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LAGUERRE POLYNOMIALS
WITH FOUR PARAMETERS

FRANTIŠEK PÚCHOVSKÝ

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ABSTRACT. The polynomials $P_n(x)$, which are orthonormal in the interval $[0, +\infty)$ with respect to the weight function $P(x) = (a+x)^{2\alpha}(x-b)^{2\beta}e^{-x}$, where $a > 0, \alpha \in \mathbb{R}, b \geq 0, \beta > \frac{1}{2}$, are investigated. There are derived relations for coefficients of these polynomials, relations for the sums of $k$th powers $s_k(n)$ of zeros of these polynomials, the relation giving a connection between the polynomials of different degrees and differential equations for the polynomials $P_n(x)$.

1. Introduction

The Laguerre polynomials $L_n(x)$ may be defined by the relation

$$L_n(x) = x^{-\alpha}e^x \frac{d^n}{dx^n}(x^{n+\alpha}e^{-x})$$

and $L_0(x) = 1, \alpha > -1$. The system of polynomials $L_n(x)$ is orthogonal in the interval $I = [0, \infty)$ with respect to the function $L(x) = x^\alpha e^{-x}$, so for $n = 0, 1, \ldots$,

$$\int_{I} L_n^2(x)L(x) \, dx = n!\Gamma(n + \alpha + 1), \quad \int_{I} L_n(x)L_m(x)L(x) \, dx = 0 \quad (2)$$

for $m \neq n$ (see [1; p. 93]).

In the work [2] J. Korous defined his polynomials $K_n(x)$, which are orthonormal in the interval $[0, +\infty)$ with respect to the weight function $K(x) = x^{\alpha}(a+x)^{\beta}e^{-x}$, where $\alpha, \beta, a$ are real numbers, $\alpha \geq -\frac{1}{2}, a > 0$.

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In the present work we will investigate the system of the polynomials \( \{ P_n(x) \}_{n=0}^{\infty} \), where

\[
P_n(x) = \sum_{k=0}^{n} a_k^{(n)} x^{n-k}, \quad a_0^{(n)} > 0 ,
\]

which are orthonormal in the interval \( I = [0, +\infty) \) with respect to the function

\[
P(x) = (a + x)^{2\alpha} (x - b)^{2\beta} e^{-x} ,
\]

where \( a > 0, \alpha \in \mathbb{R}, b \geq 0, \beta > \frac{1}{2} \), i.e.

\[
\int_0^\infty P_n(x) P_m(x) P(x) \, dx = \delta_{m,n} ,
\]

and \( \delta_{n,n} = 1, \delta_{m,n} = 0 \) for \( m \neq n \).

Laguerre polynomials \( L_n(x) \) and Korous polynomials \( K_n(x) \) are special cases of considered polynomials \( P_n(x) \).

In this work we derive some relations for coefficients of the polynomials (3), the relations (19) and (20) for the sums of \( k \)th powers \( s_k^{(n)} \) of the zeros of these polynomials, the relation (24), which expresses a connection between polynomials of different degrees and differential equations (35) and (37).

\section{Basic relations and lemmas for polynomials \( P_n(x) \)}

\textbf{Notation.} For \( n = 0, 1, 2, \ldots \) we define

\[
q_n = \frac{a_0^{(n-1)}}{a_0^{(n)}} \quad \text{for} \quad n > 0 , \quad q_n = 0 \quad \text{for} \quad n \leq 0 ,
\]

\[
r_k^{(n)} = \frac{a_k^{(n)}}{a_0^{(n)}} \quad \text{for} \quad k > 0 , \quad r_k^{(n)} = 0 \quad \text{for} \quad k < 0 , \quad r_0^{(n)} = 1 ,
\]

\[
j_n = \int_I x P_n^2(x) P(x) \, dx .
\]

\textbf{Remark 1.} \( \pi_n(x) = \pi_n \) denotes arbitrary polynomial of at most \( n \)th degree.

