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ENTROPY OF COMPLETE FUZZY PARTITIONS

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ABSTRACT. This paper deals with a fuzzy generalization of notion of a probability space. An entropy and a conditional entropy of complete fuzzy partitions are defined. The main properties of such quantities are proved.

0. Introduction

In the classical probability theory [1] probability spaces $(X, \mathcal{S}, P)$ are studied. A σ-algebra $\mathcal{S}$ of subsets of a set X is the main notion of the Kolmogorov classical model of probability theory. The Kolmogorov probability model may be uniquely represented by a system of characteristic functions of subsets of a set X from the given σ-algebra $\mathcal{S}$, which have values in the closed interval $\langle 0, 1\rangle$. When an event f, say, is described vaguely, then by a fuzzy set f (fuzzy event f) we shall understand a real-valued function $f\colon X \to \langle 0, 1\rangle$, which describes the fuzziness of the event f. This is a basic idea of Zadeh's fuzzy sets theory [2].

In this paper we shall use a fuzzy generalization of notion of a probability space. A fuzzy generalization of a notion of measurable partition from the classical probability theory is a notion of complete fuzzy partition [3]. In this paper an entropy and a conditional entropy of complete fuzzy partitions are defined. The main properties of such quantities are stated.

1. Basic definitions and facts

Here we follow mainly [3]. Let $X \neq \emptyset$. By a soft fuzzy σ-algebra M we mean the set $M \subset \langle 0, 1\rangle^X$ satisfying the following conditions:

(1.1) if $1(x) = 1$ for any $x \in X$, then $1 \in M$;
(1.2) if $f \in M$, then $f' := 1 - f \in M$;
(1.3) if $1/2(x) = 1/2$ for any $x \in X$, then $1/2 \notin M$;

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(1.4) $\bigvee_{n=1}^{\infty} f_n := \sup_n f_n \in M$ for any $\{f_n\}_{n=1}^{\infty} \subset M$.

In the set M we define the partial ordering relation in the following way: $f \leq g$ if and only if $f(x) \leq g(x)$ for each $x \in X$. Using the complementation $'\colon f \to f'$ for any fuzzy subset $f \in M$, we see that the complementation $'$ satisfies two conditions:

(1.5) $(f')' = f$ for every $f \in M$;
(1.6) if $f \leq g$, then $g' \leq f'$.

So that M is a distributive σ-lattice with the complementation $'$, for which the de Morgan laws hold:

(1.7) $\left(\bigvee_{n=1}^{\infty} f_n\right)' = \bigwedge_{n=1}^{\infty} f_n'$;
(1.8) $\left(\bigwedge_{n=1}^{\infty} f_n\right)' = \bigvee_{n=1}^{\infty} f_n'$ for any sequence $\{f_n\}_{n=1}^{\infty} \subset M$.

Of course, here $\bigwedge_n f_n = \inf_n f_n$. In the fuzzy sets theory the fuzzy subset 1 is called *universum*, the fuzzy subset $0 = 1'$ is called *empty set* and all fuzzy subsets $f, g \in M$ such that $f \wedge g = 0$ are called *separated fuzzy sets.* Analogous weak notions (W-notions) are defined in [4] as follows: Each fuzzy subset $f \in M$ such that $f \geq 1 - f$ is called a *W-universum.* Each fuzzy subset $f \in M$ such that $f \leq 1 - f$ is called a *W-empty set.* All fuzzy subsets $f, g \in M$ such that $f \leq 1 - g$ are called *W-separated fuzzy sets.*

LEMMA 1.1. *A fuzzy subset $f \in M$ is a W-universum if and only if there exists a fuzzy subset $g \in M$ such that $f = g \vee (1 - g)$* [4].

LEMMA 1.2. *Let a finite or infinite sequence $\{f_n\}$ of fuzzy subsets from M be given. Then the fuzzy subsets g_n defined by*

$$g_n = \begin{cases} f_1, & \text{if } n = 1, \\ f_n \wedge \left(\bigvee_{i=1}^{n-1} f_i\right)', & \text{if } n > 1 \end{cases} \tag{1.9}$$

are pairwise W-separated. Furthermore, if $\bigvee_n f_n$ is a W-universum, then $\bigvee_n g_n$ is a W-universum [3].

