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ON I-CONTINUITY AND I-SEMICONTINUITY POINTS

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Let $f\colon X \to R$ be a real function. The purpose of the present paper is to study the relation between the set $C(f)$ of all points at which f is continuous, the set $C_I(f)$ of all points at which f is I-continuous, the set $S_I(f)$ of all points at which f is I-upper semicontinuous and the set $S_I^1(f)$ of all points at which f is I-lower semicontinuous.

Let X be a Polish space and $\mathscr{I} \subseteq \mathscr{P}(X)$ be a σ-complete ideal which has the following properties:
(a) if $x \in X$ then $\{x\} \in \mathscr{I}$,
(b) if $\emptyset \neq U \subseteq X$ is open then $U \notin \mathscr{I}$.
We say that a subset $A \subseteq X$ is $\mathscr{I}$-small at point $p \in X$ iff there exists a neighbourhood $U(p)$ of p such that $U(p) \cap A \in \mathscr{I}$. We denote by $d_I(A)$ the set of all points at which A is not $\mathscr{I}$-small, namely

$$d_I(A) = \{p\colon\ \forall V(p) \quad V(p) \cap A \notin \mathscr{I}\}$$

($d_I(A)$ is A^* in the sense of Hashimoto [2]).

The family of subsets of X

$$\{G - I\colon\ G \text{ is open and } I \in \mathscr{I}\}$$

is a new topology on X (it is $*$-topology in the sense of Hashimoto or "$\mathscr{I}$-topology" in the sense of Vaidyanathoswamy — c. f. [2], [4], [7], [11]).

We say that a function $f\colon X \to R$ is I-continuous (semicontinuous) iff f is continuous (semicontinuous) in the $\mathscr{I}$-topology.
We use the following notation:

$$I\text{-}\liminf_{t \to x} f(t) = \sup\{y \in R\colon\ x \notin d_I(\{t\colon\ y > f(t)\})\},$$

$$I\text{-}\limsup_{t \to x} f(t) = \inf\{y \in R\colon\ x \notin d_I(\{t\colon\ y < f(t)\})\},$$

$C(f)$ is the set of all points at which f is continuous,

$$C_I(f) = \{x \in X\colon\ I\text{-}\liminf_{t \to x} f(t) = f(x) = I\text{-}\limsup_{t \to x} f(t)\},$$

$$S_I(f)=\{x\in X:\ I\text{-}\limsup_{t\to x} f(t)\leq f(x)\},$$

$$S_I^1(f)=\{x\in X:\ I\text{-}\liminf_{t\to x} f(t)\geq f(x)\},$$

$$T_I(f)=\{x\in X:\ I\text{-}\limsup_{t\to x} f(t)< f(x)\},$$

$$T_I^1(f)=\{x\in X:\ I\text{-}\liminf_{t\to x} f(t)> f(x)\}.$$

Let $\psi_I(A)$ denotes the set of all points at which the set $X-A$ is $\mathcal{I}$-small, namely

$$\psi_I(A)=\{x:\ \exists U(x)\quad U(x)-A\in\mathcal{I}\}.$$

($\psi_I(A)=X-(X-A)^*$ in the sense of Hashimoto)

Notice that

(i) for every $A\subseteq X$ the set $\psi_I(A)$ is open,

(ii) if $A\subseteq B$ then $\psi_I(A)\subseteq\psi_I(B)$,

(iii) for every $A\subseteq X$ $\psi_I(A)-A\in\mathcal{I}$.

In fact, if $(U_n)_{n\in N}$ is a basis of X and $A_n=\{x\in\psi_I(A):\ U(x)=U_n\}$ then $A_n-A\subseteq U_n-A\in\mathcal{I}$ and $\psi_I(A)-A=\bigcup_{n\in N} A_n-A\in\mathcal{I}$.

We shall use the following simple facts.

Fact 0. *For every function f: $X\to R$ we have*

$$I\text{-}\limsup_{t\to x} f(t)=-I\text{-}\liminf_{t\to x}(-f)(t).$$

Fact 1. *If function f, g: $X\to R$ are bounded then*

a) $I\text{-}\limsup_{t\to x} f(t)+I\text{-}\limsup_{t\to x} g(t)\geq I\text{-}\limsup_{t\to x}(f+g)(t)\geq I\text{-}\limsup_{t\to x} f(t)+$

$+I\text{-}\liminf_{t\to x} g(t)$,

b) $I\text{-}\liminf_{t\to x} f(t)+I\text{-}\liminf_{t\to x} g(t)\leq I\text{-}\liminf_{t\to x}(f+g)(t)\leq I\text{-}\liminf_{t\to x} f(t)+$

$+I\text{-}\limsup_{t\to x} g(t)$.

Proof. a) Assume that $I\text{-}\limsup_{t\to x} f(t)=a$ and $I\text{-}\limsup_{t\to x} g(t)=b$. Then $x\notin d_I\left(\left\{t:\ f(t)>a+\frac{\varepsilon}{2}\right\}\right)\cup d_I\left(\left\{t:\ g(t)>b+\frac{\varepsilon}{2}\right\}\right)$ for all $\varepsilon>0$. Hence $x\notin d_I(\{t:\ f(t)+g(t)>a+b+\varepsilon\})$ for all $\varepsilon>0$ and

$$I\text{-}\limsup_{t\to x}(f+g)(t)\leq a+b.$$

$$I\text{-}\limsup_{t\to x} f(t) = I\text{-}\limsup_{t\to x}\,[(f+g)(t) - g(t)] \leq I\text{-}\limsup_{t\to x}\,(f+g)(t) +$$

$$+ I\text{-}\limsup_{t\to x}\,(-g)(t) = I\text{-}\limsup_{t\to x}\,(f+g)(t) - I\text{-}\liminf_{t\to x} g(t).$$

Hence $I\text{-}\limsup_{t\to x} f(t) + I\text{-}\liminf_{t\to x} g(t) \leq I\text{-}\limsup_{t\to x}\,(f+g)(t)$.
The case (b) is similar.