\textbf{Remark 2.} \( c \) and \( c_i \) \( (i = 1, 2, \ldots) \) are positive constants independent of \( n \) and \( x \) and \( t \), respectively. The numbering of these constants in each lemma and theorem is independent of the numbering in the others.
LEMMA 1. (RECCURENCE FORMULA FOR \( P_n(x) \)) Under notations (3), (6) and (8) for every \( n \)
\[
(x - j_n)P_n(x) = q_{n+1}P_{n+1}(x) + q_nP_{n-1}(x).
\]

Proof. See [1; p. 77]. □

LEMMA 2 (CHRISTOFFEL-DARBOUX FORMULA) Let \( \{ P_n(x) \}_{n=0}^{\infty} \) be a system of orthonormal polynomials in the interval \([0, +\infty)\). Denote
\[
P_n(x, t) = \sum_{k=0}^{n} P_k(x)P_k(t).
\]

Then for \( x \neq t \)
\[
P_n(x, t) = q_{n+1}(x - t)^{-1}\{ P_{n+1}(x)P_n(t) - P_n(x)P_{n+1}(t) \}.
\]

Proof. (11) follows from (9). See [1; p. 79]. □

LEMMA 3. Every polynomial \( F_n(x) \) of the \( n \)th degree can be expressed in the form
\[
F_n(x) = \sum_{k=0}^{n} \alpha_k P_k(x),
\]
where
\[
\alpha_k = \int_{I} F_n(t)P_k(t)P(t) \, dt.
\]

Proof is well known.

LEMMA 4. Preserving the notation (8) we denote
\[
i_n = 2\alpha a \int_{I} (x + a)^{-1} P_n^2(x)P(x) \, dx,
\]
\[
h_n = 2\beta b \int_{I} (x - b)^{-1} P_n^2(x)P(x) \, dx.
\]

Then for \( n = 1, 2, \ldots \)
\[
j_n = 2n + 1 + 2\alpha + 2\beta - i_n + h_n.
\]
Proof. We use integration by parts on (8). Then
\[ j_n = \int x P^2_n(x) P(x) \, dx \]
\[ = -\int x P^2_n(x) [P(x) e^x] \, dx \]
\[ = [-x P^2_n(x) P(x)]_0^\infty + \int x P_n(x) P'_n(x) P(x) \, dx \]
\[ + \int \{1 + 2\alpha x(x + a)^{-1} + 2\beta x(x - b)^{-1}\} P^2_n(x) P(x) \, dx \]
\[ = 2n + 1 + 2\alpha + 2\beta - i_n + h_n, \]
where in the last integral we substitute \( x \) by \( x = x + a - a \) and by \( x = x - b + b \), respectively and we use (5).

\[ \text{Lemma 5. For } n \to +\infty \text{ the following relations hold} \]
\[ i_n = O(1), \]
\[ h_n = O(1). \]  \hfill (17) \hfill (18)

Proof. The relations (17) and (18) follow from (14) and (15).

\[ \text{Lemma 6. Let } s_k^{(n)} \text{ for } k = 0, 1, 2, \ldots \text{ be the sum of kth powers of zeros of the polynomials } P_n(x). \text{ Then} \]
\[ s_1^{(n)} = \sum_{k=0}^{n-1} j_k = -r_1^{(n)}, \]  \hfill (19)
\[ s_2^{(n)} = \sum_{k=0}^{n-1} (2q_k^2 + j_k^2), \]  \hfill (20)
\[ -r_1^{(n)} = n^2 + (2\alpha + 2\beta)n + \sigma_n, \]  \hfill (21)

where
\[ \sigma_n = \sum_{k=0}^{n-1} (h_k - i_k). \]  \hfill (22)

Proof.

I. There holds
\[ s_1^{(n)} = \sum_{i=1}^{n} x_i = -\frac{a_1(n)}{a_0(n)} = -r_1(n), \]  \hfill (a)
LAGUERRE POLYNOMIALS WITH FOUR PARAMETERS

where $r_k^{(n)}$ is defined in (7).

