Tomasz Natkaniec
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ON $I$-CONTINUITY
AND $I$-SEMICONTINUITY POINTS

TOMASZ NATKANIEC

Let $f: X \rightarrow \mathbb{R}$ be a real function. The purpose of the present paper is to study the
relation between the set $C(f)$ of all points at which $f$ is continuous, the set $G(f)$ of
all points at which $f$ is $I$-continuous, the set $S_i(f)$ of all points at which $f$ is $I$-upper
semicontinuous and the set $S^1_i(f)$ of all points at which $f$ is $I$-lower semicontinuous.

Let $X$ be a Polish space and $\mathcal{J} \subseteq \mathcal{P}(X)$ be a $\sigma$-complete ideal which has the
following properties:
(a) if $x \in X$ then $\{x\} \in \mathcal{J}$,
(b) if $\emptyset \neq U \subseteq X$ is open then $U \notin \mathcal{J}$.

We say that a subset $A \subseteq X$ is $\mathcal{J}$-small at point $p \in X$ iff there exists a neighbour­
hood $U(p)$ of $p$ such that $U(p) \cap A \in \mathcal{J}$. We denote by $d_I(A)$ the set of all points at
which $A$ is not $\mathcal{J}$-small, namely

$$d_I(A) = \{ p: \forall V(p) \ V(p) \cap A \in \mathcal{J} \}$$

($d_I(A)$ is $A^*$ in the sense of Hashimoto [2]).

The family of subsets of $X$

$$\{ G - I: G \text{ is open and } I \in \mathcal{J} \}$$

is a new topology on $X$ (it is $^*$-topology in the sense of Hashimoto or "$\mathcal{J}$-topology"
in the sense of Vaidyanathoswamy — c.f. [2], [4], [7], [11]).

We say that a function $f: X \rightarrow \mathbb{R}$ is $I$-continuous (semicontinuous) iff $f$ is
continuous (semicontinuous) in the $\mathcal{J}$-topology.

We use the following notation:

$$I\liminf_{t \to x} f(t) = \sup \{ y \in \mathbb{R}: x \notin d_I(\{ t: y > f(t) \}) \},$$

$$I\limsup_{t \to x} f(t) = \inf \{ y \in \mathbb{R}: x \notin d_I(\{ t: y < f(t) \}) \},$$

$C(f)$ is the set of all points at which $f$ is continuous,

$$C(f) = \{ x \in X: I\liminf_{t \to x} f(t) = f(x) = I\limsup_{t \to x} f(t) \},$$
\[ S_t(f) = \{ x \in X : \limsup_{t \to x} f(t) \leq f(x) \}, \]
\[ S^1_t(f) = \{ x \in X : \liminf_{t \to x} f(t) \geq f(x) \}, \]
\[ T_t(f) = \{ x \in X : \limsup_{t \to x} f(t) < f(x) \}, \]
\[ T^1_t(f) = \{ x \in X : \liminf_{t \to x} f(t) > f(x) \}. \]

Let \( \psi_t(A) \) denotes the set of all points at which the set \( X - A \) is \( \mathcal{J} \)-small, namely
\[ \psi_t(A) = \{ x : \exists U(x) \ U(x) - A \in \mathcal{J} \}. \]

\( \psi_t(A) = X - (X - A)^* \) in the sense of Hashimoto.

Notice that

(i) for every \( A \subseteq X \) the set \( \psi_t(A) \) is open,
(ii) if \( A \subseteq B \) then \( \psi_t(A) \subseteq \psi_t(B) \),
(iii) for every \( A \subseteq X \) \( \psi_t(A) - A \in \mathcal{J} \).

In fact, if \( (U_n)_{n \in \mathbb{N}} \) is a basis of \( X \) and \( A_n = \{ x \in \psi_t(A) : U(x) = U_n \} \) then
\[ A_n - A \subseteq U_n - A \in \mathcal{J} \quad \text{and} \quad \psi_t(A) - A = \bigcup_{n \in \mathbb{N}} A_n - A \in \mathcal{J}. \]

We shall use the following simple facts.

**Fact 0.** For every function \( f : X \to \mathbb{R} \) we have
\[ \limsup_{t \to x} f(t) = -\liminf_{t \to x} (-f)(t). \]

**Fact 1.** If functions \( f, g : X \to \mathbb{R} \) are bounded then

a) \[ \limsup_{t \to x} f(t) + \limsup_{t \to x} g(t) \geq \limsup_{t \to x} (f + g)(t) \geq \limsup_{t \to x} f(t) + \liminf_{t \to x} g(t), \]

b) \[ \liminf_{t \to x} f(t) + \liminf_{t \to x} g(t) \leq \liminf_{t \to x} (f + g)(t) \leq \liminf_{t \to x} f(t) + \limsup_{t \to x} g(t). \]

**Proof.** a) Assume that \( \limsup_{t \to x} f(t) = a \) and \( \limsup_{t \to x} g(t) = b \). Then
\[ x \notin d_t\left(\left\{ t : f(t) > a + \frac{\varepsilon}{2} \right\}\right) \cup d_t\left(\left\{ t : g(t) > b + \frac{\varepsilon}{2} \right\}\right) \quad \text{for all} \ \varepsilon > 0. \]
Hence \( x \notin d_t(\{ t : f(t) + g(t) > a + b + \varepsilon \}) \) for all \( \varepsilon > 0 \) and
\[ \limsup_{t \to x} (f + g)(t) \leq a + b. \]
\[ I\limsup_{t \to x} f(t) = I\limsup_{t \to x} [(f + g)(t) - g(t)] \leq I\limsup_{t \to x} (f + g)(t) + I\limsup_{t \to x} (-g)(t) = I\limsup_{t \to x} (f + g)(t) - I\liminf_{t \to x} g(t). \]

Hence \( I\limsup_{t \to x} f(t) + I\liminf_{t \to x} g(t) \leq I\limsup_{t \to x} (f + g)(t) \).

The case (b) is similar.