Fact 2. *If $\sum_{n\in N} f_n(t)$ is uniformly convergent in some neighbourhood U of x then*

$$I\text{-}\limsup_{t\to x} \sum_{n\in N} f_n(t) \leq \sum_{n\in N} I\text{-}\limsup_{t\to x} f_n(t)$$

and

$$\text{I-}\liminf_{t\to x} \sum_{n\in N} f_n(t) \geq \sum_{n\in N} I\text{-}\liminf_{t\to x} f_n(t).$$

This fact follows from Fact 1.

Lemma 0. *If $D\subseteq X$ is a G_δ set then there exists $E\subseteq X$ such that E is a G_δ set, $D\subseteq E$, $E-D\in \mathcal{I}$, $E=\bigcap_{n\in N} G_n$, G_n are open, $G_{n+1}\subseteq G_n$ and $E=\bigcap_{n\in N} \psi_I(G_n)$.*

Proof. Assume that $D=\bigcap_{n\in N} H_n$, H_n is open and $H_{n+1}\subseteq H_n$. Then $\psi_I(H_{n+1})\subseteq \psi_I(H_n)$, $\psi_I(H_n)$ is open and $\psi_I(H_n)-H_n\in\mathcal{I}$.
We define E as follows:

$$E=\bigcap_{n\in N} \psi_I(H_n).$$

Then $\psi_I(\psi_I(H_n))=\psi_I(H_n)$ and $E-D=\bigcap_{n\in N}\psi_I(H_n)-\bigcap_{n\in N} H_n\subseteq \bigcup_{n\in N}(\psi_I(H_n)-H_n)$.

Remark. If $\mathcal{I}$ is the ideal of the sets of first category then $\psi_I(A)=A$ means that A is a regular open set i. e. $A=\psi_I(A)$ iff Int Cl $A=A$.

Proof. If $A=\psi_I(A)$ then A is open, so $A\subseteq$ Int Cl A. If $x\in$ Int Cl A then there exists a neighbourhood U of x such that $U\subseteq$ Cl A Since A is open and dense in u, A is residual in U and $U-A\in\mathcal{I}$. Hence $x\in\psi_I(A)=A$.

If Int Cl $A=A$ then A is open and $A\subseteq\psi_I(A)$. If $x\in\psi_I(A)$ then there exists a neighbourhood U of x such that $U-A\in\mathcal{I}$. Then $U\subseteq$ Cl A and $x\in$ Int Cl $A=A$.

I.

Fact 0. *$C(f)$ is a G_δ set.*
It is the well known fact (cf. [9]).

Fact 1. $T_I(f) \cup T_I^1(f) \in \mathcal{J}$.

Proof. Let $(U_n)_{n\in N}$ (resp. $(V_n)_{n\in N}$) be a countable basis of X (resp. R). If $x \in T_I(f)$ then $f(x) > I\text{-}\limsup_{t\to x} f(t)$. Thus there exist $n(x)$, $m(x) \in N$ such that $x \in U_{n(x)}$, $f(x) \in V_{m(x)}$ and $U_{n(x)} \cap f^{-1} * V_{m(x)} \in \mathcal{J}$. Let $A(n, m) = \{x \in T_I(f)\colon n(x) = n$ and $m(x) = m\}$. Then for every $x \in A(n, m)$ we have $A(n, m) \subseteq U_{n(x)} \cap f^{-1} * V_{m(x)} \in \mathcal{J}$. Hence $T_I(f) = \bigcup_{n, m \in N} A(n, m) \in \mathcal{J}$. Similarly, $T_I^1(f) \in \mathcal{J}$. (Z. Grande in [1] proved this fact for $X = R$ and the ideal of all sets of the first category.)

Fact 2. *There exists a G_δ set D such that.*

$$C_I(f) = D - (T_I(f) \cup T_I^1(f)).$$

Proof. We define D as follows:

$$D = \{x \in X\colon I\text{-}\liminf_{t\to x} f(t) = I\text{-}\limsup_{t\to x} f(t)\}.$$

It is clear that $C_I(f) = D - (T_I(f) \cup T_I^1(f))$. We shall prove that $X - D$ is a F_σ set.

$$X - D = \{x \in X\colon I\text{-}\liminf_{t\to x} f(t) < I\text{-}\limsup_{t\to x} f(t)\}.$$

Let $A(p, q) = \{x \in X\colon I\text{-}\liminf_{t\to x} f(t) \leqslant p$ and $I\text{-}\limsup_{t\to x} f(t) \geqslant q\}$. For each $p, q \in Q$ the set $A(p, q)$ is closed.

Indeed, if $x_n \xrightarrow[n\to\infty]{} x$ and $\{x_n\colon n \in N\} \subseteq A(p, q)$ then $I\text{-}\liminf_{t\to x} f(t) \leqslant p$ and $I\text{-}\limsup_{t\to x} f(t) \geqslant q$.

Since $X - D = \bigcup_{p, q \in Q} A(p, q)$, $X - D$ is a F_σ set.

Fact 3. $C_I(f) - C(f) \subseteq \mathrm{Cl}\,(T_I(f) \cup T_I^1(f))$.

Proof. Assume that $x \in C_I(f)$ and there exists a neighbourhood U of x such that $U \cap (T_I(f) \cup T_I^1(f)) = \emptyset$ i. e. for each $t \in U$

$$I\text{-}\liminf_{s\to t} f(s) \leqslant f(t) \leqslant I\text{-}\limsup_{s\to t} f(t) \qquad \text{and}$$

$$I\text{-}\liminf_{t\to x} f(t) = f(x) = I\text{-}\limsup_{t\to x} f(t).$$

Notice that:

(i) $I\text{-}\liminf_{t\to x} f(t) \leqslant \liminf_{t\to x} \left(I\text{-}\liminf_{s\to t} f(s)\right)$ and

(ii) $I\text{-}\limsup_{t\to x} f(t) \geq \limsup_{t\to x} \left(I\text{-}\limsup_{s\to t} f(s) \right)$.