From recurrence formula (9) comparing coefficients at powers $x^n$ we get

$$a_0^{(n)} r_1^{(n)} - j_n a_0^{(n)} = q_{n+1} a_0^{(n+1)} r_1^{(n+1)}.$$

Dividing this equation by $a_0^{(n)}$ and substituting for $q_{n+1}$ according to (6) we have

$$r_1^{(n+1)} - r_1^{(n)} = -j_n. \quad (b)$$

Let us put for $k = 0, 1, 2, \ldots$ $\delta_k^{(n)} = s_k^{(n)} - s_k^{(n-1)}$. From there using (a) and (b) we have

$$\delta_1^{(i)} = s_1^{(i)} - s_1^{(i-1)} = -r_1^{(i)} + r_1^{(i-1)} = j_i. \quad (c)$$

(because $s_1^{(0)} = 0$).

From there

$$\sum_{i=1}^{n} \delta_1^{(i)} = \sum_{i=1}^{n} [s_1^{(i)} - s_1^{(i-1)}]$$

and thus

$$s_1^{(n)} = \sum_{i=0}^{n-1} j_i.$$

II. From recurrence formula (9) by comparing of coefficients at powers $x^n$ and $x^{n-1}$, respectively, we get

$$a_0^{(n)} r_1^{(n)} - j_n a_0^{(n)} = q_{n+1} a_0^{(n+1)} r_1^{(n+1)},$$
$$a_0^{(n)} r_2^{(n)} - j_n a_0^{(n)} r_1^{(n)} = q_{n+1} a_0^{(n+1)} r_2^{(n+1)} + a_0^{(n-1)} q_n.$$

Dividing these two equations by $a_0^{(n)}$ and substituting for $q_n$ and $q_{n+1}$ according to (6) and, further, using known relation

$$s_2^{(n)} = \sum_{i=1}^{n} x_i^2 = [s_1^{(n)}]^2 - 2r_2^{(n)} = \neg r_1^{(n)} s_1^{(n)} - 2r_2^{(n)},$$
i.e.

$$s_2^{(n)} + r_1^{(n)} s_1^{(n)} + 2r_2^{(n)} = 0, \quad (d)$$

we get

$$r_1^{(n+1)} - r_1^{(n)} = -j_n; \quad r_2^{(n+1)} - r_2^{(n)} = -q_n^2 - j_n r_1^{(n)}. \quad (e)$$

Further

$$s_2^{(n)} + r_1^{(n)} s_1^{(n)} + 2r_2^{(n)} = 0,$$
$$s_2^{(n-1)} + r_1^{(n-1)} s_1^{(n-1)} + 2r_2^{(n-1)} = 0.$$
Subtracting the second of these equations from the first and using (c) we have

\[ \delta_2^{(n)} + r_1^{(n-1)} \delta_1^{(n)} + (r_1^{(n)} - r_1^{(n-1)}) s_1^{(n)} + 2(r_2^{(n)} - r_2^{(n-1)}) = 0, \]

from where in virtue of (e), (c), (a)

\[ \delta_2^{(n)} = -j_{n-1} r_1^{(n-1)} - j_{n-1} r_1^{(n)} + 2j_{n-1} r_1^{(n-1)} + 2q_{n-1}^2 = j_{n-1}^2 + 2q_{n-1}^2. \]

Because \( s_2^{(0)} = 0 \), it follows from there

\[ s_2^{(n)} = \sum_{k=0}^{n-1} (2q_k^2 + j_k^2). \]

III. From the relations (19) and (16) we get

\[ -r_1^{(n)} = \sum_{k=0}^{n-1} j_k = \sum_{k=0}^{n-1} (2k + 1 + 2\alpha + 2\beta + h_k - i_k) \]

\[ = n(n - 1) + n + 2\alpha n + 2\beta n + \sigma_n = n^2 + (2\alpha + 2\beta)n + \sigma_n, \]

where \( \sigma_n \) is defined in (22).