**Fact 2.** If \( \sum_{n \in \mathbb{N}} f_n(t) \) is uniformly convergent in some neighbourhood \( U \) of \( x \) then

\[ I\limsup_{t \to x} \sum_{n \in \mathbb{N}} f_n(t) \leq \sum_{n \in \mathbb{N}} I\limsup_{t \to x} f_n(t) \]

and

\[ I\liminf_{t \to x} \sum_{n \in \mathbb{N}} f_n(t) \geq \sum_{n \in \mathbb{N}} I\liminf_{t \to x} f_n(t). \]

This fact follows from Fact 1.

**Lemma 0.** If \( D \subseteq X \) is a \( G_6 \) set then there exists \( E \subseteq X \) such that \( E \) is a \( G_6 \) set, \( D \subseteq E \), \( E - D \in \mathcal{F} \), \( E = \bigcap \{ G_n \mid G_n \text{ are open}, G_{n+1} \subseteq G_n \text{ and } E = \bigcap \psi_1(G_n) \}. \)

**Proof.** Assume that \( D = \bigcap \psi_1(H_n) \), \( H_n \) is open and \( H_{n+1} \subseteq H_n \). Then \( \psi_1(H_{n+1}) \subseteq \psi_1(H_n) \), \( \psi_1(H_n) \) is open and \( \psi_1(H_n) - H_n \in \mathcal{F} \). We define \( E \) as follows:

\[ E = \bigcap \psi_1(H_n). \]

Then \( \psi_1(\psi_1(H_n)) = \psi_1(H_n) \) and \( E - D = \bigcap \psi_1(H_n) - \bigcap \psi_1(H_n) \subseteq \bigcup (\psi_1(H_n) - H_n) \).

**Remark.** If \( \mathcal{F} \) is the ideal of the sets of first category then \( \psi_1(A) = A \) means that \( A \) is a regular open set i.e. \( A = \psi_1(A) \) iff \( \text{Int Cl } A = A \).

**Proof.** If \( A = \psi_1(A) \) then \( A \) is open, so \( A \subseteq \text{Int Cl } A \). If \( x \in \text{Int Cl } A \) then there exists a neighbourhood \( U \) of \( x \) such that \( U \subseteq \text{Cl } A \) Since \( A \) is open and dense in \( U \), \( A \) is residual in \( U \) and \( U - A \in \mathcal{F} \). Hence \( x \in \psi_1(A) = A \).

If \( \text{Int Cl } A = A \) then \( A \) is open and \( A \subseteq \psi_1(A) \). If \( x \in \psi_1(A) \) then there exists a neighbourhood \( U \) of \( x \) such that \( U - A \in \mathcal{F} \). Then \( U \subseteq \text{Cl } A \) and \( x \in \text{Int Cl } A = A \).

**I.**

**Fact 0.** \( C(f) \) is a \( G_6 \) set.

It is the well known fact (cf. [9]).
**Fact 1.** $T_1(f) \cup T_1(f) \in \mathscr{J}$.

**Proof.** Let $(U_n)_{n \in \mathbb{N}}$ (resp. $(V_n)_{n \in \mathbb{N}}$) be a countable basis of $X$ (resp. $R$). If $x \in T_1(f)$ then $f(x) > \liminf_{t \rightarrow x} f(t)$. Thus there exist $n(x), m(x) \in \mathbb{N}$ such that $x \in U_{n(x)}, f(x) \in V_{m(x)}$ and $U_{n(x)} \cap f^{-1} * V_{m(x)} \in \mathscr{J}$. Let $A(n, m) = \{ x \in T_1(f) : n(x) = n \text{ and } m(x) = m \}$. Then for every $x \in A(n, m)$ we have $A(n, m) \subseteq U_{n(x)} \cap f^{-1} * V_{m(x)} \in \mathscr{J}$. Hence $T_1(f) = \bigcup_{n, m \in \mathbb{N}} A(n, m) \in \mathscr{J}$. Similarly, $T_1(f) \in \mathscr{J}$. (Z. Grande in [1] proved this fact for $X = R$ and the ideal of all sets of the first category.)

**Fact 2.** There exists a $G_\delta$ set $D$ such that $C_1(f) = D - (T_1(f) \cup T_1(f))$.

**Proof.** We define $D$ as follows:

$$D = \{ x \in X : \liminf_{t \rightarrow x} f(t) = \limsup_{t \rightarrow x} f(t) \}.$$  

It is clear that $C_1(f) = D - (T_1(f) \cup T_1(f))$. We shall prove that $X - D$ is a $F_\sigma$ set.

$$X - D = \{ x \in X : \liminf_{t \rightarrow x} f(t) < \limsup_{t \rightarrow x} f(t) \}.$$  

Let $A(p, q) = \{ x \in X : \liminf_{t \rightarrow x} f(t) \leq p \text{ and } \limsup_{t \rightarrow x} f(t) \geq q \}$. For each $p, q \in Q$ the set $A(p, q)$ is closed. Indeed, if $x_n \rightarrow x$ and $\{ x_n : n \in \mathbb{N} \} \subseteq A(p, q)$ then $\liminf_{t \rightarrow x} f(t) \leq p$ and $\limsup_{t \rightarrow x} f(t) \geq q$.

Since $X - D = \bigcup_{p, q \in Q} A(p, q)$, $X - D$ is a $F_\sigma$ set.

**Fact 3.** $C_1(f) - C(f) \subseteq \text{Cl} (T_1(f) \cup T_1(f))$.

**Proof.** Assume that $x \in C_1(f)$ and there exists a neighbourhood $U$ of $x$ such that $U \cap (T_1(f) \cup T_1(f)) = \emptyset$ i.e. for each $t \in U$

$$\liminf_{s \rightarrow t} f(s) \leq f(t) \leq \limsup_{s \rightarrow t} f(t) \quad \text{and}$$

$$\liminf_{t \rightarrow x} f(t) = f(x) = \limsup_{t \rightarrow x} f(t).$$

Notice that:

(i) $\liminf_{t \rightarrow x} f(t) \leq \liminf_{s \rightarrow t} f(s)$ and

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\( (ii) \ I\text{-}\lim \sup_{t \to x} f(t) \geq \lim \sup_{t \to x} \left( I\text{-}\lim \sup_{t \to l} f(s) \right) \).