In fact, let $x_n \xrightarrow[n\to\infty]{} x$ such that

$$\lim_{n\to\infty} \left(I\text{-}\liminf_{s\to x_n} f(s) \right) = \liminf_{t\to x} \left(I\text{-}\liminf_{s\to t} f(s) \right) = g.$$

Suppose that $I\text{-}\liminf_{t\to x} f(t) > g$. Then for some $\varepsilon > 0$ $I\text{-}\liminf_{t\to x} f(t) > g + \varepsilon$ i. e. there exists a neighbourhood U of x such that $\{t \in U\colon f(t) < g + \varepsilon\} \in \mathcal{I}$. Since there exists $k \in N$ such that for every $n > k$ $x_n \in U$ then for $n > k$ $I\text{-}\liminf_{s\to x_n} f(s) \geq g + \varepsilon$. Hence $\lim_{n\to\infty} \left(I\text{-}\liminf_{s\to x_n} f(s) \right) \geq g + \varepsilon$ — a contradiction.

The same arguments work in the case (ii).

Thus

$$I\text{-}\liminf_{t\to x} f(t) \leq \liminf_{t\to x} \left(I\text{-}\liminf_{s\to t} f(s) \right) \leq \liminf_{t\to x} f(t) \leq \limsup_{t\to x} f(t) \leq$$
$$\leq \limsup_{t\to x} \left(I\text{-}\limsup_{s\to t} f(s) \right) \leq I\text{-}\limsup_{t\to x} f(t) = f(x).$$

Hence $\liminf_{t\to x} f(t) = \limsup_{t\to x} f(t) = f(x)$ and $x \in C(f)$.

Corollary. (a) Int $C_I(f) \subseteq C(f)$,
(b) *If f is I-continuous then f is continuous.*

II.

Let $\mathcal{B}$ denotes the family of Borel sets on X. We say that $\mathcal{I}$ is a Borel ideal on X iff for every $A \in \mathcal{I}$ there exists $B \in \mathcal{I} \cap \mathcal{B}$ such that $A \subseteq B$. (The collection of all countable subsets of X, the family of all first category subsets of X and the collection of all measure zero subsets of R^n are Borel ideals.)

In this and next parts of this paper we assume that $\mathcal{I}$ is a Borel ideal and for every open non-void subset $G \subseteq X$ card. G is continuum.

Lemma 1. *There exists a partition A, B of X such that for every $x \in X$ and every closed set $F \subseteq X$ if $x \in d_I(F)$ then $x \in d_I(F \cap A)$ and $x \in d_I(F \cap B)$.*

The construction of A and B is very similar to the construction of Bernstein's set (cf. [3], [6], see proof of Lemma 2).

Proposition 0. *If D is a G_δ set then, there exists a function $g\colon X \to R$ such that $C(g) = C_I(g) = D$.*

Proof. Let $X=A\cup B$, $A\cap B=\emptyset$, where A and B are defined in Lemma 1. Assume that $X-D=\bigcup_{n\in N} F_n$, where $F_n\subseteq F_{n+1}$ and F_n are closed. Let $(a_n)_{n\in N}$ be a sequence of positive real numbers such that $\sum_{n\in N} a_n=1$ and $a_n>2\sum_{k>n} a_k$.
For every $n\in N$, we define the function g_n: $X\to R$:

$$g_n(x)=\begin{cases} a_n & \text{for} \quad x\in F_n\cap A, \\ -a_n & \text{for} \quad x\in F_n\cap B, \\ 0 & \text{for} \quad x\in X-F_n. \end{cases}$$

Then
(a) $C_I(g_n)=C(g_n)=X-F_n$,
(b) $i\text{-}\lim\inf_{t\to x} g_n(t)\geqslant -a_n$ and $I\text{-}\lim\sup_{t\to x} g_n(t)\leqslant a_n$ for all $x\in F_n$.
Let us define g: $X\to R$ as follows:

$$g(x)=\sum_{n\in N} g_n(x).$$

The uniform convergence of this series implies the continuity of g on D. If $x\notin D$ then there exists $n\in N$ such that $x\in F_n$. Let

$$n(x)=\min\{n\in N\colon\ x\in F_n\}. \text{Then, if } x\in\psi_I(F_n) \text{ so } g(x)=\sum_{k\geqslant n(x)} a_k$$

$$I\text{-}\lim\sup_{t\to x} g(t)\geqslant I\text{-}\lim\sup_{t\to x} g_{n(x)}(t)+\sum_{k>n(x)} I\text{-}\lim\inf_{t\to x} g_k(t)\geqslant a_{n(x)}-\sum_{k>n(x)} a_k>0 \text{ and}$$

$$I\text{-}\lim\inf_{t\to x} g(t)\leqslant I\text{-}\lim\inf_{t\to x} g_{n(x)}(t)+\sum_{k>n(x)} I\text{-}\lim\sup_{t\to x} g_k(t)\leqslant -a_{n(x)}+\sum_{k>n(x)} a_k<0.$$

Hence $x\notin\{x\in X\colon\ I\text{-}\lim\inf_{t\to x} g(t)=I\text{-}\lim\sup_{t\to x} g(t)\}$. If $x\in A\cap F_n-\psi_I(F_n)$ then $g(x)=\sum_{k\geqslant n(x)} a_k>\sum_{k>n(x)} a_k\geqslant I\text{-}\lim\sup_{t\to x} g(t)$. Similarly, if $x\in B\cap F_n-\psi_I(F_n)$ then $g(x)<I\text{-}\lim\inf_{t\to x} g(t)$.

Proposition 1. *If D is a G_δ set and $I\in\mathcal{J}$ then there exists a function f: $X\to R$ such that $C_I(f)=D-I$.*

Proof. Let g: $X\to R$ be the function which is defined in Proposition 0. We define f: $X\to R$ as follows:

$$f(x)=\begin{cases} g(x)+1 & \text{for} \quad x\in I\cap D, \\ g(x) & \text{for} \quad x\in X-(I\cap D). \end{cases}$$

It is easy to show that f satisfies the above conditions.

