y_{t(n)},$$

whence $c^{-1}_{t(1)} b'_{t(1)}, \ldots, c^{-1}_{t(n)} b'_{t(n)}$ are elements of B'.

Summarizing, we conclude that each element of V is a linear combination of some elements of B'. Hence (by applying 3.3 and 3.5) we have:

PROPOSITION 3.6. *(V, B') is a correct pair.*

The fact that the generalized Boolean algebra B' was obtained by the above described construction from the indexed system of ideals $(X_t)_{t \in T}$ of B and the indexed system $(c_t)_{t \in T}$ of reals will be expressed by writing

$$B' = f\big((X_t, c_t)_{t \in T}\big).$$

LEMMA 3.7. *Let $V \in \mathcal{C}_1$ and let (V, B) be a correct pair. Assume that $0 < b \in B$ and that x, y are elements of V such that $x \wedge y = 0$, $x \vee y = b$. Then x and y belong to B.*

P r o o f . The cases when $x = 0$ or $y = 0$ are trivial; suppose that $x > 0$ and $y > 0$. There exist $0 < a_i \in \mathbb{R}$, $0 < a_j^0 \in \mathbb{R}$, $0 < b_i \in B$, $0 < b_j^0 \in B$ ($i \in \{1,2,\dots,n\} = I$, $j \in \{1,2,\dots,m\} = J$) such that the systems $(b_i)_{i\in I}$ and $(b_j^0)_{j\in J}$ are orthogonal and

$$x = a_1 b_1 + \dots + a_n b_n\,, \qquad y = a_1^0 b_1^0 + \dots + a_m^0 b_m^0\,.$$

We have $a_i b_i \leqq x$ and $a_j^0 b_j^0 \leqq y$, whence $a_i b_i \wedge a_j^0 b_j^0 = 0$. This yields that $a_i \wedge b_j^0 = 0$ for each $i \in I$ and $j \in J$. Thus the system $\{b_1, \dots, b_n, b_1^0, \dots, b_m^0\}$ is orthogonal. We have

$$b = x \vee y = x + y = a_1 b_1 + \dots + a_n b_n + a_1^0 b_1^0 + \dots + a_n^0 b_m^0\,.$$

In view of 2.3 we obtain the relations

$$b = b_1 \vee \dots \vee b_n \vee b_1^0 \vee \dots \vee b_m^0\,,$$
$$1 = a_1 = \dots = a_n = a_1^0 = \dots = a_m'\,.$$

Thus we get

$$x = b_1 + \dots + b_n = b_1 \vee \dots \vee b_n\,,$$
$$y = b_1^0 + \dots + b_m^0 = b_1^0 \vee \dots \vee b_m^0\,.$$

Therefore x and y belong to B. □

Now let us assume that $\{0\} \neq V \in \mathcal{C}_1$ and that B, B' are generalized Boolean algebras such that (V,B) and (V,B') are correct pairs.

Let $0 < b' \in B'$. Since $b' \in V$, in view of 2.1 there exist $0 < a_i \in \mathbb{R}$ and $0 < b_i \in B$ (for $i \in \{1,2,\dots,n\} = I$) such that the system $(b_i)_{i\in I}$ is orthogonal and

$$b' = \sum_{i\in I} a_i b_i\,.$$

We can also assume that if $i(1)$ and $i(2)$ are distinct elements of I, then $a_{i(1)} \neq a_{i(2)}$. Namely, if, e.g., $a_1 = a_2$, then $a_1 b_1 + a_2 b_2$ can be replaced by $a_1 b_1 + a_1 b_2 = a_1 (b_1 \vee b_2)$ and $b_1 \vee b_2 \in B$.

Further, for each $i \in I$ there exist a finite set $J_i \neq \emptyset$ and elements $0 < c_{ij} \in \mathbb{R}$, $0 < b'_{ij} \in B'$ ($j \in J_i$) such that

$$b_i = \sum_{j\in J_i} c_{ij} b'_{ij}\,.$$

Moreover, in view of 2.1 we can assume that all the systems $(b_i)_{i\in I}$, $(b'_{ij})_{j\in J_i}$ ($i \in I$) are orthogonal. Without loss of generality we can suppose that

$J_{i(1)} \cap J_{i(2)} = \emptyset$ whenever $i(1)$ and $i(2)$ are distinct elements of I. Put $J = \bigcup_{i \in I} J_i$. Then the system $(b'_j)_{j \in J}$ is orthogonal as well. We obtain

$$b' = \sum_{j \in J_1} a_1 c_{1j} b'_{1j} + \dots + \sum_{j \in J_n} a_n c_{nj} b'_{nj} . \tag{+}$$

According to Lemma 2.3 (applied for B') we conclude that $a_1 c_{1j} = 1$ for each $j \in J_1, \dots, a_n c_{nj} = 1$ for each $j \in J_n$. Hence there are $0 < c_i \in \mathbb{R}$ $(i \in I)$ such that $c_1 = c_{1j}$ for each $j \in J_1, \dots, c_n = c_{nj}$ for each $j \in J_n$. Thus

$$b_1 = c_1 \sum_{j \in J_1} b'_{1j}, \ \dots, \ b_n = c_n \sum_{j \in J_n} b'_{nj} .$$

We denote

$$\sum_{j \in J_1} b'_{1j} = b''_1, \ \dots, \ \sum_{j = J_n} b_{nj} = b''_n .$$

Since, in view of the orthogonality,

$$\sum_{j \in J_1} b'_{1j} = \bigvee_{j \in J_1 .} b'_{1j}, \ \dots, \ \sum_{j \in J_n} b'_{nj} = \bigvee_{j \in J_n} b'_{nj},$$

we get that all elements $b''_1, \dots, b''_n$ belong to B'. We have

$$b_1 = c_1 b''_1, \ \dots, \ b_n = c_n b''_n . \tag{1}$$

For $0 < r \in \mathbb{R}$ we denote by B_r the set of all elements $b \in B$ such that $rb \in B'$. In view of (1), there exist $0 < r \in \mathbb{R}$ with $B_r \neq \emptyset$; let R_0 be the set of all such reals r.