In fact, let \( x_n \to x \) such that

\[
\lim_{n \to \infty} \left( I\text{-}\lim \inf_{s \to x_n} f(s) \right) = \lim \inf_{t \to x} \left( I\text{-}\lim \inf_{t \to l} f(s) \right) = g.
\]

Suppose that \( I\text{-}\lim \inf_{t \to x} f(t) > g \). Then for some \( \epsilon > 0 \) \( I\text{-}\lim \inf_{t \to x} f(t) > g + \epsilon \) i.e. there exists a neighbourhood \( U \) of \( x \) such that \( \{ t \in U : f(t) < g + \epsilon \} \in \mathcal{J} \). Since there exists \( k \in \mathbb{N} \) such that for every \( n > k \) \( x_n \in U \) then for \( n > k \) \( I\text{-}\lim \inf_{s \to x_n} f(s) \geq g + \epsilon \). Hence

\[
\lim_{n \to \infty} \left( I\text{-}\lim \inf_{s \to x_n} f(s) \right) \geq g + \epsilon \quad \text{— a contradiction.}
\]

The same arguments work in the case (ii).

Thus

\[
I\text{-}\lim \inf_{t \to x} f(t) \leq \lim \inf_{t \to x} \left( I\text{-}\lim \inf_{t \to l} f(s) \right) \leq \lim \inf_{t \to x} f(t) \leq \lim \sup_{t \to x} f(t) \leq \lim \sup_{t \to x} \left( I\text{-}\lim \sup_{s \to t} f(s) \right) = I\text{-}\lim \sup_{t \to x} f(t) = f(x).
\]

Hence \( \lim \inf_{t \to x} f(t) = \lim \sup_{t \to x} f(t) = f(x) \) and \( x \in \mathcal{C}(f) \).

**Corollary.**

(a) \( \text{Int} \ C_i(f) \subseteq \mathcal{C}(f) \),

(b) If \( f \) is \( I \)-continuous then \( f \) is continuous.

**II.**

Let \( \mathcal{B} \) denotes the family of Borel sets on \( X \). We say that \( \mathcal{J} \) is a Borel ideal on \( X \) iff for every \( A \in \mathcal{J} \) there exists \( B \in \mathcal{J} \cap \mathcal{B} \) such that \( A \subseteq B \). (The collection of all countable subsets of \( X \), the family of all first category subsets of \( X \) and the collection of all measure zero subsets of \( R^n \) are Borel ideals.)

In this and next parts of this paper we assume that \( \mathcal{J} \) is a Borel ideal and for every open non-void subset \( G \subseteq X \) card \( G \) is continuum.

**Lemma 1.** There exists a partition \( A, B \) of \( X \) such that for every \( x \in X \) and every closed set \( F \subseteq X \) if \( x \in d_t(F) \) then \( x \in d_t(F \cap A) \) and \( x \in d_t(F \cap B) \).

The construction of \( A \) and \( B \) is very similar to the construction of Bernstein’s set (cf. [3], [6], see proof of Lemma 2).

**Proposition 0.** If \( D \) is a \( G_\delta \) set then, there exists a function \( g : X \to R \) such that \( C(g) = C_i(g) = D \).
Proof. Let $X = A \cup B$, $A \cap B = \emptyset$, where $A$ and $B$ are defined in Lemma 1. Assume that $X - D = \bigcup_{n \in N} F_n$, where $F_n \subseteq F_{n+1}$ and $F_n$ are closed. Let $(a_n)_{n \in N}$ be a sequence of positive real numbers such that $\sum_{n \in N} a_n = 1$ and $a_n > 2 \sum_{k > n} a_k$.

For every $n \in N$, we define the function $g_n : X \to \mathbb{R}$:

$$g_n(x) = \begin{cases} a_n & \text{for } x \in F_n \cap A, \\ -a_n & \text{for } x \in F_n \cap B, \\ 0 & \text{for } x \in X - F_n. \end{cases}$$

Then

(a) $C_i(g_n) = C(g_n) = X - F_n$,

(b) $\liminf_{t \to x} g_n(t) \geq -a_n$ and $\limsup_{t \to x} g_n(t) \leq a_n$ for all $x \in F_n$.

Let us define $g : X \to \mathbb{R}$ as follows:

$$g(x) = \sum_{n \in N} g_n(x).$$

The uniform convergence of this series implies the continuity of $g$ on $D$. If $x \notin D$ then there exists $n \in N$ such that $x \in F_n$. Let $n(x) = \min \{ n \in N : x \in F_n \}$. Then, if $x \in \psi_l(F_n)$ so $g(x) = \sum_{k \geq n(x)} a_k$

$$\limsup_{t \to x} g(t) \geq \limsup_{t \to x} g_n(x(t)) + \sum_{k \geq n(x)} \liminf_{t \to x} g_k(t) \geq a_{n(x)} - \sum_{k \geq n(x)} a_k > 0 \text{ and }$$

$$\liminf_{t \to x} g(t) \leq \liminf_{t \to x} g_n(x(t)) + \sum_{k \geq n(x)} \limsup_{t \to x} g_k(t) \leq -a_{n(x)} + \sum_{k \geq n(x)} a_k < 0.$$

Hence $x \notin \{ x \in X : \liminf_{t \to x} g(t) = \limsup_{t \to x} g(t) \}$. If $x \in A \cap F_n - \psi_l(F_n)$ then $g(x) = \sum_{k > n(x)} a_k \geq I \limsup_{t \to x} g(t)$. Similarly, if $x \in B \cap F_n - \psi_l(F_n)$ then $g(x) < I \liminf_{t \to x} g(t)$.

**Proposition 1.** If $D$ is a $G_\delta$ set and $I \in \mathfrak{I}$ then there exists a function $f : X \to \mathbb{R}$ such that $C_i(f) = D - I$.

**Proof.** Let $g : X \to \mathbb{R}$ be the function which is defined in Proposition 0. We define $f : X \to \mathbb{R}$ as follows:

$$f(x) = \begin{cases} g(x) + 1 & \text{for } x \in I \cap D, \\ g(x) & \text{for } x \in X - (I \cap D). \end{cases}$$

It is easy to show that $f$ satisfies the above conditions.
**Proposition 2.** Assume that $B$, $D$ are $G_\delta$ subsets of $X$ and $B \subseteq D$. Then there exists $I \in \mathcal{I}$ and there exists a function $f: X \to \mathbb{R}$ such that $C(f) = B$ and $G(f) = D - I$.