Proposition 2. *Assume that B, D are G_δ subsets of X and $B \subseteq D$. Then there exists $I \in \mathscr{I}$ and there exists a function f: $X \to R$ such that $C(f) = B$ and $C_I(f) = D - I$.*

Proof. Let g: $X \to (-1, 1)$ be a function which is defined in the proposition 0 i. e. $g|D = 0$ and

$$C_I(g) = C(g) = D.$$

Let $B = \bigcap_{n \in N} G_n$, $X - B = \bigcup_{n \in N} F_n$, $F_n = X - G_n$, $F_n \subseteq F_{n+1}$ and F_n are closed. For $x \in X - B$ let us define $n(x) = \min\{n\colon\ x \in F_n\}$.

We define inductively the sequence $(I_n)_{n \in N}$ of subsets of X such that:

(i) $I_n \subseteq F_n$,

(ii) $I_n \subseteq I_{n+1}$,

(iii) I_n is dense in $F_n - (T_I(g) \cup T_I^1(g))$,

(iv) I_n is countable.

(v) $I_n \cap (T_I(g) \cup T_I^1(g)) = \emptyset$.

Let (a_n) be a sequence of positive real numbers such that $\sum_{n \in N} a_n = 1$. For each n we define the function f_n: $X \to R$ as follows:

$$f_n(x) = \begin{cases} a_n(g(x) + 3) & \text{for } x \in I_n, \\ a_n g(x) & \text{for } x \notin I_n. \end{cases}$$

Then $C(f_n) = D - \mathrm{Cl}\,(I_n) = D - F_n$.

Let us put $f(x) = \sum_{n \in N} f_n(x)$.

Since $\{x\colon\ f(x) \neq f(x)\} = \bigcup_{n \in N} I_n = I \in \mathscr{I}$, $I\text{-}\limsup_{t \to x} f(t) = I\text{-}\limsup_{t \to x} g(t)$ and $I\text{-}\liminf_{t \to x} f(t) = I\text{-}\liminf_{t \to x} g(t)$ for all $x \in X$. Hence $T_I(g) \subseteq T_I(f)$ and $T_I^1(g) \subseteq T_I^1(f)$.

(a) If $x \in I \cap D$ then there exists n such that $x \in I_n$. Let $m(x) = \min\{n \in N\colon\ x \in I_n\}$. Then

$$f(x) = g(x) + 3 \sum_{n \geq m(x)} a_n = 3 \sum_{n \geq m(x)} a_n > 0 = I\text{-}\limsup_{t \to x} f(t).$$

Hence $I \cap D \subseteq X - C_I(f)$.

(b) If $x \in D - I$ then $f(x) = g(x) = I\text{-}\limsup_{t \to x} f(t) = I\text{-}\liminf_{t \to x} f(t)$. Thus $D - I \subseteq C_I(f)$.

(c) If $x \notin D$ and $x \notin T_I(g) \cup T_I^1(g)$ then $I\text{-}\limsup_{t \to x} f(t) = I\text{-}\limsup_{t \to x} g(t) > I\text{-}\liminf_{t \to x} f(t)$. Hence $C_I(f) = D - I$.

Assume that $x \in B$. The uniform convergence of $\sum_{n=1}^{\infty} f_n$ implies the continuity of f at x.

If $x \in D - B$ then $x \in F_{n(x)}$ and $x \in \bigcap_{k<n(x)} G_k$. The following two cases may occur:

(a) There exists a sequence (x_k) in $I_{n(x)} - \{x\}$ such that $\lim_{k\to\infty} x_k = x$. Then for each $k \in N$ $f(x_k) \geq g(x_k) + 3 \sum_{m \geq n(x)} a_m$. Thus

$$\limsup_{t\to x} f(t) \geq \lim_{k\to\infty} f(x_k) \geq 3 \sum_{m \geq n(x)} a_m.$$

Let (y_k) be a sequence of points in $X - I_{n(x)}$ such that $\lim_{k\to\infty} y_k = x$. Then $f(y_k) \leq g(y_k) + 3 \sum_{m>n(x)} a_m$. Hence $\liminf_{t\to x} f(t) \leq \lim_{k\to\infty} f(y_k) \leq 3 \sum_{m>n(x)} a_m$. Thus $x \notin C(f)$.

(b) Assume that the point (a) do not hold. Then $x \in I_{n(x)}$ and there exists a neighbourhood U of x such that $U - \{x\} \subseteq G_{n(x)}$. Then $f(x) = g(x) + 3 \sum_{m \geq n(x)} a_m$ and $\limsup_{t\to x} f(t) \leq 3 \sum_{m>n(x)} a_m$, hence $x \notin C(f)$. Thus $C(f) = B$ and $C_I(f) = D - I$.

Question, 0. Assume that B, D are G_δ subsets of X, $I \in \mathcal{I}$, $B \subseteq D - I$ and $D - B \subseteq \mathrm{Cl}\, I$. Is there a function $f\colon X \to R$ such that $C(f) = B$ and $C_I(f) = D - I$?

III.

We say that an ideal $\mathcal{I} \subseteq P(x)$ is uniform iff $\{A \subseteq X\colon \text{card}\, A < 2^{\omega}\} \subseteq \mathcal{I}$ Notice that if CH or Martin's Axiom are assumed then the ideal $\mathcal{I} \subseteq P(X)$ of all sets of first category and $\mathcal{I} \subseteq P(R^n)$ of all measure zero subsets of R^n are uniform (cf. [10]).

Lemma 2. *Assume furthermore that an ideal $\mathcal{I}$ is uniform. Let $(A_n)_{n\in N}$ be a sequence of subsets of X. Then there exists a partition $(K_n)_{n\in N}$ of X such that*

$$\forall x \in X \quad \forall m \in N \quad [x \in d_I(A_m) \Rightarrow \forall n \in N \quad x \in d_I(A_m \cap K_n)].$$

Proof. The construction of K_n is very similar to the construction of Bernstein's set (cf. [3], [6]).

Let a sequence $(G_n)_{n\in N}$ be a countable basis of X. For every $n \in N$ let $(H_{n\xi})$ $(\xi < 2^{\omega_0})$ be an enumeration of the family $\{A \subseteq X\colon \exists I \in \mathcal{I} \cap \mathcal{B}\ A = G_n - I\}$ ($\mathcal{B}$ denotes the family of all Borel sets of X). It is possible for card $\mathcal{B} = 2^{\omega_0}$. Since $\mathcal{I}$ is uniform and $G_n \notin \mathcal{I}$, card $H_{n\xi} = 2^{\omega_0}$.