**Lemma 7.** For \( n \to +\infty \)

\[ \sigma_n = O(1). \quad (23) \]

**Proof.** (23) follows from (17), (18) and (22).

**Lemma 8.** For \( n = 1, 2, \ldots \) we have

\[ XP_n'(x) = nP_n(x) + q_n P_{n-1}(x) + \sum_{k=0}^{n-1} \gamma_k P_k(x), \quad (24) \]

where

\[ \gamma_k = i_n^{(k)} - h_n^{(k)}, \]

\[ i_n^{(k)} = 2aa \int_I (x + a)^{-1} P_n(x) P_k(x) P(x) \, dx, \]

\[ h_n^{(k)} = 2\beta b \int_I (x - b)^{-1} P_n(x) P_k(x) P(x) \, dx. \]

**Proof.** According to (12)

\[ XP_n'(x) = \sum_{k=0}^{n} \beta_k P_k(x), \quad (a) \]
where
\[ \beta_k = \int_I x P'_n(x) P_k(x) P(x) \, dx. \] (b)

We use integration by parts to (b) and then
\[
\beta_k = [x P_n(x) P_k(x) P(x)]_0^\infty - \int_I P_n(x) P_k(x) P(x) \, dx
- \int_I x P_n(x) P'_k(x) P(x) \, dx + \int_I x P_n(x) P_k(x) P(x) \, dx
- \int_I x [2\alpha(x + a)^{-1} + 2\beta(x - b)^{-1}] P_n(x) P_k(x) P(x) \, dx
\]
\[= \int_I x P_n(x) [P_k(x) - P'_k(x)] P(x) \, dx + (-2\alpha - 2\beta) \int_I P_n(x) P_k(x) P(x) \, dx
- \int_I P_n(x) P_k(x) P(x) \, dx + i^{(k)}_n - h_n^{(k)}. \] (c)

1. Let \( k = n \).
Then in virtue of (8), (16), (14) and (15) from (c) we get (because \( i^{(n)}_n = i_n, h^{(n)}_n = h_n \))
\[ \beta_n = j_n - n - 1 - 2\alpha - 2\beta + i_n - h_n \]
\[= 2n + 1 + 2\alpha + 2\beta - i_n + h_n - n - 1 - 2\alpha - 2\beta + i_n - h_n = n. \] (d)

2. Let \( k = n - 1 \).
Then from (c) in virtue of (6) and (5) there is
\[ \beta_{n-1} = q_n + i^{(n-1)}_n - h_n^{(n-1)}. \] (e)

3. Let \( k < n - 1 \).
Then from (c) in virtue of (5) there is
\[ \beta_k = i^{(k)}_n - h_n^{(k)} = \gamma_k. \] (f)

From (c)–(f) we get (24). \( \square \)

**Lemma 9.** Let \( n = 1, 2, \ldots \). Then
\[ q_n^2 - 2\delta_n q_n + r_1^{(n)} = 0, \] (28)
where \( q_n \) is defined in (6), \( r_1^{(n)} \) in (7) and \( 2\delta_n = h_n^{(n-1)} - i_n^{(n-1)} \).