**Proof.** Let $g: X \to (-1, 1)$ be a function which is defined in the proposition 0 i.e. $g|D = 0$ and $C(g) = C(f) = D$.

Let $B = \bigcap_{n \in \mathbb{N}} G_n$, $X - B = \bigcup_{n \in \mathbb{N}} F_n$, $F_n = X - G_n$, $F_n \subseteq F_{n+1}$ and $F_n$ are closed. For $x \in X - B$ let us define $n(x) = \min \{ n : x \in F_n \}$. We define inductively the sequence $(I_n)_{n \in \mathbb{N}}$ of subsets of $X$ such that:

- (i) $I_n \subseteq F_n$,
- (ii) $I_n \subseteq I_{n+1}$,
- (iii) $I_n$ is dense in $F_n - (T_i(g) \cup T_i'(g))$,
- (iv) $I_n$ is countable.

Let $(a_n)$ be a sequence of positive real numbers such that $\sum_{n \in \mathbb{N}} a_n = 1$. For each $n$ we define the function $f_n: X \to \mathbb{R}$ as follows:

$$f_n(x) = \begin{cases} a_n(g(x) + 3) & \text{for } x \in I_n, \\ a_n g(x) & \text{for } x \notin I_n. \end{cases}$$

Then $C(f_n) = D - \text{Cl}(I_n) = D - F_n$.

Let us put $f(x) = \sum_{n \in \mathbb{N}} f_n(x)$.

Since $\{ x : f(x) \neq f_0(x) \} = \bigcup_{n \in \mathbb{N}} I_n \in \mathcal{I}$, $I$-lim sup $f(t) = I$-lim sup $g(t)$ and $I$-lim inf $f(t) = I$-lim inf $g(t)$ for all $x \in X$. Hence $T_i(g) \subseteq T_0(f)$ and $T_i'(g) \subseteq T_0'(f)$.

(a) If $x \in I \cap D$ then there exists $n$ such that $x \in I_n$. Let $m(x) = \min \{ n \in \mathbb{N} : x \in I_n \}$. Then

$$f(x) = g(x) + 3 \sum_{n > m(x)} a_n = 3 \sum_{n > m(x)} a_n > 0 = I$$-lim sup $f(t)$.

Hence $I \cap D \subseteq X - C(f)$.

(b) If $x \in D - I$ then $f(x) = g(x) = I$-lim sup $f(t) = I$-lim inf $f(t)$. Thus $D - I \subseteq C(f)$.

(c) If $x \notin D$ and $x \notin T_i(g) \cup T_i'(g)$ then $I$-lim sup $f(t) = I$-lim sup $g(t) > I$-lim inf $f(t)$. Hence $C(f) = D - I$. 303
Assume that $x \in B$. The uniform convergence of $\sum_{n=1}^{\infty} f_n$ implies the continuity of $f$ at $x$.

If $x \in D - B$ then $x \in F_{n(x)}$ and $x \in \bigcap_{k < n(x)} G_k$. The following two cases may occur:

(a) There exists a sequence $(x_k)$ in $I_{n(x)} - \{x\}$ such that $\lim_{k \to \infty} x_k = x$. Then for each $k \in N$

$$f(x_k) \geq g(x_k) + 3 \sum_{m \supseteq n(x)} a_m.$$  

Thus

$$\limsup_{t \to x} f(t) \geq \lim_{k \to \infty} f(x_k) \geq 3 \sum_{m \supseteq n(x)} a_m.$$  

Let $(y_k)$ be a sequence of points in $X - I_{n(x)}$ such that $\lim y_k = x$. Then $f(y_k) \leq g(y_k) + 3 \sum_{m \supseteq n(x)} a_m$. Hence $\liminf_{t \to x} f(t) \leq \lim_{k \to \infty} f(y_k) \leq 3 \sum_{m \supseteq n(x)} a_m$. Thus $x \notin C(f)$.

(b) Assume that the point (a) do not hold. Then $x \in I_{n(x)}$ and there exists a neighbourhood $U$ of $x$ such that $U - \{x\} \subseteq G_{n(x)}$. Then $f(x) = g(x) + 3 \sum_{m \supseteq n(x)} a_m$ and $\limsup_{t \to x} f(t) \leq 3 \sum_{m \supseteq n(x)} a_m$, hence $x \notin C(f)$. Thus $C(f) = B$ and $C_l(f) = D - I$.

Question 0. Assume that $B$, $D$ are $G_\delta$ subsets of $X$, $I \in \mathcal{J}$, $B \subseteq D - I$ and $D - B \subseteq \text{Cl} I$. Is there a function $f$: $X \to \mathbb{R}$ such that $C(f) = B$ and $C_l(f) = D - I$?

III.

We say that an ideal $\mathcal{I} \subseteq P(X)$ is uniform iff $\{A \subseteq X: \text{card } A < 2^\alpha\} \subseteq \mathcal{I}$. Notice that if CH or Martin's Axiom are assumed then the ideal $\mathcal{I} \subseteq P(X)$ of all sets of first category and $\mathcal{I} \subseteq P(R^n)$ of all measure zero subsets of $R^n$ are uniform (cf. [10]).

**Lemma 2.** Assume furthermore that an ideal $\mathcal{I}$ is uniform. Let $(A_n)_{n \in N}$ be a sequence of subsets of $X$. Then there exists a partition $(K_n)_{n \in N}$ of $X$ such that

$$\forall x \in X \forall m \in N \quad [x \in d_l(A_m) \Rightarrow \exists n \in N \quad x \in d_l(A_m \cap K_n)].$$

**Proof.** The construction of $K_n$ is very similar to the construction of Bernstein's set (cf. [3], [6]).

Let a sequence $(G_n)_{n \in N}$ be a countable basis of $X$. For every $n \in N$ let $(H_n)$ $(\xi < 2^\omega)$ be an enumeration of the family $\{A \subseteq X: \exists I \in \mathcal{I} \cap \mathcal{B} A = G_n - I\}$ ($\mathcal{B}$ denotes the family of all Borel sets of $X$). It is possible for card $\mathcal{B} = 2^\omega$. Since $\mathcal{I}$ is uniform and $G_n \notin \mathcal{I}$, card $H_n = 2^{<\omega}$. Notice that if $G_n \cap A_m \notin \mathcal{I}$ then $H_n \cap A_m \notin \mathcal{I}$.  