Notice that if $G_n \cap A_m \notin \mathcal{I}$ then $H_n \cap A_m \notin \mathcal{I}$.

We define

$$H_{n\xi}^{m}=\begin{cases} H_{n\xi} & \text{iff} \quad G_n\cap A_m\in\mathcal{J}, \\ H_{n\xi}\cap A_m & \text{iff} \quad G_n\cap A_m\notin\mathcal{J}. \end{cases}$$

Let (H_ξ) $(\xi<2^{\omega_0})$ be an enumeration of all sets $H_{n\xi}^m$, m, $n\in N$, $\xi<2^{\omega_0}$ and (r_ξ) an enumeration of X.

We shall construct inductively a sequence $(x_{\xi,n})$ of the type $2^{\omega_0}.\omega_0$

$$x_{\eta 0}=\min_{\xi}\{x_\xi:\ x_\xi\in H_\eta-\{x_{\gamma k}:\ k<\omega_0,\ \gamma<\eta\}\},$$

$$x_{\eta n}=\min_{\xi}\{x_\xi:\ x_\xi\in H_\eta-\{x_{\gamma k}:\ \gamma<\eta\vee(\gamma=\eta\ \&\ k<n)\}\}.$$

This construction is possible since card $H_\eta=2^{\omega_0}$.

Let us define sets K_n as follows:

$$K_n=\begin{cases} \{x_{\eta n}:\ \eta<2^{\omega_0}\} & \text{for } n>0, \\ X-\bigcup_{n\in N-\{0\}} K_n & \text{for } n=0. \end{cases}$$

The family $\{K_n\}$ satisfies the above condition. Indeed, if $x\in d_I(A_m)$ then $G_k\cap A_m\in\mathcal{J}$ for some k. If $G_k\cap A_m\cap K_n\in\mathcal{J}$, then there exists $B\in\mathcal{B}\cap\mathcal{J}$ such that $B\subseteq G_k$ and $G_k\cap A_m\cap K_n\subseteq B$. This is impossible since the set $H_{k\gamma}^{m}=(G_k-B)\cap A_m$ satisfies the condition $H_{k\gamma}^m\cap K_n\neq\emptyset$.

Theorem. *Let us assume that $\mathcal{J}$ is a uniform ideal on X. Let A, A_1, B_1, C, C_1 are subsets of X such that*

(i) $C\cup C_1\in\mathcal{J}$,

(ii) $B_1=A\cap A_1$,

(iii) $C\subseteq A-B_1$, $\quad C_1\subseteq A_1-B_1$,

(iv) *there exists $D\subseteq X$ such that D is a G_δ set and $B_1=D-(C\cup C_1)$,*

(v) *the sets $A-B_1$, A_1-B_1 do not contain subsets of the form $U-I$, where U is open and non-empty and $I\in\mathcal{J}$,*

(vi) *$E-D$ is a F_σ set (where E is defined in Lemma 0).*

Then

(x) *there exists a function f: $X\to R$ such that*

$$C_I(f)=B_1,\quad S_I(f)=A,\quad S_I^1(f)=A_1,\quad T_I(f)=C,\ \text{and}\ T_I^1(f)=C_1.$$

If we assume furthermore that B is a subset of X such that

(vii) *$B\subseteq B_1$, B is a G_δ set, $B_1-B\subseteq \mathrm{Cl}\,(C\cup C_1)$ and*

(viii) *$B\cap \mathrm{Cl}\,(C\cup C_1)=\emptyset$ then $C(f)=B$.*

Proof. The set $R-E$ is a F_σ set i.e. $R-E=\bigcup_{n\in N} F_n$, F_n are closed and $F_n\subseteq F_{n+1}$.
By lemma 2 there exists a partition $(K_n)_{n\in N}$ of X such that:
(0) $\forall x\in X\ \forall n\in N\ x\in d_I(K_n)$,
(1) $\forall x\in X\ [x\in d_I([-(A\cup A_1)]\cup B_1)$
$$\Rightarrow \forall n\in N\ x\in d_I(([X-(A\cup A_1)]\cup B_1)\cap K_n)],$$
(2) $\forall x\in X\ \forall m\in N\ [x\in d_I(F_m-A)\Rightarrow \forall n\in N\ x\in d_I(F_m\cap K_n-A)]$,
(3) $\forall x\in X\ \forall m\in N\ [x\in d_I(F_m-A_1)\Rightarrow \forall n\in N\ x\in d_I(F_m\cap K_n-A_1)]$.

I. In the first step we shall construct a function g: $X\to R$ such that

$$I\text{-}\liminf_{t\to x} g(t)=-1 \text{ and } I\text{-}\limsup_{t\to x} g(t)=1 \text{ for all } a\in X$$

The function g is defined as follows:

$$g(x)=(-1)^n\frac{2n}{2n+1} \qquad \text{for } x\in K_n.$$

It is easy to show that g satisfies the above conditions.

II. In the next step we shall construct a function h: $X\to R$ such that $C_I(h)=T_I(h)=T_I^1(h)=\emptyset$, $S_I(h)=A-B_1$ and $S_I^1(h)=A_1-B_1$. Let

$$h(x)=\begin{cases} I\text{-}\liminf_{t\to x} g(t)=-1 & \text{for } x\in A_1-B_1,\\ I\text{-}\limsup_{t\to x} g(t)=1 & \text{for } x\in A-B_1,\\ g(x) & \text{for } x\in[X-(A\cup A_1)]\cup B_1.\end{cases}$$

The following two cases may occur:
(a) $x\in d_I(A_1-B_1)\cap d_I(A-B_1)$,
(b) since (v) holds, if (a) do not hold then $x\in d_I([X-(A\cup A_1)]\cup B_1)$. Hence $I\text{-}\liminf_{t\to x} h(t)=I\text{-}\liminf_{t\to x} g(t)=-1$, $I\text{-}\limsup_{t\to x} h(t)=I\text{-}\limsup_{t\to x} g(t)=1$ and the function h has the above properties.