LEMMA 3.8. *Let $r \in R_0$, $b \in B_r$ and $b_1^0 \in B$, $0 < b_1^0 < b$. Then $b_1^0 \in B_r$.*

Proof. There exists $b_1^1 \in B$ such that $b_1^0 \wedge b_1^1 = 0$, $b_1^0 \vee b_1^1 = b$. Then we have

$$rb_1^0 \wedge rb_1^1 = 0, \qquad rb_1^0 \vee rb_1^1 = rb \in B' .$$

According to 3.7 (applied for B') we get $rb_1^0 \in B'$, whence $b_1^0 \in B_r$. □

LEMMA 3.9. *Let $r \in R_0$, $b \in B_r$, $b^0 \in B_r$. Then $b \vee b^0 \in B_r$.*

Proof. Since $rb \in B'$ and $rb^0 \in B'$ we get $r(b \vee b') = rb \vee rb^0 \in B'$. Thus $b \vee b^0 \in B_r$. □

Put $B_r^0 = B_r \cup \{0\}$. In view of 3.8 and 3.9 we get:

LEMMA 3.10. *Let $r \in R_0$. Then B_r^0 is an ideal of B.*

From the definition of B_r we immediately obtain:

Lemma 3.11. *Let $r_1, r_2 \in R_0$, $r_1 \neq r_2$. Then $B^0_{r_1} \cap B^0_{r_2} = \{0\}$.*

For each $r \in R_0$ we denote

$$Z_r = \{rb_r : \ b_r \in B^0_r\}.$$

If r_1, r_2 are distinct elements of R_0, and $b_{r_1} \in B_{r_1}$, $b_{r_2} \in B_{r_2}$, then in view of 3.10 and 3.11 we have $b_{r_1} \wedge b_{r_2} = 0$, whence $r_1 b_{r_1} \wedge r_2 b_{r_2} = 0$. Therefore $Z_{r_1} \cap Z_{r_2} = \{0\}$.

Let $b' \in B'$ be as above. Since the elements $a_1, \dots, a_n$ are distinct, the elements $c_1, \dots, c_n$ are distinct as well and so are the elements $r_1 = c_1^{-1}$, ..., $r_n = c_n^{-1}$. In view of (1), $\{r_1, \dots, r_n\} \subseteq R_0$ and $b''_1 \in B_{r_1}$, ..., $b''_n \in B_{r_n}$. Hence, under the notation as above we have (cf. the relation (+))

$$b' = y''_1 + \cdots + y''_n = y''_1 \vee \cdots \vee y''_n. \tag{2}$$

This is analogous to the relation $(*)$.

We will speak about a positive linear combination meaning a linear combination with positive coefficients.

Lemma 3.12. *Let $0 < b \in B$. There exist distinct elements $r_1, \dots, r_n$ of R_0, nonzero elements $b^0_i \in B_{r_i}$ such that $b = b^0_1 \vee \cdots \vee b^0_n$.*

Proof. The element b can be expressed as a positive linear combination of a system S of nonzero elements of B' such that the system S is orthogonal. Further, each element b' of S can be expressed as a positive linear combination of an orthogonal system of nonzero elements belonging to $\bigcup_{r \in R_0} B_r = Y$.

In view of the orthogonality and of the fact that all elements of Y belong to B, by using Lemma 2.3 we infer that all the coefficients in the expression of b obtained in this way must be equal to 1. Applying again the orthogonality we get that the sum can be replaced by the operation of join. According to 3.9 we obtain the desired result. □

Proposition 3.13. *Let $V \in \mathcal{C}_1$ and suppose that (V, B) is a correct pair. If (V, B') is an another correct pair, then under the notation as above, we have $B' = f\big((B_r, r)_{r \in R_0}\big)$.*

Proof. This is a consequence of 3.10, 3.11 and 3.12. □

In other words, given a correct pair (V, B), each generalized Boolean algebra B' yielding a correct pair (V, B') can be obtained by the construction f.

Proposition 3.14. *Let $V \in \mathcal{C}_1$. Assume that (V, B) and (V, B') are correct pairs. Then $B \simeq B'$.*

Proof. We apply 3.10–3.13 and the notation as above. Let $b \in B$. For $b = 0$ we put $\varphi(b) = 0$. Let $0 < b$. Consider the representation of b described in 3.12. Put $R_1 = \{r_1, \dots, r_n\}$. Hence

$$b = \bigvee_{r \in R_1} b_r^0 . \tag{3}$$

We put

$$\varphi(b) = \bigvee_{r \in R_1} r b_r^0 . \tag{4}$$

Then we have $r b_r^0 \in B'$ for each $r \in R_1$, whence $\varphi(b) \in B'$.