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We define
\[ H_m^n = \begin{cases} \quad \text{iff } G_n \cap A_m \in \mathcal{I}, \\ H_n^k \cap A_m \quad \text{iff } G_n \cap A_m \notin \mathcal{I}. \end{cases} \]

Let \((H_n^k) (\xi < 2^{\omega_0})\) be an enumeration of all sets \(H_m^n, m, n \in \mathbb{N}, \xi < 2^{\omega_0}\) and \((r_n)\) an enumeration of \(X\).

We shall construct inductively a sequence \((x_\xi, n)\) of the type \(2^{\omega_0} \cdot \omega_0\)

\[ x_{n0} = \min \{ x_\xi: x_\xi \in H_n - \{ x_{yk}: k < \omega_0, \gamma < \eta \} \}, \]

\[ x_{nn} = \min \{ x_\xi: x_\xi \in H_n - \{ x_{yk}: \gamma < \eta \lor (\gamma = \eta & k < n) \} \}. \]

This construction is possible since \(\text{card } H_n = 2^{\omega_0}\).

Let us define sets \(K_n\) as follows:

\[ K_n = \begin{cases} \{ x_m : \eta < 2^{\omega_0} \} & \text{for } n > 0, \\ X - \bigcup_{n \in \mathbb{N} - \{0\}} K_n & \text{for } n = 0. \end{cases} \]

The family \(\{K_n\}\) satisfies the above condition. Indeed, if \(x \in d_i(A_m)\) then \(G_k \cap A_m \in \mathcal{I}\) for some \(k\). If \(G_k \cap A_m \cap K_n \in \mathcal{I}\), then there exists \(B \in \mathcal{B} \cap \mathcal{I}\) such that \(B \subseteq G_k\) and \(G_k \cap A_m \cap K_n \subseteq B\). This is impossible since the set \(H_n^k = (G_k - B) \cap A_m\) satisfies the condition \(H_n^k \cap K_n \neq \emptyset\).

**Theorem.** Let us assume that \(\mathcal{I}\) is a uniform ideal on \(X\). Let \(A, A_1, B_1, C, C_1\) are subsets of \(X\) such that

(i) \(C \cup C_1 \in \mathcal{I}\),
(ii) \(B_1 = A \cap A_1\),
(iii) \(C \subseteq A - B_1, \quad C_1 \subseteq A_1 - B_1\),
(iv) there exists \(D \subseteq X\) such that \(D\) is a \(G_\delta\) set and \(B_1 = D - (C \cup C_1)\),
(v) the sets \(A - B_1, A_1 - B_1\) do not contain subsets of the form \(U - I\), where \(U\) is open and non-empty and \(I \in \mathcal{I}\),
(vi) \(E - D\) is a \(F_\sigma\) set (where \(E\) is defined in Lemma 0).

Then

(x) there exists a function \(f: X \rightarrow R\) such that

\[ G_1(f) = B_1, \quad S_1(f) = A, \quad S_1(f) = A_1, \quad \bar{c}_1(f) = C, \text{ and } T_1(f) = C_1. \]

If we assume furthermore that \(B\) is a subset of \(X\) such that

(vii) \(B \subseteq B_1, B\) is a \(G_\delta\) set, \(B_1 - B \subseteq \text{Cl} (C \cup C_1)\) and
(viii) \(B \cap \text{Cl} (C \cup C_1) = \emptyset\) then \(C(f) = B\).
Proof. The set $R - E$ is a $F_{\sigma}$ set i.e. $R - E = \bigcup_{n \in \mathbb{N}} F_n$, $F_n$ are closed and $F_n \subseteq F_{n+1}$.

By lemma 2 there exists a partition $(K_n)_{n \in \mathbb{N}}$ of $X$ such that:

0. $\forall x \in X \ \forall n \in \mathbb{N} \ x \in d_i(K_n)$,
1. $\forall x \in X \ [x \in d_i([- (A \cup A_i)] \cup B_i) \Rightarrow \forall n \in \mathbb{N} \ x \in d_i((X - (A \cup A_i)] \cup B_i) \cap K_n)]$,
2. $\forall x \in X \ \forall m \in \mathbb{N} \ [x \in d_i(F_m - A) \Rightarrow \forall n \in \mathbb{N} \ x \in d_i(F_m \cap K_n - A)]$,
3. $\forall x \in X \ \forall m \in \mathbb{N} \ [x \in d_i(F_m - A_i) \Rightarrow \forall n \in \mathbb{N} \ x \in d_i(F_m \cap K_n - A_i)]$.

I. In the first step we shall construct a function $g: X \rightarrow R$ such that

$I\lim_{t \rightarrow x} g(t) = -1$ and $I\lim_{t \rightarrow x} g(t) = 1$ for all $a \in X$

The function $g$ is defined as follows:

$$g(x) = (-1)^n \frac{2n}{2n+1} \quad \text{for } x \in K_n.$$ 

It is easy to show that $g$ satisfies the above conditions.

II. In the next step we shall construct a function $h: X \rightarrow R$ such that $C_i(h) = T_i(h) = T_i(h) = \emptyset$, $S_i(h) = A - B_i$ and $S_i(h) = A_1 - B_i$. Let

$$h(x) = \begin{cases} 
I\lim_{t \rightarrow x} g(t) = -1 & \text{for } x \in A_1 - B_i, \\
I\lim_{t \rightarrow x} g(t) = 1 & \text{for } x \in A - B_i, \\
g(x) & \text{for } x \in [X - (A \cup A_i)] \cup B_i.
\end{cases}$$

The following two cases may occur:

(a) $x \in d_i(A_i - B_i) \cap d_i(A - B_i)$,
(b) since (v) holds, if (a) do not hold then $x \in d_i([- (A \cup A_i)] \cup B_i)$. Hence

$I\lim_{t \rightarrow x} h(t) = I\lim_{t \rightarrow x} g(t) = -1$, $I\lim_{t \rightarrow x} h(t) = I\lim_{t \rightarrow x} g(t) = 1$ and the function $h$ has the above properties.