III. Let for $x\in X-E$, $n(x)=\min\{n\in N\colon x\in F_n\}$. We define a following function k: $X\to R$

$$k(x)=\begin{cases} 2^{-n(x)}h(x) & \text{for } x\in X-E,\\ 0 & \text{for } x\in E.\end{cases}$$

The following cases may occur:

(a) Let $x\in E$. Then k is continuous at x. In fact, if $x_n\xrightarrow[n\to\infty]{} x$ then $\forall m\in N\ \exists k\in N\ \forall l>k\ (x_l\in G_m)$ i.e.

$\forall m \in N \; \exists k \in N \; \forall l > k \; |k(x_l)| < 2^{-m}$. Thus $k(x_n) \xrightarrow[n \to \infty]{} 0 = k(x)$ and $x \in C(k)$

(b) Let $x \notin E$. Since the assumption (v) holds, $x \in d_I(X-(A-B_1))$ and $x \in d_I(X-(A_1-B_1))$. Let

$$K = \{n \in N:\; x \in d_I(F_n - (A - B_1))\}, \quad L = \{n \in N:\; x \in d_I(F_n - (A_1 - B_1))\},$$

$$n_0 = \begin{cases} \min K & \text{if } K \neq \emptyset, \\ \infty & \text{if } K = \emptyset, \end{cases} \qquad m_0 = \begin{cases} \min L & \text{if } L \neq \emptyset, \\ \infty & \text{if } L = \emptyset, \end{cases}$$

Notice that $n_0 \in N$ or $m_0 \in N$. Indeed, if for all $n \in N$ $x \notin d_I(F_n - (A - B_1))$ and $x \notin d_I(F_n - (A_1 - B_1))$ then for each n the set F_n is I-small at point x. Then $x \in \bigcap_{n \in N} \psi_I(X - F_n) = E$ — a contradiction. If $n_0 \in N$ and $m_0 \in N$ then

$$I\text{-}\limsup_{t \to x} k(t) = 2^{-m_0} \; I\text{-}\limsup_{t \to x} h(t) = 2^{-m_0} \quad \text{and}$$

$$I\text{-}\liminf_{t \to x} k(t) = 2^{-n_0} \; I\text{-}\liminf_{t \to x} h(t) = -2^{-n_0}.$$

Assume that $n_0 \in N$ and $m_0 = \infty$. Hence

$$I\text{-}\limsup_{t \to x} k(t) = 2^{-n_0} \; I\text{-}\limsup_{t \to x} h(t) = 2^{-n_0} \quad \text{and}$$

$$I\text{-}\liminf_{t \to x} k(t) = 2^{-n_0} \; I\text{-}\liminf_{t \to x} h(t) = -2^{-n_0}.$$

Similarly, if $m_0 \in N$ and $n_0 = \infty$ then $I\text{-}\limsup_{t \to x} k(t) = 2^{-m_0}$ and $I\text{-}\liminf_{t \to x} k(t) = -2^{-n_0}$. Since the sets F_{n_0}, F_{m_0} are closed, $x \in F_{n_0} \cap F_{m_0}$. Therefore $n(x) \leqslant \min(n_0, m_0)$.

If $x \in A - B_1$ then $k(x) = 2^{-n(x)} \geqslant I\text{-}\limsup_{t \to x} k(t)$. So $A - E \subseteq S_I(k) - C_I(k)$.

If $x \in A_1 - B_1$ then $k(x) \leqslant I\text{-}\liminf_{t \to x} k(t)$. Hence $A_1 - E \subseteq S_I^1(k) - C_I(k)$.

If $x \in X - (A \cup A_1)$ then $k(x) \neq I\text{-}\limsup_{t \to x} k(t)$, $k(x) \neq I\text{-}\liminf_{t \to x} k(t)$ and

$I\text{-}\liminf_{t \to x} k(t) \neq I\text{-}\limsup_{t \to x} k(t)$.

Thus $[X - (A \cup A_1)] - E \subseteq [X - (S_I(k) \cup S_I^1(k))] \cup T_I(k) \cup T_I^1(k)$.

IV. In the fourth step we define a function l: $X \to R$ such that $C_I(l) = E$, $T_I(l) = T_I^1(l) = \emptyset$, $S_I(l) = A \cup E$ and $S_I^1(l) = A_1 \cup E$.

Let us define l as follows:

$$l(x)=\begin{cases} I\text{-}\limsup_{t\to x} k(t) & \text{for } x\in(A-B_1)\cap T_I(k), \\ I\text{-}\liminf_{t\to x} k(t) & \text{for } x\in(A_1-B_1)\cap T_I^1(k), \\ \frac{1}{2}\left(I\text{-}\liminf_{t\to x} k(t)+I\text{-}\limsup_{t\to x} k(t)\right) & \text{for } x\in[T_I(k)\cup T_I^1(k)]-(A\cup A_1) \\ k(x) & \text{elsewhere.} \end{cases}$$

Since $\{x\in X\colon\ l(x)\neq k(x)\}\in\mathcal{I}$, for each $x\in X$

$$I\text{-}\liminf_{t\to x} l(t)=I\text{-}\liminf_{t\to x} k(t) \text{ and } I\text{-}\limsup_{t\to x} l(t)=I\text{-}\limsup_{t\to x} k(t).$$

It is clear that l satisfies the above conditions.

V. Since $E-D$ is a F_σ set and $E-D\in\mathcal{I}$, there exists a sequence of closed sets $(H_n)_{n\in N}$ such that $E-D=\bigcup_{n\in N} H_n$, $H_n\subseteq H_{n+1}$ and $H_n\in\mathcal{I}$. Let $(a_n)_{n\in N}$ be a sequence of positive real numbers such that $\sum_{n\in N} a_n=1$ and $a_n\geqslant 2\sum_{i\geqslant n+1} a_i$. For every $n\in N$ there exists a function $m_n\colon X\to\langle -a_n, a_n\rangle$ such that:

(0) m_n is continuous at every point $x\notin H_n$,

(1) $\forall x\in H_n\quad I\text{-}\liminf_{t\to x} m_n(t)=-I\text{-}\limsup_{t\to x} m_n(t)=a_n$,

(2) $\forall x\in H_n\quad m_n(x)=0$.