A fuzzy P-measure on M is a mapping $m\colon M \to \langle 0, \infty)$ fulfilling the following conditions:

(1.10) $m\big(f \vee (1 - f)\big) = 1$ for every $f \in M$;
(1.11) if $\{f_n\}_{n=1}^{\infty}$ is a finite or infinite sequence of pairwise W-separated fuzzy subsets from M, then $m\left(\bigvee_{n=1}^{\infty} f_n\right) = \sum_{n=1}^{\infty} m(f_n)$.

Each above described triplet (X, M, m) is called in the fuzzy theory a *soft fuzzy probability space.*

E x a m p l e 1.1. Let $(X, \mathcal{S}, P)$ be a probability space in the sense of the classical probability theory. Put $M = \{\chi_A ;\ A \in \mathcal{S}\}$ (χ_A is the characteristic function of the set $A \in \mathcal{S}$). If we define the mapping $m\colon M \to \langle 0,1\rangle$ by the equality $m(\chi_A) = P(A)$, then the triplet (X, M, m) is a soft fuzzy probability space.

E x a m p l e 1.2. Let $X = \langle 0,1\rangle$, $M = \{f, f', f \vee f', f \wedge f', 0, 1\}$, where $f\colon X \to \langle 0,1\rangle$, $f(x) = x$ for each $x \in X$. If we define the mapping $m\colon M \to \langle 0,1\rangle$ by the equalities $m(f) = m(f') = 1/2$, $m(1) = m(f \vee f') = 1$, $m(0) = m(f \wedge f') = 0$, then the triplet (X, M, m) is a soft fuzzy probability space.

It is easy to see that any fuzzy P-measure m has the following properties:

(1.12) $m(f') = 1 - m(f)$ for every $f \in M$.

(1.13) m is a nondecreasing function, i.e. if $f, g \in M$, $f \leq g$, then $m(f) \leq m(g)$.

(1.14) Let $g \in M$ be given. Then $m(f \wedge g) = m(f)$ for all $f \in M$ if and only if $m(g) = 1$.

(1.15) If $f, g \in M$ are W-separated, then $m(f \wedge g) = 0$.

The mapping $m(\cdot/g)\colon M \to \langle 0, \infty)$ defined for each $g \in M$, $m(g) > 0$, by the equality $m(f/g) = \dfrac{m(f \wedge g)}{m(g)}$, is a P-measure on M (see [3]).

The monotonicity of fuzzy P-measure implies that this measure transforms M into the interval $\langle 0,1\rangle$.

2. Entropy of complete fuzzy partitions

Let any soft fuzzy probability space (X, M, m) be given. K a b a l a and W r o c i n s k i [5] mean by a *complete partition* each finite or infinite sequence of pairwise separated fuzzy subsets $\{f_n\}$ such that $\bigvee_n f_n$ is a universum. If $\{f_n\} \subset M$ is a complete partition, then for every $x \in X$ there exists i_0 such that $f_{i_0}(x) = 1$ and for every $j \neq i_0$, $f_j(x) = 0$ holds. This means that $\{f_n\}$ contains only crisp subsets of the set X and hence the mentioned definition is not useful for considerations on fuzzy subsets. Therefore in this contribution we shall work with the following notion:

DEFINITION 2.1. [3] *Each finite or infinite sequence of pairwise W-separated fuzzy subsets $\{f_n\} \subset M$ such that $\bigvee_n f_n$ is a W-universum is called a complete fuzzy partition.*

It is easy to see that a partition described in Definition 2.1 contains uncrisp subsets, in general. Namely, if we have any sequence $\{f_n\}$ such that $\bigvee_n f_n$ is a W-universum, then we can find the complete fuzzy partition $\{g_n\}$ defined by (1.9). The sequence $\{f_n\}$ described above always exists. So, if it does not contain crisp subsets only, then the generated partition $\{g_n\}$ contains uncrisp subsets.