In view of 3.1, the expression of b in the form (3) is unique, thus the mapping φ is correctly defined. Further, from the properties of Z_r $(r \in R_0)$ and from (2) we obtain (by using 3.1 again) that φ is a monomorphism. Next, in view of (2) we conclude that the mapping φ is surjective. It is easy to verify that for $b' \in B$ we have

$$b \leqq b' \iff \varphi(b) \leqq \varphi(b') .$$

□

We also remark that if $V_1, V_2 \in \mathcal{C}_1$ and (V_1, B_1), (V_2, B_2) are correct pairs such that $B_1 \simeq B_2$, then $V_1 \simeq V_2$. The proof will be omitted.

4. Direct product decompositions

The direct product of vector lattices is defined in the usual way. We apply the notation $\prod_{i \in I} V_i$ (or $V_1 \times V_2 \times \dots \times V_n$ if the number of direct factors is finite). Let V and V_i $(i \in I)$ be vector lattices. If

$$V \simeq \prod_{i \in I} V_i , \tag{1}$$

then we say that the relation (1) is a direct product decomposition of V.

The consideration of this section would be trivial in the case $V = \{0\}$. Thus we suppose that V has more than one element. Also, the one-element direct factors in (1) can be omitted. The expression "direct factor" below will mean a non-zero direct factor.

The aim of the present section is to show that each direct product decomposition of a vector lattice belonging to $\mathcal{C}$ has only a finite number of direct factors.

PROPOSITION 4.1. *Let V and $V_1, V_2, \dots, V_n$ be vector lattices, $V \simeq V_1 \times \dots \times V_n$. Then the following conditions are equivalent:*

(i) $V \in \mathcal{C}$.

(ii) *All V_i $(i = 1, 2, \dots, n)$ belong to $\mathcal{C}$.*

Proof. There exists an isomorphism φ of V onto $V_1 \times \dots \times V_n$. For $x \in V$ and $i \in \{1, 2, \dots, n\} = I$ we denote by x_i the component of $\varphi(x)$ in V_i.

a) Assume that $V \in \mathcal{C}$. Without loss of generality we can suppose that there is a generalized Boolean algebra B such that $V = C(B)$. For $i \in I$ put $B_i = \{b_i : b \in B\}$. It is easy to verify that B_i is a generalized Boolean algebra. Moreover, the zero element of V_i belongs to B_i; also, B_i is a sublattice of V_i. Let $0 \neq y \in V_i$. There exists $0 \neq x \in V$ with $x_i = y$. Further, there are $0 \neq a_j \in \mathbb{R}$ and $0 \neq b_j \in B$ $(j = 1, 2, \dots, m)$ such that

$$x = a_1 b_1 + \dots + a_m b_m .$$

Then

$$y = x_i = a_1 (b_1)_i + \dots + a_m (b_m)_i .$$

Therefore V_i is equal (up to isomorphism) to $C(B_i)$ and hence $V_i \in \mathcal{C}$.

b) Conversely, assume that V_i belong to $\mathcal{C}$ for each $i \in I$. Hence we may suppose that there are generalized Boolean algebras B_i with $V_i = C(B_i)$. We denote by B the set of all elements $z \in V$ such that $z_i \in B_i$ for each $i \in I$. Then B is a generalized Boolean algebra, $0 \in B$ and B is a sublattice of the lattice V.

For each $i \in I$ and $y \in V_i$ we denote by $\overline{y}$ the element of V with $\overline{y}_i = y$ and $y_{i(1)} = 0$ for $i(1) \in I$, $i(1) \neq i$.

Let $z \in V$. Then we have

$$z = \overline{z}_1 + \dots + \overline{z}_n .$$

For each $i \in I$ there are $a^i_1, \dots, a^i_{m(i)} \in \mathbb{R}$, $b^i_1, \dots, b^i_{m(i)} \in B_i$ such that

$$z_i = a^i_1 b^i_1 + \dots + a^i_{m(i)} b^i_{m(i)} .$$

Hence

$$\overline{z}_i = a^i_1 \overline{b^i_1} + \dots + a^i_{m(i)} \overline{b^i_{m(i)}} .$$

We obtain

$$z = a^1_1 \overline{b^1_1} + \dots + a^1_{m(1)} \overline{b^1_{m(1)}} + \dots + a^n_1 \overline{b^n_1} + \dots + a^n_{m(n)} \overline{b^n_{m(n)}} .$$

All elements $\overline{b^1_1}, \dots, \overline{b^n_{m(n)}}$ belong to B. Hence V is isomorphic to $C(B)$. □

LEMMA 4.2. *Let $0 < x \in C(B)$. There exists $m \in \mathbb{N}$ such that for each $0 < b \in B$ we have $mb \nleqq x$.*

Proof. The element x can be expressed in the form

$$x = a_1 b_1 + \cdots + a_n b_n$$

such that $0 < a_i \in \mathbb{R}$, $0 < b_i \in B$ for each $i \in \{1, 2, \ldots, n\} = I$ and that the systems $\{b_i\}_{i \in I}$ is orthogonal. Let $0 < b \in B$. Choose $m \in \mathbb{N}$ such that $m > a_i$ for each $i \in I$.

Denote $(b_1 \vee b_2 \vee \cdots \vee b_n) \wedge b = b_0$ and let b_{01} be the complement of b_0 in the interval $[0, b]$ of B.