We shall define the function m_n as follows. There exists a sequence (w_k) or natural numbers such that $w_{k+1}>w_k$ and the sets $U_k=\{x\in X\colon\ w_k^{-1}>\text{dist}\,(x, H_n)>w_{k+1}^{-1}\}$ are open and non-empty.

Let

$$m_n(x)=\begin{cases} a_n & \text{for } x\in \text{Cl } U_{4k}, \\ -a_n & \text{for } x\in \text{Cl } U_{4k+2}. \end{cases}$$

By Tietze- Urysohn Theorem we shall extend m_n to the function continuous on $X-H_n$.

We define a function $m\colon X\to R$ such that $S_I(m)=S_I^1(m)=C_I(m)=X-\bigcup_{n\in N} H_n$ and $T_I(m)=T_I^1(m)=\emptyset$.

Let $m(x)=\sum_{n\in N} m_n(x)$.

The verification that m has the above properties is very similar to the verification that the adequate properties posses the function g which is defined in Proposition 0.

Let j: $X \to R$ be the following function:

$$j(x) = \begin{cases} I\text{-}\liminf_{t \to x} m(t) & \text{for } x \in A_1 \cap \bigcup_{n \in N} H_n, \\ I\text{-}\limsup_{t \to x} m(t) & \text{for } x \in A \cap \bigcup_{n \in N} H_n, \\ m(x) & \text{elsewhere.} \end{cases}$$

Since $\{x \in X \colon j(x) \neq m(x)\} \in \mathcal{I}$, for each $x \in X$ we have

$$I\text{-}\liminf_{t \to x} j(t) = I\text{-}\liminf_{t \to x} m(t) \text{ and } I\text{-}\limsup_{t \to x} j(t) = I\text{-}\limsup_{t \to x} m(t).$$

Hence $C_I(j) = X - \bigcup_{n \in N} H_n$, $S_I(j) = \left(X - \bigcup_{n \in N} H_n\right) \cup \left(A \cap \bigcup_{n \in N} H_n\right)$,

$$S_I^1(j) = \left(X - \bigcup_{n \in N} H_n\right) \cup \left(A_1 \cap \bigcup_{n \in N} H_n\right) \text{ and } T_I(j) = T_I^1(j) = \emptyset.$$

VI. The final step consists in the construction of a function f: $X \to R$ such that $S_I(f) = A$, $S_I^1(f) = A_1$, $C_I(f) = B_1$, $T_1(f) = C$ and $T_I^1(f) = C_1$.
Let us define a function f as follows:

$$f(x) = \begin{cases} 3 & \text{for } x \in C, \\ -3 & \text{for } x \in C_1, \\ j(x) + l(x) & \text{for } x \notin C \cup C_1. \end{cases}$$

(a) It is clear that $C \subseteq T_I(f)$ and $C_1 \subseteq T_I^1(f)$.

(b) Assume that $x \in X - \left(\bigcup_{n \in N} H_n \cup C \cup C_1\right)$. since the function j is continuous at x,

$$I\text{-}\liminf_{t \to x} f(t) = I\text{-}\liminf_{t \to x} l(t) + j(x) \quad \text{and}$$

$$I\text{-}\limsup_{t \to x} f(t) = I\text{-}\limsup_{t \to x} l(t) + j(x). \quad \text{Hence,}$$

if $x \notin \bigcup_{n \in N} H_n \cup C \cup C_1$ then

(0) $x \in C_I(f)$ iff $x \in C_I(l)$,
(1) $x \in S_I(f)$ iff $x \in S_I(l)$,
(2) $x \in S_I^1(f)$ iff $x \in S_I^1(l)$.

Similarly, if $x \in \bigcup_{n \in N} H_n - (C \cup C_1)$ then

(0) $x \in S_I(f)$ iff $x \in S_I(j)$ and
(1) $x \in S_I^1(f)$ iff $x \in S_I^1(j)$.
Thus the function f has the following property:

$$C_I(f)=[C_I(l)-(C\cup C_1)]-\bigcup_{n\in N} H_n=[E-(C\cup C_1)]-(E-D)=D-(C\cup C_1)=B_1,$$

$$S_I(f)=\left[S_I(l)-\bigcup_{n\in N} H_n\right]\cup\left[S_I(j)\cap\bigcup_{n\in N} H_n\right]\cup C=A,$$

$$S_I^1(f)=\left[S_I^1(l)-\bigcup_{n\in N} H_n\right]\cup\left[S_I^1(j)\cap\bigcup_{n\in N} H_n\right]\cup C_1=A_1,\quad T_I(f)=C \text{ and } T_I^1(f)=C_1.$$

Remark. (MA) If $X=R$ and $\mathcal{I}$ is the ideal of all sets of the first category then the conditions (i)—(v) and (x) are equivalent (see [5]).

Questions. 1. Let us assume that for A, A_1, B_1, C, $C_1\subseteq X$ the conditions (i)—(v) and (vii) hold. Does then the statement (x) hold?

2. Let us assume that for A, A_1, B, B_1, C, $C_1\subseteq X$ the conditions (i)—(v) and (vii) hold. Is there a function $f\colon X\to R$ such that

$$C(f)=B,\quad C_I(f)=B_1,\quad S_I(f)=A,\quad S_I^1(f)=A_1,\quad T_I(f)=C \text{ and } T_I^1(f)=C_1?$$

IV.

In this part we shall consider the followig question: is the condition (v) from Theorem essential?

Let $\mathcal{N}$ denotes the ideal of all sets of the first category in X.