LEMMA 2.1. *Let $\mathcal{A} = \{f_i\}$ and $\mathcal{B} = \{g_j\}$ be two complete fuzzy partitions. Then the set $\mathcal{A} \vee \mathcal{B} := \{f_i \wedge g_j\,;\ f_i \in \mathcal{A},\ g_j \in \mathcal{B}\}$ is a complete fuzzy partition, too.*

P r o o f . It is easy to see that $\mathcal{A} \vee \mathcal{B}$ is a set of pairwise W-separated elements (see [6]). Moreover, $\bigvee_i \bigvee_j (f_i \wedge g_j) = \left(\bigvee_i f_i\right) \wedge \left(\bigvee_j g_j\right) \geq 1/2$, so that $\bigvee_i \bigvee_j (f_i \wedge g_j)$ is a W-universum.

In the set $\mathcal{F}$ of all complete fuzzy partitions we can define the relation $\leq$ in the following way: for every $\mathcal{A}, \mathcal{B} \in \mathcal{F}$, $\mathcal{A} \leq \mathcal{B}$ if and only if for every $g \in \mathcal{B}$ there exists $f \in \mathcal{A}$ such that $g \leq f$. In this case we say that $\mathcal{B}$ is the *refinement of* $\mathcal{A}$. Since $\mathcal{A} \leq \mathcal{A} \vee \mathcal{B}$, $\mathcal{B} \leq \mathcal{A} \vee \mathcal{B}$, we shall read the symbol $\mathcal{A} \vee \mathcal{B}$ a *common refinement of* $\mathcal{A}$ *and* $\mathcal{B}$. Each $\mathcal{A} = \{f_1, f_2, \dots\} \in \mathcal{F}$ represents in the sense of classical probability theory the random experiment with finite or countable number of outcomes with the probability distribution $p_i = m(f_i)$, $f_i \in \mathcal{A}$, since $p_i \geq 0$ and $\sum_i p_i = \sum_i m(f_i) = m\left(\bigvee_i f_i\right) = 1$ (see Lemma 1.1 and (1.10)).

We define an *entropy of any experiment* $\mathcal{A} = \{f_1, f_2, \dots\} \in \mathcal{F}$ by Shannon's formula:

$$H_m(\mathcal{A}) = -\sum_i F\big(m(f_i)\big)\,, \qquad \text{where} \quad F\colon \langle 0, \infty) \to \mathbb{R}, \tag{2.1}$$

$$F(x) = \begin{cases} x \log x\,, & \text{if} \quad x > 0\,, \\ 0\,, & \text{if} \quad x = 0\,, \end{cases}$$

$H_m(\mathcal{A})$ is not necessarily finite.

If $\mathcal{A}, \mathcal{B} \in \mathcal{F}$, $\mathcal{A} = \{f_i\}$, $\mathcal{B} = \{g_j\}$, we define a *conditional entropy*

$$H_m\left(\mathcal{B}/_{\mathcal{A}}\right) = -\sum_i \sum_j m(f_i) F\left(\mathring{\mathrm{m}}(g_j/f_i)\right), \tag{2.2}$$

where

$$\mathring{\mathrm{m}}(g_j/f_i) = \begin{cases} m(g_j/f_i)\,, & \text{if} \quad m(f_i) > 0\,, \\ 0\,, & \text{if} \quad m(f_i) = 0\,. \end{cases}$$

The following example shows that the notion of entropy of complete fuzzy partition is a generalization of S h a n n o n 's entropy of a measurable partition [7].

E x a m p l e 2.1. Let $(X, \mathcal{S}, P)$ be a probability space in the sense of the classical probability theory. Let us consider the soft fuzzy probability space (X, M, m) from Example 1.1. Then the system $\mathcal{F}$ contains all partitions of the type $\{\chi_{A_1}, \dots, \chi_{A_k}\}$, where $A_i \in \mathcal{S}$ $(i = 1, \dots, k)$, $A_i \cap A_j = \emptyset$ $(i \neq j)$ and $\bigcup_{i=1}^{k} A_i = X$. The entropy of a complete fuzzy partition $\mathcal{A} = \{\chi_{A_1}, \dots, \chi_{A_k}\}$ is the number $H_m(\mathcal{A}) = -\sum_{i=1}^{k} F(m(\chi_{A_i})) = -\sum_{i=1}^{k} F(P(A_i))$, which is the Shannon entropy of measurable partition $\{A_1, \dots, A_k\}$ of a space $(X, \mathcal{S}, P)$.