For $i \in I$ put $b_i^1 = b_i \wedge b$ and let b_i^2 be the complement of b_i^1 in the interval $[0, b_i]$ of B. Then we have

$$b = b_0 \vee b_{01}, \quad b_0 = (b_1 \vee \cdots \vee b_n) \wedge b = b_1^1 \vee \cdots \vee b_n^1, \quad b_i = b_i^1 \vee b_i^2 = b_i^1 + b_i^2 .$$

Hence we obtain

$$\begin{aligned} x &= a_1 b_1^1 + a_1 b_1^2 + \cdots + a_n b_n^1 + a_n b_n^2 + 0 b_{01}, \\ b &= 1 b_1^1 + 0 b_1^2 + \cdots + 1 b_n^1 + 0 b_n^2 + 1 b_{01}, \\ mb &= m b_1^1 + 0 b_1^2 + \cdots + m b_n^1 + 0 b_n^2 + m b_{01} . \end{aligned}$$

If $mb \leqq x$, then according to 2.6 we would have $m \leqq a_1, \ldots, m \leqq a_n$, which is a contradiction. □

LEMMA 4.3. *Let $0 < x \in C(B)$, $C(B) = V_1 \times V_2$, $x(V_2) = 0$. Let $x = a_1 b_1 + \cdots + a_n b_n$, $0 < b_i \in B$, $0 < a_i \in \mathbb{R}$ and suppose that the system $(b_i)_{i=1,2,\ldots,n}$ is orthogonal. Then $b(V_2) = 0$ for $i = 1, 2, \ldots, n$.*

Proof. By way of contradiction, assume that $b_i(V_2) \neq 0$ for some $i \in \{1, 2, \ldots, n\} = I$. Then $b_i(V_2) > 0$. Also, $(a_i b_i)(V_2) = a_i(b_i(V_2)) > 0$. Since $(a_j b_j)(V_2) \geqq 0$ for each $j \in I$, $j \neq i$, we get $x(V_2) > 0$, which is a contradiction. □

PROPOSITION 4.4. *Let $V \in \mathcal{C}$. Then V cannot be expressed as a direct product of infinitely many direct factors.*

Proof. Without loss of generality we can assume that $V = C(B)$, where B is a generalized Boolean algebra. By way of contradiction, suppose that the relation (1) is valid and that the set I is infinite. We apply the analogous notation as in the proof of 4.1. We can write

$$V \simeq V_1 \times V_2 \times \cdots \times V' .$$

For each $n \in \mathbb{N}$ there exists $y^n \in V_n$ with $0 < y^n$. Then $0 < \overline{y^n} \in V$ and the system $(\overline{y^n})_{n \in \mathbb{N}}$ is orthogonal. We recall that if $m \in \mathbb{N}$, $m \neq n$, then the component of $\overline{y^n}$ in V_m is 0; also, the component of $\overline{y^n}$ in V' is equal to 0. The component of $\overline{y^n}$ in V_n is y^n.

The element $\overline{y^n}$ can be expressed in the form

$$\overline{y^n} = a_1^n b_1^n + \cdots + a_{m(n)}^n b_{m(n)}^n$$

such that $0 < a_1^n \in \mathbb{R}, \ldots, 0 < a_{m(n)}^n \in \mathbb{R}, 0 < b_1^n \in B, \ldots, 0 < b_{m(n)}^n \in B$ and the system $\{b_1^n, \ldots, b_{m(n)}^n\}$ is orthogonal.

Thus from 4.3 we conclude that $b_1^n(V') = 0$ and $b_1^n(V_m) = 0$ for each $m \in \mathbb{N}$, $m \neq n$. Therefore we must have $b_1^n(V_n) > 0$. Further, the system $(b_1^n)_{n\in\mathbb{N}}$ is orthogonal.

Since $n(b_1^n)(V_n) = (nb_1^n)(V_n)$ we infer that there exists $x \in V$ such that

$$x(V_n) = nb_1^n(V_n) \qquad \text{for} \quad n = 1, 2, \ldots, \quad \text{and} \quad x(V') = 0.$$

Let m be as in 4.2; choose $n > m$ and let $b = b_1^n$. Then

$$(mb)(V_n) = mb_1^n(V_n) < nb_1^n(V_n) = x(V_n),$$
$$mb(V') = 0 = x(V'),$$

and for $k \in \mathbb{N}$, $k \neq n$, we have $mb(V_k) = 0$, $x(V_k) > 0$. Thus $mb \leqq x$, which is a contradiction. □

By an analogous method (using only integer coefficients) we can prove

PROPOSITION 4.5. *Let G be a Specker lattice ordered group. Then G cannot be expressed as a direct product of infinitely many nonzero direct factors.*

5. Internal direct product decompositions

Internal direct product decompositions of lattice ordered groups and lattices were dealt with, e.g., in [6] and [9].

For vector lattices, the notion of an internal direct product decomposition can be defined as follows.

Assume that V is a vector lattice and let the relation (1) from the previous section be valid. Also, let φ be as above and let $i \in I$. We denote by V^{i0} the set of all $y \in V$ such that $\varphi(y)_j = 0$ for each $j \in I$, $j \neq i$. Then, in view of the induced operations, V^{i0} is a vector lattice. For $x_i \in V_i$ let x^{i0} be the element of V^{i0} such that $(\varphi(x^{i0}))_i = x_i$. Then the mapping

$$\varphi_i \colon V_i \to V^{i0} \qquad (*)$$

defined by $\varphi_i(x_i) = x^{i0}$ is an isomorphism of V_i onto V_i^0. For each $x \in V$ we put

$$\varphi_0(x) = (\varphi_i(x_i))_{i\in I}.$$

Then, in view of (1), the mapping

$$\varphi_0 : V \to \prod_{i\in I} V_i^0 \tag{1'}$$

is a direct product decomposition of V. We say that φ_0 is an *internal* direct product decomposition.