Proposition 3. *If $\mathcal{I}$ is a σ-ideal and $\mathcal{I}\subseteq\mathcal{N}$ or $\mathcal{N}\subseteq\mathcal{I}$ then for every function $f\colon X\to R$ the set $S_I(f)-C_I(f)$ do not contain subsets of the form $G-I$ where G is non-empty and open and $I\in\mathcal{I}$.*

Proof. Assume that U is an open and non-empty subset of X, $I\in\mathcal{I}$ and $U-I\subseteq S_I(f)$. Then $I\text{-}\limsup_{t\to x} f(t)\leq f(x)$ for all $x\in U-I$. Hence for each $y>f(x)$ there exists a neighbourhood V of x such that $\{t\in V\colon\ f(t)\geq y\}\in\mathcal{I}$. Let (p_n, q_n) be a sequence of all open, non-empty intervals such that p_n, $q_n\in Q$. Then for each $n\in N$ there exist a F_σ subset $A_n\subseteq U$ and $J_n\in\mathcal{I}$ such that

$$(f|U)^{-1}*(p_n, q_n)=A_n\triangle J_n.$$

Let $J=I\cup\bigcup_{n\in N} J_n$ and $B=U-J$. Then $f|B$ belongs to the first class of Baire. Since $J\in\mathcal{I}$, $B\notin\mathcal{I}$. If $\mathcal{I}\subseteq\mathcal{N}$ then $J\in\mathcal{N}$ and $B\notin\mathcal{N}$. Similarly if $\mathcal{N}\subseteq\mathcal{I}$ then $B\notin\mathcal{N}$. By the Baire Theorem the set of all points at which $f|B$ not continuous is of the first category in B. (cf. [9]) Thus there exists a point $x\in U\cap C_I(f|U)=U\cap C_I(f)$.

Proposition 4. *Let $X=R$ and $\mathcal{I}$ be the ideal of all measure zero subsets of X. Then there exists a function $f\colon R\to R$ such that*

$$S_I(f)=R \quad \textit{and} \quad C_I(f)=\emptyset.$$

Proof. Assume that A and B is a partition of R such that A is a set of the first category and $B \in \mathcal{Y}$. (cf. [6]). It is possible to assume that A, is a F_σ set, $A = \bigcup_{n \in N} F_n$, the sets F_n are pairwise disjoint, closed and nowhere dense (see [8]). Notice that infinite many of F_n have a positive measure. In fact, suppose that there exists $m \in N$ such that $F_n \in \mathcal{I}$ for $n > m$ Then the set $F = \bigcup_{n \leqslant m} F_n$ is closed, nowhere dense and $R - F \in \mathcal{I}$ — a contradiction. Hence it is possible to assume that $F_n \notin \mathcal{I}$ for each $n \in N$.

Let us define a function f: $R \to R$:

$$f(x) = \begin{cases} n^{-1} & \text{for} \quad x \in F_n, \\ 2 & \text{for} \quad x \in B. \end{cases}$$

Then f satisfies the conditions of this proposition.
If $x \in B$ then $x \in T_I(f) \subseteq S_I(f)$.

If $x \in F_n$ and (x_k) is a sequence in A then almost all terms of (x_k) belong to $\bigcup_{k \geqslant n} F_k$. Thus $\limsup_{k \to \infty} (x_k) \leqslant n^{-1}$ and $I\text{-}\limsup_{t \to x} f(t) \leqslant f(x)$. Since for each $m \in N$ the set $\bigcup_{k \leqslant m} F_k$ is nowhere dense, in every neighbourhood U of x there exist an open, non-empty subset $V \subseteq R - \bigcup_{k \leqslant m} F_k$. Thus $I\text{-}\liminf_{t \to x} f(t) \leqslant m^{-1}$ for all $m \in N$ and consequelly, $I\text{-}\liminf_{t \to x} f(t) = 0$. Hence $C_I(f) = \emptyset$ and $S_I(f) = R$.

For every $A \subseteq X$ we define $\mathrm{Int}_I\, A$ as follows:

$$\mathrm{Int}_I\, A = \{x \in A\colon\ \exists V (x \in V,\ V \text{ is open and } V - A \in \mathcal{I})\}.$$

Proposition 5. *For every subset A of X there exists a function f: $X \to R$ such that $S_I(f) = A$.*

Proof. Let $B = \mathrm{Int}_I\, A$. By Lemma 0 there exists an open set G and $I \in \mathcal{I}$ such that $B = G - I$ and $G = \psi_I(G)$.
Let $(K_n)_{n \in N}$ be a partition of the set $X - G$ such that for each $x \in X$ if $x \in d_I(X - G)$ then $x \in d_I(K_n - G)$.
We define f as follows:

$$f(x) = \begin{cases} 1 & \text{for} \quad x \in A, \\ (-1)^n \dfrac{n}{n+1} & \text{for} \quad x \in K_n - A, \\ -1 & \text{for} \quad x \in I - A. \end{cases}$$

For this function $S_I(f) = A$.

If $x \in A$ then $x \in d_I(X-G)$ or $x \in I$. Indeed, suppose that $x \notin A$ and $x \notin I$. Then $x \notin B$ and $x \notin G$. Since $\psi_I(G)=G$, $x \in d_I(X-G)$.
If $x \in I-A$ then $x \in S_I^1(f) \subseteq X - S_I(f)$.

If $x \in d_I(X-G)$ then $f(x) \leq I\text{-}\limsup_{t \to x} f(t)$. Thus $A = S_I(f)$.

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О ТОЧКАХ *I*-НЕПРЕРЫВНОСТИ И *I*-ПОЛУНЕПРЕРЫВНОСТИ

Tomasz Natkaniec

Резюме

Пусть $(X, \mathcal{T})$ — польское пространство и $\mathcal{I} \subseteq 2^X$ — есть σ-идеал. I-топологией на X будем называть семейство $\{A-B\colon A \in \mathcal{T},\ B \in \mathcal{I}\}$. В работе исследованы связи между множествами точек непрерывности, точек I-непрерывности и точек I-полунепрерывности вещественной функции $f\colon X \to R$. В частности, рассмотрен случай, когда $X = R$ и $\mathcal{I}$ есть идеал всех множеств с мерой Лебега равной нулю. В случае, когда $\mathcal{I}$ является идеалом множеств первой категории, обобшены результаты 3. Грандэ.