III. Let for $x \in X - E$, $n(x) = \min \{n \in \mathbb{N} : x \in F_n\}$. We define a following function $k: X \rightarrow R$

$$k(x) = \begin{cases} 2^{-n(x)}h(x) & \text{for } x \in X - E, \\
0 & \text{for } x \in E.
\end{cases}$$

The following cases may occur:

(a) Let $x \in E$. Then $k$ is continuous at $x$. In fact, if $x_n \xrightarrow{\text{a}} x$ then

$\forall m \in \mathbb{N} \ \exists l \in \mathbb{N} \ \forall l > k \ (x_i \in G_m)$ i.e.
∀m ∈ N ⇒ k ∈ N ∀l > k \left| k(x_l)\right| < 2^{-m}. Thus k(x_n) \xrightarrow{n \to \infty} 0 = k(x) and x ∈ C(k).

(b) Let x ∈ E. Since the assumption (v) holds, x ∈ d_1(X - (A - B_1)) and x ∈ d_1(X - (A_1 - B_1)). Let
\[ K = \{ n ∈ N: x ∈ d_1(F_n - (A - B_1))\}, \quad L = \{ n ∈ N: x ∈ d_1(F_n - (A_1 - B_1))\}, \]
\[ n_0 = \begin{cases} \min K & \text{if } K \neq \emptyset, \\ \infty & \text{if } K = \emptyset, \end{cases} \quad m_0 = \begin{cases} \min L & \text{if } L \neq \emptyset, \\ \infty & \text{if } L = \emptyset, \end{cases} \]
Notice that n_0 ∈ N or m_0 ∈ N. Indeed, if for all n ∈ N x ∉ d_1(F_n - (A - B_1)) and x ∉ d_1(F_n - (A_1 - B_1)) then for each n the set F_n is I-small at point x. Then x ∈ \bigcap_{n ∈ N} \psi_t(X - F_n) = E — a contradiction. If n_0 ∈ N and m_0 ∈ N then
\[ I\lim_{t \to x} \sup k(t) = 2^{-m_0} I\lim_{t \to x} \inf k(t) = 2^{-n_0}. \]
Assume that n_0 ∈ N and m_0 = ∞. Hence
\[ I\lim_{t \to x} \sup k(t) = 2^{-m_0} I\lim_{t \to x} \inf k(t) = 2^{-n_0}. \]
Similarly, if m_0 ∈ N and n_0 = ∞ then I\lim_{t \to x} \sup k(t) = 2^{-m_0} and I\lim_{t \to x} \inf k(t) = -2^{-n_0}. Since the sets F_{m_0}, F_{n_0} are closed, x ∈ F_{m_0} ∩ F_{n_0}. Therefore n(x) ≤ \min (n_0, m_0).

If x ∈ A - B_1 then k(x) = 2^{-n_0} \geq I\lim_{t \to x} \sup k(t). So A - E ⊆ S_I(k) - C_I(k).

If x ∈ A_1 - B_1 then k(x) ≤ I\lim_{t \to x} \inf k(t). Hence A_1 - E ⊆ S^I_I(k) - C_I(k).

If x ∈ X - (A ∪ A_1) then k(x) ≠ I\lim_{t \to x} \sup k(t), k(x) ≠ I\lim_{t \to x} \inf k(t) and
\[ I\lim_{t \to x} \sup k(t) \neq I\lim_{t \to x} \inf k(t). \]
Thus \[ X - (A ∪ A_1) - E \subseteq [X - (S_I(k) ∪ S^I_I(k))] ∪ T_I(k) ∪ T^I_I(k). \]

IV. In the fourth step we define a function I: X → R such that C_I(l) = E, T_I(l) = T^I_I(l) = ∅, S_I(l) = A ∪ E and S^I_I(l) = A_1 ∪ E.
Let us define \( l \) as follows:

\[
l(x) = \begin{cases} 
  \liminf_{t \to x} k(t) & \text{for } x \in (A - B_i) \cap T_i(k), \\
  \limsup_{t \to x} k(t) & \text{for } x \in (A_1 - B_i) \cap T_i(k), \\
  \frac{1}{2} \left( \liminf_{t \to x} k(t) + \limsup_{t \to x} k(t) \right) & \text{for } x \in [T_i(k) \cup T_i(k)] - (A \cup A_1), \\
  k(x) & \text{elsewhere.}
\end{cases}
\]

Since \( \{ x \in X : l(x) \neq k(x) \} \in \mathcal{F} \), for each \( x \in X \)

\[
I\liminf_{t \to x} l(t) = I\liminf_{t \to x} k(t) \text{ and } I\limsup_{t \to x} l(t) = I\limsup_{t \to x} k(t).
\]

It is clear that \( l \) satisfies the above conditions.

V. Since \( E - D \) is a \( F_a \) set and \( E - D \in \mathcal{F} \), there exists a sequence of closed sets \((H_n)_{n \in \mathbb{N}}\) such that \( E - D = \bigcup_{n \in \mathbb{N}} H_n \), \( H_n \subseteq H_{n+1} \) and \( H_n \in \mathcal{F} \). Let \((a_n)_{n \in \mathbb{N}}\) be a sequence of positive real numbers such that \( \sum_{n \in \mathbb{N}} a_n = 1 \) and \( a_n \geq 2 \sum_{l \geq n+1} a_l \). For every \( n \in \mathbb{N} \) there exists a function \( m_n : X \to (-a_n, a_n) \) such that:

1. \( \forall x \in H_n \liminf_{t \to x} m_n(t) = \limsup_{t \to x} m_n(t) = a_n \),
2. \( \forall x \in H_n m_n(x) = 0 \).

We shall define the function \( m_n \) as follows. There exists a sequence \((w_k)\) of natural numbers such that \( w_{k+1} > w_k \) and the sets \( U_k = \{ x \in X : w_k^{-1} > \text{dist} (x, H_n) > w_{k+1}^{-1} \} \) are open and non-empty.