Thus to each direct decomposition φ of V there corresponds an internal direct product decomposition φ_0 of V such that, up to isomorphism, φ and φ_0 are not essentially different.

All direct factors in an internal direct product decomposition are subsets of V. This yields that the collection of all internal direct product decompositions of V is a set. On the other hand, the collection of all direct product decompositions of V is a proper class.

The definition of the internal direct product decompositions for lattice ordered groups and for lattices having the least element are analogous.

For a vector lattice V we denote by $s(V)$ the system of all nonempty subsets X of V such that, whenever $x, y \in X$ and $r \in \mathbb{R}$, then all the elements $x - y$, $x \wedge y$, $x \vee y$ and rx belong to X. Under the operations induced from V, each $X \in s(V)$ is a vector lattice.

From the definition of the internal direct product decomposition we infer that (1) is internal if and only if the following conditions are satisfied:

(i) all V_i belong to $s(V)$;
(ii) whenever $i \in I$ and $x \in V_i$, then $x_i = x$ and $x_j = 0$ for $j \in I$, $j \neq i$.

In view of 4.4 and 4.5, we are interested in finite internal direct product decompositions of elements of $\mathcal{C}$. If $V \in \mathcal{C}$, then without loss of generality we can suppose that $V = C(B)$, where B is a generalized Boolean algebra.

LEMMA 5.1. *Let V be a vector lattice and let $V_1, \dots, V_n$ be nonzero elements of $s(V)$. Then the following conditions are equivalent:*

(i) *V is an internal direct product of $V_1, \dots, V_n$.*
(ii) *If $x \in V$, then x can be uniquely expressed in the form $x = x_1 + \dots + x_n$ with $x_1 \in V_1$, ..., $x_n \in V_n$. If y is another element of V having the analogous expression $y = y_1 + \dots + y_n$, then $x \leqq y$ if and only if $x_1 \leqq y_1$, ..., $x_n \leqq y_n$.*

Proof.

a) Assume that (i) holds. Let φ be the corresponding isomorphism of V onto $V_1 \times \dots \times V_n$. From the definition of the internal direct product decomposition we conclude that whenever $i \in \{1, 2, \dots, n\} = I$ and $z \in V_i$, then $\varphi(z)_i = z$ and $\varphi(z)_j = 0$ for $j \in I$, $j \neq i$.

Let $x \in V$, $\varphi(x) = (x_1, x_2, \dots, x_n)$. Denote $x' = x_1 + \dots + x_n$. We have $\varphi(x_1) = (x_1, 0, \dots, 0)$, ..., $\varphi(x_n) = (0, \dots, 0, x_n)$, whence

$$\varphi(x') = \varphi(x_1) + \dots + \varphi(x_n) = (x_1, x_2, \dots, x_n) = \varphi(x).$$

Therefore $x = x_1 + \dots + x_n$.

Assume that, at the same time, $x = x^1 + \dots + x^n$ with $x^1 \in V_1, \dots, x^n \in V_n$. Then

$$\varphi(x) = \varphi(x^1) + \dots + \varphi(x^n) = (x^1, \dots, x^n).$$

Hence $x^1 = x_1, \dots, x^n = x_n$ and thus the expression of x under consideration is unique.

Let $y \in V$ have an analogous expression $y = y_1 + \dots + y_n$. If $x_1 \leqq y_1, \dots, x_n \leqq y_n$, then clearly $x \leqq y$. Conversely, assume that $x \leqq y$. Thus $x \vee y = y$. From

$$\varphi(x) = (x_1, \dots, x_n), \quad \varphi(y) = (y_1, \dots, y_n)$$

we obtain

$$\varphi(y) = \varphi(x \vee y) = (x_1 \vee y_1, \dots, x_n \vee y_n),$$

whence $y_1 = x_1 \vee y_1, \dots, y_n = x_n \vee y_n$. Therefore $x_1 \leqq y_1, \dots, x_n \leqq y_n$. We verified that (i) $\Longrightarrow$ (ii).

b) Assume that (ii) is valid. Let $x \in V$. There exists uniquely determined elements $x_1, \dots, x_n$ with $x_1 \in V_1, \dots, x_n \in V_n$ such that $x = x_1 + \dots + x_n$. Put $\varphi(x) = (x_1, \dots, x_n)$. Hence $\varphi(x) \in V_1 \times \dots \times V_n$ and φ is a bijection. Then $rx_i \in V_i$ for each $i \in I$, whence $\varphi(rx) = r\varphi(x)$. Further, if $x, y \in V$, then $\varphi(x + y) = \varphi(x) + \varphi(y)$. Therefore (i) holds. □

For a lattice ordered group G we denote by $s(G)$ the system of all ℓ-subgroups of G. Then the result analogous to 5.1 is valid for G (with the same idea of the proof).