**Proof.** There holds
\[ \int_I x P_n(x) P_{n-1}(x) P(x) \, dx = \frac{a_0^{(n-1)}}{a_0^{(n)}} = q_n. \] (a)
Integrating left-hand side by parts we get
\[ q_n = -\int xP_n(x)P_{n-1}(x)\left[P(x)e^x\right] \, \text{d}e^{-x} \]
\[ = \left[-xP_n(x)P_{n-1}(x)P(x)\right]_{0}^{\infty} + \int xP'_n(x)P_{n-1}(x)P(x) \, \text{d}x \]
\[ + \int x \left[2\alpha(x+a)^{-1} + 2\beta(x-b)^{-1}\right]P_n(x)P_{n-1}(x)P(x) \, \text{d}x \]
\[ = \int xP'_n(x)P_{n-1}(x)P(x) \, \text{d}x - 2aa \int (x+a)^{-1}P_n(x)P_{n-1}(x)P(x) \, \text{d}x \]
\[ + 2\beta b \int (x-b)^{-1}P_n(x)P_{n-1}(x)P(x) \, \text{d}x \]
\[ = J_n - i_n^{(n-1)} + h_n^{(n-1)}, \]
where
\[ J_n = \int xP'_n(x)P_{n-1}(x)P(x) \, \text{d}x \]
\[ = \int \left[nP_n(x) - r_1^{(n)}q_n^{-1}P_{n-1}(x) + \pi_{n-2}\right]P_{n-1}(x)P(x) \, \text{d}x \]
\[ = -r_1^{(n)}q_n^{-1}, \]  \hspace{1cm} (c)
\[ i_n^{(n-1)} = 2aa \int (x+a)^{-1}P_n(x)P_{n-1}(x)P(x) \, \text{d}x, \]  \hspace{1cm} (d)
\[ h_n^{(n-1)} = 2\beta b \int (x-b)^{-1}P_n(x)P_{n-1}(x)P(x) \, \text{d}x. \]  \hspace{1cm} (e)

From (b) in virtue of (c)–(e) we get
\[ q_n = J_n + h_n^{(n-1)} - i_n^{(n-1)} = -r_1^{(n)}q_n^{-1} + 2\delta_n. \]

From there (28) follows. □

**Lemma 10.** For \( n \to +\infty \)
\[ q_n = n + O(1). \]  \hspace{1cm} (29)

**Proof.** (29) follows from (28), (17), (18), (21) and (22). □
LAGUERRE POLYNOMIALS WITH FOUR PARAMETERS

**Lemma 11.** Let $P_n(x)$ be a polynomial of the degree $n$ and $s_k$ be the sum of $k$th powers of its zeros. Then for a natural number $r$

$$x^r P'_n(x) = \sum_{k=0}^{r-1} s_k x^{r-k-1} P_n(x) + \pi_{n-1}.$$  \hspace{1cm} (30)

**Proof.** See [3; p. 352]. \hfill \Box

**Lemma 12.** Let $P_n(x)$ be a polynomial of the degree $n$ and $s_k$ be the sum of $k$th powers of its zeros. Then for a natural number $r$

$$x^r P''_n(x) = P_n(x) \sum_{k=0}^{r-2} \sigma_k x^{r-k-2} + \pi_{n-1},$$

where

$$\sigma_k = \sum_{m=0}^{k} s_m s_{k-m} - (k+1)s_k.$$ \hspace{1cm} (32)

**Proof.** (31) follows from (30). \hfill \Box

3. Differential equations for polynomials $P_n(x)$

**Theorem 1.** Let us denote

$$\epsilon = \left\{ \begin{array}{c} \alpha \\ \beta \end{array} \right\} \quad \delta = \left\{ \begin{array}{c} \beta \\ \alpha \end{array} \right\} \quad \Rightarrow \quad \epsilon + \delta = \alpha + \beta,$$

$$e = \left\{ \begin{array}{c} a \\ -b \end{array} \right\} \quad f = \left\{ \begin{array}{c} -b \\ a \end{array} \right\} \quad \Rightarrow \quad e + f = a - b.$$ \hspace{1cm} (33) \hspace{1cm} (34)

Let $n = 0, 1, 2, \ldots$. Then

$$P^{-1}(x) \frac{d}{dx} [(x + e)x P'_n(x)P(x)] - 2\delta f(e - f)(x + f)^{-1} P'_n(x)$$

$$+ [n(x + e - 1) + \sigma_n] P_n(x)$$

$$= \sum_{k=0}^{n-1} \chi_k P_k(x) + q_n P_{n-1}(x),$$

where $\sigma_n$ is defined in (22),

$$\chi_k = -2\delta(e - f)f^{-1} P_k(0)P_n(0)$$ \hspace{1cm} (36)
and

\[ xP_n''(x) + [2\delta - f(x + f)^{-1}]P_n'(x) + \left[ n + \frac{\sigma_n - n}{x + e} \right]P_n(x) = \frac{1}{x + e}R_n(x), \tag{37} \]

where \( R_n(x) = \pi_{n-1} \).