Let

\[
m_n(x) = \begin{cases} 
  a_n & \text{for } x \in \text{Cl } U_{k}, \\
  -a_n & \text{for } x \in \text{Cl } U_{k+2}.
\end{cases}
\]

By Tietze- Urysohn Theorem we shall extend \( m_n \) to the function \( m \) continuous on \( X - H_n \).

We define a function \( m : X \to \mathbb{R} \) such that \( S_i(m) = S_i'(m) = C_i(m) = X - \bigcup_{n \in \mathbb{N}} H_n \) and \( T_i(m) = T_i'(m) = 0 \).

Let \( m(x) = \sum_{n \in \mathbb{N}} m_n(x) \).

The verification that \( m \) has the above properties is very similar to the verification that the adequate properties possess the function \( g \) which is defined in Proposition 0.
Let $j : X \to \mathbb{R}$ be the following function:

$$
    j(x) = \begin{cases} 
    \liminf_{t \to x} m(t) & \text{for } x \in A \cap \bigcup_{n \in \mathbb{N}} H_n, \\
    \limsup_{t \to x} m(t) & \text{for } x \in A \cap \bigcup_{n \in \mathbb{N}} H_n, \\
    m(x) & \text{elsewhere.}
    \end{cases}
$$

Since $\{x \in X : j(x) \neq m(x)\} \in \mathcal{F}$, for each $x \in X$ we have

$$
    \liminf_{t \to x} j(t) = \liminf_{t \to x} m(t) \quad \text{and} \quad \limsup_{t \to x} j(t) = \limsup_{t \to x} m(t).
$$

Hence $C_i(j) = X - \bigcup_{n \in \mathbb{N}} H_n$, $S_i(j) = \left( X - \bigcup_{n \in \mathbb{N}} H_n \right) \cup \left( A \cap \bigcup_{n \in \mathbb{N}} H_n \right)$,

$$
    S'_i(j) = \left( X - \bigcup_{n \in \mathbb{N}} H_n \right) \cup \left( A \cap \bigcup_{n \in \mathbb{N}} H_n \right) \quad \text{and} \quad T_i(j) = T'_i(j) = \emptyset.
$$

**VI.** The final step consists in the construction of a function $f : X \to \mathbb{R}$ such that $S_i(f) = A$, $S'_i(f) = A_1$, $C_i(f) = B_1$, $T_i(f) = C$ and $T'_i(f) = C_1$. Let us define a function $f$ as follows:

$$
    f(x) = \begin{cases} 
    3 & \text{for } x \in C, \\
    -3 & \text{for } x \in C_1, \\
    j(x) + l(x) & \text{for } x \notin C \cup C_1.
    \end{cases}
$$

(a) It is clear that $C \subseteq T_i(f)$ and $C_1 \subseteq T'_i(f)$.

(b) Assume that $x \in X - \left( \bigcup_{n \in \mathbb{N}} H_n \cup C \cup C_1 \right)$. Since the function $j$ is continuous at $x$,

$$
    \liminf_{t \to x} f(t) = \liminf_{t \to x} l(t) + j(x) \quad \text{and} \quad \limsup_{t \to x} f(t) = \limsup_{t \to x} l(t) + j(x).
$$

Hence, if $x \notin \bigcup_{n \in \mathbb{N}} H_n \cup C \cup C_1$ then

1. $x \in C_i(f)$ iff $x \in C_i(l)$,
2. $x \in S_i(f)$ iff $x \in S_i(l)$,
3. $x \in S'_i(f)$ iff $x \in S'_i(l)$.

Similarly, if $x \in \bigcup_{n \in \mathbb{N}} H_n - (C \cup C_1)$ then

1. $x \in S_i(f)$ iff $x \in S_i(j)$ and
2. $x \in S'_i(f)$ iff $x \in S'_i(j)$.

Thus, the function $f$ has the following property:
\[ C_i(f) = [C_i(l) - (C \cup C_i)] - \bigcup_{n \in N} H_n = [E - (C \cup C_i)] - (E - D) = D - (C \cup C_i) = B_i, \]
\[ S_i(f) = \left[ S_i(l) - \bigcup_{n \in N} H_n \right] \cup \left[ S_i(j) \cap \bigcup_{n \in N} H_n \right] \cup C = A, \]
\[ S_i(f) = \left[ S_i(l) - \bigcup_{n \in N} H_n \right] \cup \left[ S_i(j) \cap \bigcup_{n \in N} H_n \right] \cup C = A_i, \quad T_i(f) = C \quad \text{and} \quad T_i(f) = C_i. \]

**Remark.** (MA) If \( X = R \) and \( \mathcal{I} \) is the ideal of all sets of the first category then the conditions (i)—(v) and (x) are equivalent (see [5]).

**Questions.** 1. Let us assume that for \( A, A_1, B_1, C, C_1 \subseteq X \) the conditions (i)—(v) and (vii) hold. Does then the statement (x) hold?

2. Let us assume that for \( A, A_1, B, B_1, C, C_1 \subseteq X \) the conditions (i)—(v) and (vii) hold. Is there a function \( f : X \to R \) such that

\[ C(f) = B, \quad C_i(f) = B_i, \quad S_i(f) = A, \quad S_i(f) = A_i, \quad T_i(f) = C \quad \text{and} \quad T_i(f) = C_i? \]

**IV.**

In this part we shall consider the following question: is the condition (v) from Theorem essential?

Let \( \mathcal{N} \) denotes the ideal of all sets of the first category in \( X \).

**Proposition 3.** If \( \mathcal{I} \) is a \( \sigma \)-ideal and \( \mathcal{I} \subseteq \mathcal{N} \) or \( \mathcal{N} \subseteq \mathcal{I} \) then for every function \( f : X \to R \) the set \( S_i(f) - C_i(f) \) do not contain subsets of the form \( G - I \) where \( G \) is non-empty and open and \( I \in \mathcal{I} \).