Let B be a generalized Boolean algebra. We denote by $s(B)$ the system of all sublattices X of B such that $0 \in X$ and X is a Boolean algebra. If B is expressed as an internal direct product of a system $(B_i)_{i \in I}$, then clearly $B_i \in s(B)$ for each $i \in I$.

LEMMA 5.2. *Let $B_1, \dots, B_n$ be nonzero elements of $s(B)$. The following conditions are equivalent:*

(i) *B is an internal direct product of $B_1, \dots, B_n$.*

(ii) *If $x \in B$, then x can be uniquely expressed in the form $x = x_1 \vee \dots \vee x_n$ with $x_1 \in B_1, \dots, x_n \in B_n$.*

Proof.

a) Let (i) be valid and suppose that φ is corresponding isomorphism. Let $x \in B$ and $\varphi(x) = (x_1, \dots, x_n)$. We have

$$\varphi(x_1) = (x_1, 0, \dots, 0), \ \dots, \ \varphi(x_n) = (0, \dots, 0, x_n).$$

Thus we obtain

$$\varphi(x_1 \vee \cdots \vee x_n) = \varphi(x_1) \vee \cdots \vee \varphi(x_n) = (x_1, \ldots, x_n) = \varphi(x).$$

Therefore $x = x_1 \vee \cdots \vee x_n$.

If $x^1 \in B_1$, ..., $x^n \in B_n$ and $x = x^1 \vee \cdots \vee x^n$, then we obtain

$$\varphi(x) = \varphi(x^1) \vee \cdots \vee \varphi(x^n) = (x^1, \ldots, x^n),$$

whence $x^1 = x_1$, ..., $x^n = x_n$, and so the condition (ii) is satisfied.

b) Assume that (ii) holds. At first we verify that whenever $i, j \in I$, $i \neq j$, then $B_i \cap B_j = \{0\}$. Obviously, $0 \in B_i \cap B_j$. By way of contradiction, assume that there exists $0 < z \in B_i \cap B_j$. Put $x^i = z$, $x^j = z$ and $x^k = 0$ for $k \in I$, $i \neq k \neq j$. Further, we set $y^i = z$ and $y^k = 0$ for $k \in I$, $k \neq i$. Then we obtain $z = x^1 \vee \cdots \vee x^n = y^1 \vee \cdots \vee y^n$, which is a contradiction.

Under the notation as in (ii) we put $\varphi(x) = (x_1, \ldots, x_n)$. Then the set $\{x_1, \ldots, x_n\}$ is orthogonal. The mapping φ is a bijection of B onto $B_1 \times \cdots \times B_n$.

Let $y \in B$, $\varphi(y) = (y_1, \ldots, y_n)$. If $x_1 \leqq y_1$, ..., $x_n \leqq y_n$, then $x \leqq y$. Suppose that $x \leqq y$ and let $i \in I$. Then

$$x_i \leqq x \leqq y = y_1 \vee \cdots \vee y_n,$$
$$x_i = x_i \wedge y = (x_i \wedge y_1) \vee \cdots \vee (x_i \wedge y_n).$$

Let $j \in I$. If $j \neq i$, then $x_i \wedge y_j = 0$. Thus $x_i = x_i \wedge y_i$ and therefore $x_i \leqq y_i$. This yields that the mapping φ is an isomorphism.

Let $i \in I$, $x \in B_i$. Put $y^i = x$, $y^j = 0$ for $j \in I$, $j \neq i$. Then $x = y^1 \vee \cdots \vee y^n$, whence $x_i = x$ and $x_j = 0$ for $j \in I$, $j \neq i$. Therefore the mapping φ determines an internal direct product decomposition of B; hence (i) is valid. □

Remark 5.2.1. Looking at the proof of the implication (ii) $\Longrightarrow$ (i) in 5.2 we see that the internal direct product decomposition mentioned in (i) is given by the mapping $\varphi(x) = (x_1, \ldots, x_n)$, where $x = x_1 \vee \cdots \vee x_n$ (under the notation as in (ii)).

LEMMA 5.3. *Let B be a generalized Boolean algebra. Assume that*

$$\varphi \colon C(B) \to V_1 \times \cdots \times V_n$$

is an internal direct product decomposition of $C(B)$. Put $\varphi_1 = \varphi|_B$ (the corresponding partial mapping defined on B) and $B_i = V_i \cap B$ for $i \in I = \{1, 2, \ldots, n\}$. Then

$$\varphi_1 \colon B \to B_1 \times \cdots \times B_n$$

is an internal direct product of B.

P r o o f . Let $i \in I$. In view of φ we infer that V_i is a convex sublattice of V containing the element 0. Thus $0 \in B_i$ and B_i is a convex sublattice of B. Hence B_i is a generalized Boolean algebra. Further, from the definition of φ we easily obtain that $V_i \cap V_j = \{0\}$ whenever i and j are distinct elements of I; hence in such case we also have $B_i \cap B_j = \{0\}$.

Let $x \in B$, $x > 0$. If $\varphi_1(x) = (x_1, \dots, x_n)$, then $x = x_1 + \dots + x_n$ and $x_1 \geqq 0, \dots, x_n \geqq 0$. Since the set $\{x_1, \dots, x_n\}$ is orthogonal, we obtain $x = x_1 \vee \dots \vee x_n$. If $x^1 \in B_1, \dots, x^n \in B^n$ and $x = x^1 \vee \dots \vee x^n$, then $x = x^1 + \dots + x^n$. In view of the properties of φ we get $x^1 = x_1, \dots, x^n = x_n$.