**Proof.**

a) Let

\[ B_n(x) = P^{-1}(x) \frac{d}{dx} [x(x + e)P_n'(x)P(x)], \tag{a} \]

i.e. according to (4)

\[ B_n(x) = P^{-1}(x) \frac{d}{dx} \left\{ x(x + e)P_n'(x) [(x + e)^{2\varepsilon}(x + f)^{2\delta}] e^{-x} \right\}. \]

Then from the previous relation we get

\[ B_n(x) = x(x + e)P_n''(x) + \left[ -x(x + e) + (x + e) + x \right. \]
\[ + x(x + e)2\varepsilon(x + e)^{-1} + 2\delta x(x + e)(x + f)^{-1} \left] P_n'(x) \right. \]
\[ = x(x + e)P_n''(x) \]
\[ + \left[ -x^2 + (2 + 2\varepsilon - e)x + 2\delta x(x + e)(x + f)^{-1} + e \right] P_n'(x). \tag{b} \]

At first we take the expression \( x(x + e)(x + f)^{-1} \) which can be written in the form

\[ x(x + e)(x + f)^{-1} = [(x + f)(x + e) - f(x + e)](x + f)^{-1} \]
\[ = x + e - f(x + e)(x + f)^{-1} \]
\[ = x + e - f[(x + f) + (e - f)](x + f)^{-1} \]
\[ = x + e - f - f(e - f)(x + f)^{-1}. \tag{c} \]

Substituting (c) into (b) we get

\[ B_n(x) = x(x + e)P_n''(x) + \left\{ -x^2 + (2 + 2\varepsilon - e)x \right. \]
\[ + 2\delta \left[ x + e - f - f(e - f)(x + f)^{-1} \right] + e \left] P_n'(x) \right. \]
\[ = x(x + e)P_n''(x) + \left\{ -x^2 + (2 + 2\varepsilon + 2\delta - e)x \right. \]
\[ + e + 2\delta(e - f) + 2\delta f(e - f)(x + f)^{-1} \left] P_n'(x). \tag{d} \]

Because according to (30) and (31) there is

\[ x(x + e)P_n''(x) = n(n - 1)P_n(x) + \pi_{n-1}, \tag{e} \]
\[ -x^2P_n'(x) = (-nx + r_1^{(n)})P_n(x) + \pi_{n-1}, \tag{f} \]
\[ xP_n'(x) = nP_n(x) + \pi_{n-1}, \tag{g} \]