**Proof.** Assume that \( U \) is an open and non-empty subset of \( X \), \( I \in \mathcal{I} \) and \( U - I \subseteq S_i(f) \). Then \( I \text{-lim sup } f(t) \leq f(x) \) for all \( x \in U - I \). Hence for each \( y > f(x) \) there exists a neighbourhood \( V \) of \( x \) such that \( \{ t \in V : f(t) \geq y \} \in \mathcal{I} \). Let \( (p_n, q_n) \) be a sequence of all open, non-empty intervals such that \( p_n, q_n \in Q \). Then for each \( n \in N \) there exist a \( F_o \) subset \( A_n \subseteq U \) and \( J_n \in \mathcal{I} \) such that

\[ (f|U)^{-1} \ast (p_n, q_n) = A_n \triangle J_n. \]

Let \( J = I \cup \bigcup_{n \in N} J_n \) and \( B = U - J \). Then \( f|B \) belongs to the first class of Baire. Since \( J \in \mathcal{I} \), \( B \notin \mathcal{I} \). If \( \mathcal{I} \subseteq \mathcal{N} \) then \( J \notin \mathcal{N} \) and \( B \notin \mathcal{N} \). Similarly if \( \mathcal{N} \subseteq \mathcal{I} \) then \( B \notin \mathcal{N} \). By the Baire Theorem the set of all points at which \( f|B \) not continuous is of the first category in \( B \). (cf. [9]) Thus there exists a point \( x \in U \cap C_i(f|U) = U \cap C_i(f) \).

**Proposition 4.** Let \( X = R \) and \( \mathcal{I} \) be the ideal of all measure zero subsets of \( X \). Then there exists a function \( f : R \to R \) such that

\[ S_i(f) = R \quad \text{and} \quad C_i(f) = \emptyset. \]
Proof. Assume that $A$ and $B$ is a partition of $R$ such that $A$ is a set of the first
category and $B \in \mathcal{Y}$. (cf. [6]). It is possible to assume that $A$, is a $F_\alpha$ set, $A = \bigcup_{n \in \mathbb{N}} F_n$,the sets $F_n$ are pairwise disjoint, closed and nowhere dense (see [8]). Notice that
infinite many of $F_n$ have a positive measure. In fact, suppose that there exists $m \in \mathbb{N}$
such that $F_n \in \mathcal{I}$ for $n > m$ Then the set $F = \bigcup_{m \leq k} F_k$ is closed, nowhere dense and
$R - F \in \mathcal{I}$ — a contradiction. Hence it is possible to assume that $F_n \notin \mathcal{I}$ for each
$n \in \mathbb{N}$.

Let us define a function $f: \mathbb{R} \to \mathbb{R}$:

$$f(x) = \begin{cases} n^{-1} & \text{for } x \in F_n, \\ 2 & \text{for } x \in B. \end{cases}$$

Then $f$ satisfies the conditions of this proposition.

If $x \in B$ then $x \in T_1(f) \subseteq S_1(f)$.

If $x \in F_n$ and $(x_k)$ is a sequence in $A$ then almost all terms of $(x_k)$ belong to $\bigcup_{k \leq m} F_k$.

Thus \(\limsup_{k \to \infty} (x_k) \leq n^{-1}\) and $I$-\(\limsup_{t \to x} f(t) \leq f(x)\). Since for each $m \in \mathbb{N}$ the set
$\bigcup_{k \leq m} F_k$ is nowhere dense, in every neighbourhood $U$ of $x$ there exist an open,
non-empty subset $V \subseteq \mathbb{R} - \bigcup_{k \leq m} F_k$. Thus $I$-\(\liminf_{t \to x} f(t) \leq m^{-1}\) for all $m \in \mathbb{N}$ and
consequently, $I$-\(\liminf_{t \to x} f(t) = 0\). Hence $C_1(f) = \emptyset$ and $S_1(f) = \mathbb{R}$.

For every $A \subseteq \mathbb{X}$ we define $\text{Int}_I A$ as follows:

$$\text{Int}_I A = \{ x \in A : \exists V(x \in V, V \text{ is open and } V - A \in \mathcal{Y}) \}.$$ 

Proposition 5. For every subset $A$ of $\mathbb{X}$ there exists a function $f: \mathbb{X} \to \mathbb{R}$ such
that $S_I(f) = A$.

Proof. Let $B = \text{Int}_I A$. By Lemma 0 there exists an open set $G$ and $I \in \mathcal{I}$ such
that $B = G - I$ and $G = \psi_I(G)$.

Let $(K_n)_{n \in \mathbb{N}}$ be a partition of the set $\mathbb{X} - G$ such that for each $x \in \mathbb{X}$ if $x \in d_I(\mathbb{X} - G)$
then $x \in d_I(K_n - G)$.

We define $f$ as follows:

$$f(x) = \begin{cases} 1 & \text{for } x \in A, \\ (-1)^n \frac{n}{n + 1} & \text{for } x \in K_n - A, \\ -1 & \text{for } x \in I - A. \end{cases}$$

For this function $S_I(f) = A$. 

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If \( x \in A \) then \( x \in d_f(X - G) \) or \( x \in I \). Indeed, suppose that \( x \notin A \) and \( x \notin I \). Then \( x \notin B \) and \( x \notin G \). Since \( \psi_f(G) = G \), \( x \in d_f(X - G) \).

If \( x \in I - A \) then \( x \in S_t(f) \subseteq X - S_t(f) \).

If \( x \in d_f(X - G) \) then \( f(x) \leq \limsup_{t \to x} f(t) \). Thus \( A = S_t(f) \).

REFERENCES


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Department of Mathematics
WSP Bydgoszcz
ul. Chodkiewicza 30
85-064 Bydgoszcz
POLAND

О ТОЧКАХ I-НЕПРЕРЫВНОСТИ И I-ПОЛУНЕПРЕРЫВНОСТИ

Tomasz Natkaniec

Резюме

Пусть \((X, \mathcal{F})\) — полское пространство и \(\mathcal{I} \subseteq 2^X\) — есть \(\sigma\)-идеал. I-топологией на \(X\) будем называть семейство \((A - B: A \in \mathcal{I}, B \in \mathcal{I})\). В работе исследованы связи между множествами точек непрерывности, точек I-непрерывности и точек I-полунепрерывности вещественной функции \(f: X \rightarrow R\). В частности, рассмотрен случай, когда \(X = R\) и \(\mathcal{I}\) есть идеал всех множеств с мерой Лебега равной нулю. В случае, когда \(\mathcal{I}\) является идеалом множеств первой категории, обобщены результаты З. Гранде.

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