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CONGRUENCE OF ANKENY-ARTIN-CHOWLA TYPE FOR CYCLIC FIELDS

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ABSTRACT. In this paper the congruence of Ankeny-Artin-Chowla type for real fields of prime conductor p is proved.

Introduction

Ankeny-Artin-Chowla obtained several congruences for the class number h_K of a quadratic field K, some of which were also obtained by Kiselev. In particular, if the discriminant of K is a prime number $p \equiv 1 \pmod 4$ and $\varepsilon = \frac{t+u\sqrt{p}}{2}$ is the fundamental unit of K, then

$$h_K \frac{u}{t} \equiv B_{\frac{p-1}{2}} \pmod p, \tag{1}$$

where B_n means the nth Bernoulli number.

Further results for more general fields K were obtained later by Carlitz, Slavutskij, Lang and Schertz, and Lu Hong Wen. Zhang Xian Ke [8] solved an analogous question for general cyclic quartic fields.

The solution of an analogous question for pure cubic fields obtained by H. Ito in [2] and for pure quartic and sixtic field by M. Kamei in [5].

In 1982 Feng Ke Qin in [1] obtained an analogue of (1) for the cyclic cubic fields.

Let β_0, β_1, β_2 be a basis of the field K formed by Gauss periods. There is a unit δ of the form $x\beta_0 + y\beta_1 + z\beta_2$, such that $\{\delta, \sigma\delta\}$ are fundamental units of K. (The unit δ is called the strong Minkowski unit.) Feng Ke Qin proved the following congruence. Let $k = \frac{p-1}{3}$, then

$$ch_K \equiv \frac{3}{4} B_k B_{2k} \pmod p, \tag{2}$$

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where

$$c = \frac{1}{p}\left[1 + \left(\frac{3}{x+y+z}\right)^3\right] + 3\frac{xy+xz+yz}{(x+y+z)^2} - 1 .$$

For cyclic fields of a prime conductor and a prime degree the congruence of Ankeny-Artin-Chowla type is given in [4].

The purpose of this paper is a slight modification of the proof published in [4].

Since it is not known whether for every cyclic field K there is a strong Minkowski unit, we shall make use of another unit. Note that the existence of the strong Minkowski unit is proved for cyclic fields of prime degree l for $l < 23$.

Let p, $p \equiv 1 \pmod n$, be a prime and K be a real subfield of $\mathbb{Q}(\zeta_p)$, where $\zeta_p = \cos\frac{2\pi}{p} + \mathrm{i}\sin\frac{2\pi}{p}$. Denote $n = [K : \mathbb{Q}]$ and $k = \frac{p-1}{n}$. Let a be a primitive root modulo p and g an integer satisfying $g \equiv a^k \pmod p$. We consider the automorphism σ of $\mathbb{Q}(\zeta_p)$ determined by $\sigma(\zeta_p) = \zeta_p^a$. We set $\beta_0 = \mathrm{Tr}_{\mathbb{Q}(\zeta_p)/K}(\zeta_p)$, $\beta_i = \sigma^i(\beta_0)$ for $i = 1, 2, \dots, n-1$.

According to [6] there is a unit δ of K such that $[U_K : \langle\delta\rangle] = f$ with $(p, f) = 1$, where U_K is the group of units of K and $\langle\delta\rangle$ means its subgroup generated by all conjugates of δ. Since the Gauss periods $\beta_0, \beta_1, \dots, \beta_{n-1}$ form an integral basis of $K/\mathbb{Q}$, there are integers $x_1, x_2, \dots, x_{n-1}$ satisfying

$$\delta = x_0\beta_0 + x_1\beta_1 + \cdots + x_{n-1}\beta_{n-1} .$$

Associate to the unit δ the polynomial $f(X)$ as follows:

$$f(X) = X^{n-1} + d_1X^{n-2} + d_2X^{n-3} + \cdots + d_{n-1} ,$$

where

$$d_i = \frac{1}{(ki)!}\frac{x_0 + x_1g^i + x_2g^{2i} + \cdots + x_{n-1}g^{i(n-1)}}{x_0 + x_1 + \cdots + x_{n-1}} .$$