522
then from (d) under notation
\[ C_n(x) = 2\delta f(e - f)(x + f)^{-1}P_n'(x) \] (h)
we have
\[ B_n(x) - C_n(x) + \left[-n(n - 1) + nx - r^{(n)}_1 - (2 + 2\varepsilon + 2\delta - e)n\right]P_n(x) = \pi_{n-i}. \]
From there by using (21) (because \(2\alpha + 2\beta = 2\varepsilon + 2\delta\))
\[ B_n(x) - C_n(x) + [n(x + e) - n + \sigma_n]P_n(x) = \pi_{n-i} = A_n(x). \] (i)
According to (12) and (13)
\[ A_n(x) = \sum_{k=0}^{n-1} \chi_k P_k(x), \] (j)
where
\[ \chi_k = \int_{I} A_n(x)P_k(x)P(x) \, dx, \] (k)
so by using (i)
\[ \chi_k = \int_{I} \left[ B_n(x) - C_n(x) \right]P_k(x)P(x) \, dx + \int_{I} [nx + en - n + \sigma_n]P_n(x)P_k(x)P(x) \, dx. \] (l)
Using (a) for every \(k\):
\[ \int_{I} B_n(x)P_k(x)P(x) \, dx \]
\[ = \int_{I} P_k(x) \, d[x(x + e)P_n'(x)P(x)] \]
\[ = \left[ P_k(x)x(x + e)P_n'(x)P(x) \right]_{0}^{\infty} - \int_{I} P_n'(x)P_{n}(x)x(x + e)P(x) \, dx \]
\[ = \int_{I} P_n(x) \frac{d}{dx} \left[ x(x + e)P_k'(x)P(x) \right] \, dx \]
\[ = \int_{I} P_n(x)B_k(x)P(x) \, dx, \]
so
\[ \int_{I} B_n(x)P_k(x)P(x) \, dx = \int_{I} P_n(x)B_k(x)P(x) \, dx. \] (m)
Because according to (i)
\[ A_k(x) = B_k(x) - C_k(x) + kxP_k(x) + \pi_k(x) = \pi_{k-1} , \]
from there
\[ B_k(x) = C_k(x) - kxP_k(x) + \pi_k(x) . \]  

Then from (m) using (n)
\[ \int B_n(x)P_k(x) \, dx = \int P_n(x)C_k(x)P(x) \, dx - \int P_n(x)[kxP_k(x) + \pi_k]P(x) \, dx . \]

For \( k < n - 1 \) from (l) by using (o) and (5)
\[ \chi_k = \int [P_n(x)C_k(x) - C_n(x)P_k(x)]P(x) \, dx . \]

For \( k = n - 1 \) from (l) using (o), (5) and (28a)
\[ \chi_{n-1} = \int P_n(x)C_{n-1}(x)P(x) \, dx - (n-1)q_n + q_n - \int P_{n-1}(x)C_n(x)P(x) \, dx , \]
and thus
\[ \chi_{n-1} = \int [P_n(x)C_{n-1}(x) - C_n(x)P_{n-1}(x)]P(x) \, dx + q_n . \]

Let us denote in (h)
\[ K = 2\delta(e-f)f . \]

Then from (p) using (h) after integration by parts we have
\[ \chi_k = K \int (x + f)^{-1} [P_k'(x)P_n(x) - P_n'(x)P_k(x)]P(x) \, dx \]
\[ = K \int (x + f)^{-1} \frac{d}{dx} \left[ \frac{P_k(x)}{P_n(x)} \right] P_n^2(x)P(x) \, dx \]
\[ = -Kf^{-1}P_k(0)P_n(0)P(0) + K \int (x + f)^{-2}P_k(x)P_n(x)P(x) \, dx \]
\[ + K \int (x + f)^{-1}P_k(x)P_n(x)P(x) \, dx \]
\[ + K \int (x + f)^{-1}[2\delta(x+a)^{-1} + 2\delta(x-b)^{-1}]P_k(x)P_n(x)P(x) \, dx \]
\[ - 2K \int (x + f)^{-1}P_k(x)P_n'(x)P(x) \, dx . \]
LAGUERRE POLYNOMIALS WITH FOUR PARAMETERS

The integrals on the right-hand side of (t) equal to zero according to (5) besides the last one.

Substituting the last nonzero integral into (j) and using (10) we have

$$\sum_{k=0}^{n-1} P_k(x) \int (t+f)^{-1} P_k(t) P_n(t) P(t) \, dt = \int (t+f)^{-1} P_n(t) P_{n-1}(x,t) P(t) \, dt = 0,$$

because $P_{n-1}(x,t) = \pi_{n-1}$ at $t$ and $P_{n-1}(x,t)(t+f)^{-1} = \pi_{n-2}$.

Hence for $k < n-1$ there is

$$x_k = -K f^{-1} P_k(0) P_n(0) P(0), \quad f > 0.$$

Substituting (24) into (35) and dividing by positive expression $(x+e)$, we get (37).

REFERENCES


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