E x a m p l e 2.2. Let (X, M, m) be a soft fuzzy probability space from Example 1.2. Then the set $\mathcal{A} = \{f, f'\}$ is a complete fuzzy partition with the non-zero entropy $H_m(\mathcal{A}) = \log 2$.

THEOREM 2.1. *The entropy H_m has the following properties:*

(2.3) $H_m(\mathcal{A}) \geq 0$ *for each* $\mathcal{A} \in \mathcal{F}$;
(2.4) *if* $\mathcal{A}, \mathcal{B} \in \mathcal{F}$, $\mathcal{A} \leq \mathcal{B}$, *then* $H_m(\mathcal{A}) \leq H_m(\mathcal{B})$;
(2.5) $H_m(\mathcal{A}) \leq H_m(\mathcal{A} \vee \mathcal{B})$, *for every* $\mathcal{A}, \mathcal{B} \in \mathcal{F}$.

P r o o f . The property (2.3) is evident. Let $\mathcal{A}, \mathcal{B} \in \mathcal{F}$, $\mathcal{A} = \{f_i\}$, $\mathcal{B} = \{g_j\}$, $\mathcal{A} \leq \mathcal{B}$. Then for every $g_j \in \mathcal{B}$ there exists $f_{i_0} \in \mathcal{A}$ such that $g_j \leq f_{i_0}$. Since $\mathcal{A}$ is a system of pairwise W-separated elements we have $g_j = g_j \wedge f_{i_0} \leq f_{i_0} \leq 1 - f_i$ for every $i \neq i_0$. Therefore by (1.15) we obtain $F(m(g_j)) = \sum_i F(m(g_j \wedge f_i))$.

Put $\alpha = \{(i,j);\ m(f_i \wedge g_j) > 0\}$, $\beta = \{i;\ m(f_i) > 0\}$. Then we have

$$\begin{aligned}
H_m(\mathcal{B}) &= -\sum_j F(m(g_j)) = -\sum_j \sum_i F(m(g_j \wedge f_i)) \\
&= -\sum_{(i,j)\in\alpha} m(f_i \wedge g_j) \cdot \log m(f_i \wedge g_j) \\
&= -\sum_{(i,j)\in\alpha} m(f_i \wedge g_j) \cdot \log m(g_j / f_i) - \sum_{(i,j)\in\alpha} m(f_i \wedge g_j) \cdot \log m(f_i) \\
&\geq -\sum_{i\in\beta} \log m(f_i) \sum_j m(f_i \wedge g_j) \\
&= -\sum_{i\in\beta} m(f_i) \log m(f_i) = -\sum_i F(m(f_i)) = H_m(\mathcal{A}).
\end{aligned}$$

Since $\mathcal{A} \le \mathcal{A} \vee \mathcal{B}$, the inequality (2.5) is a simple consequence of (2.4).

THEOREM 2.2. *$H_m(\mathcal{B} \vee \mathcal{C}/_{\mathcal{A}}) = H_m(\mathcal{C}/_{\mathcal{A} \vee \mathcal{B}}) + H_m(\mathcal{B}/_{\mathcal{A}})$ for every $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \mathcal{F}$.*

Proof. Let $\mathcal{A} = \{f_i\}$, $\mathcal{B} = \{g_j\}$, $\mathcal{C} = \{h_k\}$. If $m(f_i \wedge g_j) > 0$, then we have

$$\begin{aligned} m(g_j \wedge h_k / f_i) &= \frac{m(g_j \wedge h_k \wedge f_i)}{m(f_i)} = \frac{m(g_j \wedge h_k \wedge f_i)}{m(f_i \wedge g_j)} \frac{m(f_i \wedge g_j)}{m(f_i)} \\ &= m(h_k / f_i \wedge g_j) \cdot m(g_j / f_i) . \end{aligned}$$

Moreover, it is easy to see that the function F satisfies the condition

$$F(x \cdot y) = x \cdot F(y) + y \cdot F(x) \qquad \text{for each} \quad x, y \in \langle 0, \infty) . \tag{2.6}$$