Put

$$S_j = S_j(d_1, d_2, \dots, d_{n-1}) = \text{sum of } j\text{th powers of roots of polynomial } f(X) .$$

Hence

$$S_1 = -d_1, \quad S_2 = d_1^2 - 2d_2, \quad S_3 = -d_1^3 + 3d_1d_2 - 3d_3, \quad \dots .$$

We shall prove the following theorem:

THEOREM. *Let K be a subfield of the field $\mathbb{Q}(\zeta_p)$, $[K : \mathbb{Q}] = n$. Let $\delta = x_0\beta_0 + x_1\beta_1 + \cdots + x_{n-1}\beta_{n-1}$ be a unit such that $[U_K : \langle\delta\rangle] = f$, $(f, p) = 1$. The following congruence holds:*

(i) *for n odd*

$$\frac{h_K}{f}S_1S_2\cdots S_{n-1} \equiv (-1)^{\frac{n-1}{2}}\frac{n}{2^{n-1}}B_kB_{2k}\cdots B_{(n-1)k} \pmod p ,$$

(ii) *for* n *even*

$$\pm\frac{h_K}{f}S_1S_2\cdots S_{n-1}\equiv\frac{1}{\frac{p-1}{2}!}\frac{n}{2^{n-1}}B_kB_{2k}\cdots B_{(n-1)k}\pmod p,$$

where $k=\frac{p-1}{n}$.

Proof. Consider the determinant

$$\mathbf{B}=\begin{pmatrix}\boldsymbol{a}\\ \boldsymbol{a}\mathbf{A}\\ \boldsymbol{a}\mathbf{A}^2\\ \vdots\\ \boldsymbol{a}\mathbf{A}^{n-2}\end{pmatrix},$$

where

$$\mathbf{A}=\begin{pmatrix}0&1&0&\dots&0\\ 0&0&1&\dots&0\\ \dots&\dots&\dots&\dots&\dots\\ -1&-1&-1&\dots&-1\end{pmatrix}$$

and $\boldsymbol{a}=(a_0,a_1,\dots,a_{n-2})$. According to [6] there holds

$$\det\mathbf{B}=\prod_{i=1}^{n-1}\left(a_0+a_1\zeta_n^i+a_2\zeta_n^{2i}+\dots+a_{n-2}\zeta_n^{i(n-2)}\right).$$

The rest of the proof is the same as in [4]. □

Remark. The reason of the unknown sign in the case of n being even is as follows. In [4] we have used $e=\det\mathbf{B}$ since $\det\mathbf{B}>0$ in this case. But if n is even we have $e=|\det\mathbf{B}|$ and $\operatorname{sign}\det\mathbf{B}=\operatorname{sign}\sum_{i=0}^{n-2}(-1)^ia_i$, which we were not able to determine.

REFERENCES

[1] FENG, KE QIN: *The Ankeny-Artin-Chowla formula for cubic cyclic number fields*, J. China Univ. Sci. Tech. **12** (1982), 20–27.

[2] ITO, H.: *Congruence relations of Ankeny-Artin-Chowla type for pure cubic field*, Nagoya Math. J. **96** (1984), 95–112.

[3] JAKUBEC, S.: *The congruence for Gauss's period*, J. Number Theory **48** (1994), 36–45.

[4] JAKUBEC, S.: *Congruence of Ankeny-Artin-Chowla type for cyclic fields of prime degree* l, Math. Proc. Cambridge Philos. Soc. **119** (1996), 17–22.

[5] KAMEI, M.: *Congruences of Ankeny-Artin-Chowla type for pure quartic and sectic fields*, Nagoya Math. J. **108** (1987), 131–144.

[6] MARKO, F.: *On the existence of p-units and Minkowski units in totally real cyclic fields*, Abh. Math. Sem. Univ. Hamburg (To appear).

[7] SCHERTZ, R.: *Über die analitische Klassenzahlformel für realle abelsche Zahlkorper*, J. Reine Angew. Math. **307-308** (1979), 424–430.

[8] ZHANG, XIAN KE: *Ten formulae of type Ankeny-Artin-Chowla for class number of general cyclic quartic fields*, Sci. China Ser. A **32** (1989), 417–428.

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