Thus according to 5.2 and 5.2.1, φ_1 determines an internal direct product decomposition of B. □

Let X be a subset of a vector lattice V. We put

$$X^\delta = \{y \in V :\ |y| \wedge |x| = 0 \text{ for each } x \in X\}.$$

Then X^δ is a *polar* of V. It is well-known that $X^\delta \in s(V)$.

Again, let B be a generalized Boolean algebra. For $\emptyset \neq X \subseteq B$ we denote by $\overline{X}$ the set of all elements $x \in C(B)$ which can be expressed in the form $x = a_1 b_1 + \dots + a_n b_n$, where $a_i \in \mathbb{R}$ and $b_i \in X$ for $i = 1, 2, \dots, n$.

Assume that $\psi\colon B \to B_1 \times \dots \times B_n$ is an internal direct product decomposition of B.

Lemma 5.4. *Let* $i \in \{1, 2, \dots, n\}$. *Then* $\overline{B_i} = B_i^{\delta\delta}$.

P r o o f . We have $B_i \subseteq B_i^{\delta\delta}$. Since $B_i^{\delta\delta} \in s(C(B))$, we get $\overline{B_i} \subseteq B_i^{\delta\delta}$.

In view of ψ we have

$$B_j \subseteq B_i^\delta \qquad \text{for each } \ i \in \{1, 2, \dots, n\},\ \ j \neq i. \tag{$*$}$$

Let $0 \neq x \in B_i^{\delta\delta}$. The element x can be expressed in the form $x = a_1 b_1' + \dots + a_m b_m'$, where $b_1', \dots, b_m'$ are nonzero elements of B, $a_1, \dots, a_m$ are nonzero elements of $\mathbb{R}$ and (in view of 2.1) the system $\{b_1', \dots, b_m'\}$ is orthogonal. Then we have

$$|x| = |a_1| b_1' + \dots + |a_m| b_m' = (|a_1| b_1') \vee \dots \vee (|a_m| b_m').$$

Consider the element $|a_1| b_1'$. We obtain $|a_1| b_1' \leqq |x|$, thus $|a_1| b_1' \in B_i^{\delta\delta}$. Since $B_i^{\delta\delta} \in s(C(B))$, we get $b_1' \in B_i^{\delta\delta}$.

In view of ψ and 5.2, b_1' can be expressed in the form

$$b_1' = b_1^0 \vee \dots \vee b_n^0,$$

where $b_1^0 \in B_1, \dots, b_n^0 \in B_n$. If $j \in I = \{1, 2, \dots, n\}$, $j \neq i$ and $b_j^0 > 0$, then in view of $(*)$ we arrive at a contradiction. Thus $b_1' = b_i^0 \in B_i$. Analogously we have $b_2' \in B_i, \dots, b_m' \in B_i$. Hence $x \in \overline{B_i}$. Therefore $B_i^{\delta\delta} \subseteq \overline{B_i}$. □

COROLLARY 5.4.1. *Let $i \in I$. Then $\overline{B_i} \in s(C(B))$.*

LEMMA 5.5. *$C(B)$ is an internal direct product of $\overline{B_1}, \dots, \overline{B_n}$.*

Proof. Let $0 \neq x \in C(B)$. Then there are $a_1, \dots, a_m \in \mathbb{R}$ and $b_1, \dots, b_m \in B$ such that $x = a_1 b_1 + \dots + a_m b_m$. Put $J = \{1, 2, \dots, m\}$ and let $j \in J$. In view of ψ, b_j can be expressed in the form $b_j = \bigvee_{i \in I} b_{ji}$, where $b_{ji} \in B_i$ for $i \in \{1, 2, \dots, n\} = I$. The system $(b_{ji})_{i \in I}$ is orthogonal, whence $\bigvee_{i \in I} b_{ji} = \sum_{i \in I} b_{ji}$. Thus we obtain

$$x = \sum_{j \in J} a_j b_j = \sum_{j \in J} \sum_{i \in I} a_j b_{ji} = \sum_{i \in I} \sum_{j \in J} a_j b_{ji} .$$

Put $\sum_{j \in J} a_j b_{ji} = x_i$. Then $x_i \in \overline{B_i}$ for each $i \in I$ and $x = x_1 + \dots + x_n$.

If $x > 0$, then in view of 3.2.1 we have $a_1 > 0, \dots, a_m > 0$. This yields that $x_1 \geqq 0, \dots, x_n \geqq 0$.

For any $x \in V$ we put $\psi_1(x) = (x_1, \dots, x_n)$. Hence ψ is a mapping of $C(B)$ into $\prod_{i \in I} \overline{B_i}$, $I = \{1, 2, \dots, n\}$.

Assume that $x_1' \in \overline{B_1}, \dots, x_n' \in \overline{B_n}$ such that, at the same time, we have $x = x_1' + \dots + x_n'$. Then

$$x_1 - x_1' = (x_2' - x_2) + \dots + (x_n' - x_n)$$

and $x_1 - x_1' \in \overline{B_1}$, $x_2' - x_2 \in \overline{B_2}, \dots, x_n' - x_n \in \overline{B_n}$. By applying the facts known from proof of 5.4 we obtain

$$x_1 - x_1' \in B_1^{\delta\delta}, \qquad x_2' - x_2 \in B_1^{\delta}, \ \dots, \ x_n' - x_n \in B_1^{\delta} .$$

Then $(x_2' - x_2) + \dots + (x_n' - x_n) \in B_1^{\delta}$. Since $B_1^{\delta\delta} \cap B_1^{\delta} = \{0\}$, we get $x_1 - x_1' = 0$, thus $x_1 = x_1'$. Similarly we obtain $x_2 = x_2', \dots, x_n = x_n'$.