Therefore we obtain

$$\begin{aligned} H_m(\mathcal{B} \vee \mathcal{C}/_{\mathcal{A}}) &= -\sum_i \sum_j \sum_k m(f_i) \cdot F(\mathring{m}(g_j \wedge h_k / f_i)) \\ &= -\sum_i \sum_j \sum_k m(f_i) \cdot F(\mathring{m}(h_k / f_i \wedge g_j) \cdot \mathring{m}(g_j / f_i)) \\ &= -\sum_i \sum_j \sum_k m(f_i) \Big(\mathring{m}(g_j / f_i) \cdot F(\mathring{m}(h_k / f_i \wedge g_j)) \\ &\qquad + \mathring{m}(h_k / f_i \wedge g_j) \cdot F(\mathring{m}(g_j / f_i)) \Big) \\ &= -\sum_i \sum_j \sum_k m(f_i) \cdot \mathring{m}(g_j / f_i) \cdot F(\mathring{m}(h_k / f_i \wedge g_j)) \\ &\quad - \sum_i \sum_j \sum_k m(f_i) \cdot \mathring{m}(h_k / f_i \wedge g_j) \cdot F(\mathring{m}(g_j / f_i)) \\ &= -\sum_i \sum_j \sum_k m(f_i \wedge g_j) \cdot F(\mathring{m}(h_k / f_i \wedge g_j)) \\ &\quad - \sum_i \sum_j m(f_i) \sum_k \mathring{m}(h_k / f_i \wedge g_j) \cdot F(\mathring{m}(g_j / f_i)) \\ &= H_m(\mathcal{C}/_{\mathcal{A} \vee \mathcal{B}}) + H_m(\mathcal{B}/_{\mathcal{A}}) . \end{aligned}$$

THEOREM 2.3. *Let* $\mathcal{A}, \mathcal{B} \in \mathcal{F}$, $\mathcal{A} \le \mathcal{B}$. *Then* $H_m(\mathcal{A}/_{\mathcal{C}}) \le H_m(\mathcal{B}/_{\mathcal{C}})$ *for each* $\mathcal{C} \in \mathcal{F}$.

Proof. Put $\mathcal{A} = \{f_i\}$, $\mathcal{B} = \{g_j\}$, $\mathcal{C} = \{h_k\}$. Since $\mathcal{A} \le \mathcal{B}$ we have $F\big(\mathring{\mathrm{m}}(g_j/h_k)\big) = \sum\limits_i F\big(\mathring{\mathrm{m}}(g_j \wedge f_i/h_k)\big)$. This fact along with (2.6) implies

$$\begin{aligned}
H_m(\mathcal{B}/_{\mathcal{C}}) &= -\sum_j \sum_k m(h_k) \cdot F\big(\mathring{\mathrm{m}}(g_j/h_k)\big) \\
&= -\sum_j \sum_k m(h_k) \sum_i F\big(\mathring{\mathrm{m}}(g_j \wedge f_i/h_k)\big) \\
&= -\sum_j \sum_k m(h_k) \sum_i F\big(\mathring{\mathrm{m}}(g_j/f_i \wedge h_k) \cdot \mathring{\mathrm{m}}(f_i/h_k)\big) \\
&= -\sum_j \sum_k m(h_k) \sum_i \mathring{\mathrm{m}}(f_i/h_k) \cdot F\big(\mathring{\mathrm{m}}(g_j/f_i \wedge h_k)\big) \\
&\quad -\sum_j \sum_k m(h_k) \sum_i \mathring{\mathrm{m}}(g_j/f_i \wedge h_k) \cdot F\big(\mathring{\mathrm{m}}(f_i/h_k)\big) \\
&\ge -\sum_i \sum_k m(h_k) \sum_j \mathring{\mathrm{m}}(g_j/f_i \wedge h_k) \cdot F\big(\mathring{\mathrm{m}}(f_i/h_k)\big) = H_m(\mathcal{A}/_{\mathcal{C}}) .
\end{aligned}$$

LEMMA 2.2. *Let* $\mathcal{A}, \mathcal{B} \in \mathcal{F}$, $\mathcal{A} = \{f_i\}$, $\mathcal{B} = \{g_j\}$, $\mathcal{A} \le \mathcal{B}$. *Then for every* $h \in M$ *it holds that* $m\Big(h \wedge \Big(\bigvee\limits_{j \in \delta_i} g_j\Big)\Big) = m(h \wedge f_i)$, *where* $\delta_i = \{j\, ;\ g_j \le f_i\}$, $i = 1, 2, \dots$.

Proof. Since $\bigvee\limits_{j \in \delta_i} g_j \le f_i$, the monotonicity of fuzzy P-measure implies the inequality

$$m\Big(h \wedge \Big(\bigvee_{j \in \delta_i} g_j\Big)\Big) \le m(h \wedge f_i) \qquad (i = 1, 2, \dots) .$$

Let us suppose that the assertion of the proved lemma is not true. This means that there exists i_0 such that $m\Big(h \wedge \Big(\bigvee\limits_{j \in \delta_{i_0}} g_j\Big)\Big) < m(h \wedge f_{i_0})$. Then we get

$$\sum_i m\Big(h \wedge \Big(\bigvee_{j \in \delta_i} g_j\Big)\Big) < \sum_i m(h \wedge f_i) .$$

This conclusion is contradictory, because by (1.14) we have

$$\sum_i m\Big(h \wedge \Big(\bigvee_{j\in\delta_i} g_j\Big)\Big) = m\Big(h \wedge \Big(\bigvee_j g_j\Big)\Big) = m(h)$$

and

$$\sum_i m(h \wedge f_i) = m\Big(h \wedge \Big(\bigvee_i f_i\Big)\Big) = m(h).$$

THEOREM 2.4. *Let $\mathcal{A}, \mathcal{B}$ be two complete fuzzy partitions, $\mathcal{A} \le \mathcal{B}$. Then $H_m(\mathcal{C}/_{\mathcal{A}}) \ge H_m(\mathcal{C}/_{\mathcal{B}})$ for each $\mathcal{C} \in \mathcal{F}$.*

Proof. The function F is convex and therefore for any convex combination $\sum_j \alpha_j x_j$ (i.e. such that $\alpha_j \ge 0$, $j = 1,2,\dots$, $\sum_j \alpha_j = 1$) of elements $x_j \in \langle 0,1\rangle$ there holds

$$F\Big(\sum_j \alpha_j x_j\Big) \le \sum_j \alpha_j F(x_j). \tag{2.7}$$

Let $\mathcal{A} = \{f_i\}$, $\mathcal{B} = \{g_j\}$, $\mathcal{C} = \{h_k\}$. Denote by

$$\alpha = \{i\,;\ m(f_i) > 0\}, \qquad \beta = \{j\,;\ m(g_j) > 0\}, \qquad \gamma = \{k\,;\ m(h_k) > 0\},$$

$\delta_i = \{j\,;\ g_j \le f_i\}$, $i = 1,2,\dots$. Put $\alpha_j = \mathring{m}(g_j/f_i)$, $x_j = \mathring{m}(h_k/g_j)$, i, k – fixed, $j = 1,2,\dots$. Let $i \in \alpha$. Then

$$\sum_{j\in\beta} \alpha_j = \sum_{j\in\beta} m(g_j/f_i) = \sum_j m(g_j/f_i) = \sum_j \frac{m(g_j \wedge f_i)}{m(f_i)}$$
$$= \frac{m\Big(\Big(\bigvee_j g_j\Big) \wedge f_i\Big)}{m(f_i)} = 1.$$

By the preceding lemma we get

$$\sum_{j\in\beta} \alpha_j x_j = \sum_{j\in\beta} m(g_j/f_i)\cdot m(h_k/g_j) = \sum_{j\in\beta} \frac{m(g_j \wedge f_i)}{m(f_i)} \frac{m(h_k \wedge g_j)}{m(g_j)}$$
$$= \sum_{j\in\delta_i} \frac{m(g_j)}{m(f_i)} \frac{m(h_k \wedge g_j)}{m(g_j)} = \frac{m\Big(h_k \wedge \Big(\bigvee_{j\in\delta_i} g_j\Big)\Big)}{m(f_i)}$$
$$= \frac{m(h_k \wedge f_i)}{m(f_i)} = \mathring{m}(h_k/f_i).$$