Let y be another element of $C(B)$ and $\psi_1(y) = (y_1, \dots, y_n)$. Thus $y = y_1 + \dots + y_n$. If $x_1 \leqq y_1, \dots, x_n \leqq y_n$, then clearly $x \leqq y$. Conversely, suppose that $x \leqq y$. Put $z = y - x$. Let $\psi_1(z) = (z_1, \dots, z_n)$. Then $z_i = y_i - x_i$ for $i \in I$. Since $z \geqq 0$, we must have (in view of the case $x > 0$ considered above) $z_1 \geqq 0, \dots, z_n \geqq 0$. Hence $y_i \geqq x_i$ for $i \in I$.

Now according to 5.1, ψ_1 determines an internal direct product decomposition of $C(B)$. □

Let φ and φ_1 be as in 5.3. The basic idea in constructing φ_1 from φ can be described by the correspondence

$$V_1 \xrightarrow{a} B_1 = V_1 \cap B, \tag{a}$$

where V_1 is an internal direct factor of $V = C(B)$ and B_1 is an internal direct factor of B.

Further, the main idea of 5.5 consists in applying the correspondence

$$B_1 \xrightarrow{b} \overline{B_1}, \tag{b}$$

where B_1 is an internal direct factor of B and $\overline{B_1}$ is an internal direct factor of $C(B)$.

By applying the correspondence (a) for $\overline{B_1}$ we get

$$\overline{B_1} \xrightarrow{a} \overline{B_1} \cap B.$$

LEMMA 5.6. *Under the denotation as above, we have $\overline{B_1} \cap B = B_1$.*

Proof. We have clearly $B_1 \subseteq \overline{B_1} \cap B$. Let $x \in \overline{B_1} \cap B$. Thus $x \in B$ and hence $x \geqq 0$. The case $x = 0$ yields $x \in B_1$. Let $x > 0$. We have $x \in \overline{B_1}$ thus there are nonzero mutually orthogonal elements $b_1, \dots, b_n$ of B_1 and nonzero elements $a_1, \dots, a_n \in \mathbb{R}$ such that

$$x = a_1 b_1 + \cdots + a_n b_n .$$

According to 2.3 we have $x = b_1 \vee \cdots \vee b_n$. Thus $x \in B_1$. Therefore $\overline{B_1} \cap B \subseteq B_1$. □

From 5.6 we immediately obtain:

COROLLARY 5.7. *There exists a one-to-one correspondence between internal direct factors of B and internal direct factors of $C(B)$.*

Summarizing, 5.3, 5.5 and 5.7 yield:

PROPOSITION 5.8. *Let B be a generalized Boolean algebra. There exists a one-to-one correspondence between internal direct product decompositions of the vector lattice $C(B)$ and finite internal direct product decompositions of B.*

By the same method as in 5.1 we obtain:

LEMMA 5.9. *Let $G_1, \dots, G_n$ be nonzero ℓ-subgroups of a lattice ordered group G. The following conditions are equivalent:*

(i) *G is an internal direct product of $G_1, \dots, G_n$.*
(ii) *If $x \in G$, then x can be uniquely expressed in the form $x = x_n + \cdots + x_n$ with $x_1 \in G_1, \dots, x_n \in G_n$. Whenever $y = y_1 + \cdots + y_n$ is such an expression for $y \in G$, then $x \leqq y$ if and only if $x_1 \leqq y_1, \dots, x_n \leqq y_n$.*

Now let us consider the lattice ordered group $S(B)$, where B is a generalized Boolean algebra. Several results on $S(B)$ can be proved by methods analogous to those which were applied for $C(B)$. We have: If

$$\varphi\colon S(B) \to G_1 \times \cdots \times G_n$$

is an internal direct product of $S(B)$ and $B_i = G_i \cap B$ $(i = 1, 2, \ldots, n)$, then

$$\varphi_0\colon B \to B_1 \times \cdots \times B_n$$

is an internal direct product decomposition of B. (Cf. 5.3.)

For each internal direct factor B_1 of B we denote by B_1^* the set of all $x \in S(B)$ which can be expressed in the form

$$x = a_1 b_1 + \cdots + a_n b_n \,,$$

where $b_1, \ldots, b_n \in B_1$ and $a_1, \ldots, a_n$ are integers. If

$$\psi\colon B \to B_1 \times \cdots \times B_n$$

is an internal direct product decompositions of B, then

$$\psi_1\colon S(B) \to B_1^* \times \cdots \times B_n^*$$

is an internal direct product decomposition of $S(B)$. (Cf. 5.5; the corresponding coefficients in the proof are now assumed to be integers.)

If B_1 is an internal direct factor of B, then $B_1^* \cap B = B_1$. (Cf. 5.6; again, we have to apply integral coefficients.)

Therefore, similarly as in 5.8, we conclude that there exists a one-to-one correspondence between internal direct decompositions of $S(B)$ and finite internal direct product decompositions of B.

As a consequence we obtain that there is a one-to-one correspondence between internal direct product decompositions of $C(B)$ and internal direct product decompositions of $S(B)$.

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