Evidently $\sum_{j\in\beta} \alpha_j x_j = \mathring{\mathrm{m}}(h_k/f_i)$ also for $i \notin \alpha$. By (2.7) we obtain $F\big(\mathring{\mathrm{m}}(h_k/f_i)\big) \leq \sum_j \mathring{\mathrm{m}}(g_j/f_i) \cdot F\big(\mathring{\mathrm{m}}(h_k/g_j)\big)$. If we multiply this inequality with $-m(f_i)$, we get

$$-m(f_i)\cdot F\big(\mathring{\mathrm{m}}(h_k/f_i)\big) \geq -m(f_i)\sum_j \mathring{\mathrm{m}}(g_j/f_i)\cdot F\big(\mathring{\mathrm{m}}(h_k/g_j)\big), \qquad i,k = 1,2,\dots .$$

Hence

$$\begin{aligned} H_m\big(\mathcal{C}/_{\mathcal{A}}\big) &= -\sum_i\sum_k m(f_i)\cdot F\big(\mathring{\mathrm{m}}(h_k/f_i)\big) \\ &\geq -\sum_i\sum_k\sum_j m(f_i)\cdot \mathring{\mathrm{m}}(g_j/f_i)\cdot F\big(\mathring{\mathrm{m}}(h_k/g_j)\big) \\ &= -\sum_j\sum_k m(g_j)\cdot F\big(\mathring{\mathrm{m}}(h_k/g_j)\big) = H_m\big(\mathcal{C}/_{\mathcal{B}}\big). \end{aligned}$$

THEOREM 2.5. *$H_m\big(\mathcal{A}/_{\mathcal{B}}\big) \leq H_m(\mathcal{A})$ for each $\mathcal{A},\mathcal{B} \in \mathcal{F}$.*

Proof. Put $\mathcal{E} = \{1\}$, $\mathcal{A} = \{f_i\}$. Then

$$H_m\big(\mathcal{A}/_{\mathcal{E}}\big) = -\sum_i m(1)\cdot F\big(m(f_i/1)\big) = -\sum_i F\big(m(f_i)\big) = H_m(\mathcal{A}).$$

Since any complete fuzzy partition $\mathcal{B}$ is a refinement of the partition $\mathcal{E} = \{1\}$, by means of Theorem 2.4 we obtain $H_m(\mathcal{A}) = H_m\big(\mathcal{A}/_{\mathcal{E}}\big) \geq H_m\big(\mathcal{A}/_{\mathcal{B}}\big)$.

THEOREM 2.6. *For each $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C} \in \mathcal{F}$, we have:*

$$H_m\big(\mathcal{B}\vee\mathcal{C}/_{\mathcal{A}}\big) \leq H_m\big(\mathcal{B}/_{\mathcal{A}}\big) + H_m\big(\mathcal{C}/_{\mathcal{A}}\big).$$

Proof. Since $\mathcal{A} \leq \mathcal{A}\vee\mathcal{B}$ for each $\mathcal{A}$, $\mathcal{B} \in \mathcal{F}$, according to Theorem 2.4 we have the inequality

$$H_m\big(\mathcal{C}/_{\mathcal{A}\vee\mathcal{B}}\big) \leq H_m\big(\mathcal{C}/_{\mathcal{A}}\big).$$

This along with Theorem 2.2 implies

$$H_m\big(\mathcal{B}\vee\mathcal{C}/_{\mathcal{A}}\big) = H_m\big(\mathcal{C}/_{\mathcal{A}\vee\mathcal{B}}\big) + H_m\big(\mathcal{B}/_{\mathcal{A}}\big) \leq H_m\big(\mathcal{C}/_{\mathcal{A}}\big) + H_m\big(\mathcal{B}/_{\mathcal{A}}\big).$$

We have seen that the conditional entropy of complete fuzzy partitions defined here fulfils all properties analogous to the properties of entropy of measurable partitions in the crisp case.

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