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On almost continuous additive functions


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ON ALMOST CONTINUOUS ADDITIVE FUNCTIONS¹

ZBIGNIEW GRANDE

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ABSTRACT. It is proved that every additive function is the sum of two almost continuous (in Stallings' sense) additive functions and the limit of a sequence (of a transfinite sequence) of almost continuous additive functions. Moreover, it is shown that the maximal additive family for the set of all almost continuous additive functions having the graphs of the second category is contained in the class of continuous additive functions.

Let \( \mathbb{R} \) be the set of all reals. A function \( g: (a, b) \to \mathbb{R} \) is said to be almost continuous (in Stallings' sense \([5]\)) if for every open set \( D \subset \mathbb{R}^2 \) containing the graph \( G(g) \) of the function \( g \) there is a continuous function \( h: (a, b) \to \mathbb{R} \) with \( G(h) \subset D \).

A function \( f: \mathbb{R} \to \mathbb{R} \) is called additive (see, e.g., \([4]\)) if it satisfies Cauchy's equation

\[
f(x + y) = f(x) + f(y), \quad x, y \in \mathbb{R}.
\]

It is well known that there exists additive almost continuous function \( f: \mathbb{R} \to \mathbb{R} \) which is not continuous (\([3]\), see also \([2]\)). In this article, I prove that every additive function is the sum of two additive almost continuous functions and the limit of a sequence (of a transfinite sequence) of additive almost continuous functions, and I investigate the maximal additive family for the class of all additive almost continuous functions with the graphs of the second category.

Throughout the article, I assume the Continuum Hypothesis CH and all considered functions are real and of real variable.

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Let
\[ Add = \{ f : f \text{ is additive} \}, \]
\[ AC = \{ f : f \text{ is almost continuous} \}, \]
\[ C = \{ f : f \text{ is continuous} \}, \]
\[ \Theta = \{ f : \text{for every } g \in Add \cap AC, \ f + g \in Add \cap AC \}. \]

Denote by \( \mathbb{Q} \) the set of all rationals. Every linear basis in \( \mathbb{R} \) over \( \mathbb{Q} \) is called a Hamel basis in \( \mathbb{R} \). Let
\begin{equation}
I_1, \ldots, I_n, \ldots
\end{equation}
be a sequence of all open intervals with rational endpoints, let \( \omega_1 \) denote the first uncountable ordinal number, and let
\[ x_1, \ldots, x_\alpha, \ldots, \quad \alpha < \omega_1, \]
be a transfinite sequence of all reals.

**Remark 1.** There is a Hamel basis \( H \subset \mathbb{R} \) such that for every open interval \( I \subset \mathbb{R} \) the intersection \( I \cap H \) is of the cardinality continuum.

**Proof.** Let
\[ 0 \neq x_0^1 \in \mathbb{R} \]
be arbitrary, and for every \( \alpha < \omega_1 \) and \( n \geq 1 \) let \( x_\alpha^n \) be the first element \( x_\beta \) of the sequence (2) such that
\[ x_\beta \in I_n \]
and the set
\[ \{x_\beta^1\}_{\beta<\alpha, \ n \geq 1} \cup \{x_\alpha^k\}_{k \leq n} \]
is linearly independent over \( \mathbb{Q} \). Then the set
\[ H = \{x_\alpha^n : \alpha < \omega_1, \ n \geq 1\} \]
is a Hamel basis such that
\[ \text{card}(H \cap I) = c \]
for every open interval \( I \).

If \( f \) is a function, we mean by a **blocking set** of \( f \) a closed set \( K \subset \mathbb{R}^2 \) such that \( G(f) \cap K = \emptyset \) and \( G(g) \cap K \neq \emptyset \) for every continuous function \( g : \mathbb{R} \to \mathbb{R} \). An **irreducible blocking set** (IBS) \( K \) of \( f \) is a blocking set of \( f \) such that no proper subset of \( K \) is a blocking set ([3]).

It is known that \( f \) is almost continuous if and only if it has no blocking set. Moreover, if \( f \) is not almost continuous, then there is an (IBS) \( K \) of \( f \), and the \( x \)-projection \( \text{pr}_x(K) \) of \( K \) is a non-degenerate connected set ([3]).

Let
\begin{equation}
K_1, \ldots, K_\alpha, \ldots, \quad \alpha < \omega_1,
\end{equation}
be a transfinite sequence of all irreducible blocking sets in \( \mathbb{R}^2 \).
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THEOREM 1. If \( f \in \text{Add} \), then there are \( g, h \in \text{Add} \cap \mathcal{AC} \) such that \( f = g + h \).

Proof. Let \( H \) be a Hamel basis from Remark 1. For every \( \alpha < \omega_1 \) there are points
\[
x_\alpha, y_\alpha \in H \cap \text{pr}_x(K_\alpha)
\]
such that
\[
x_\alpha \neq y_\beta, \quad y_\alpha \neq x_\beta, \quad \text{for } \beta \leq \alpha
\]
and
\[
x_\alpha \neq x_\beta, \quad y_\alpha \neq y_\beta, \quad \text{for } \beta < \alpha.
\]

For every \( \alpha < \omega_1 \) let \( u_\alpha, v_\alpha \in \mathbb{R} \) be points such that
\[
(x_\alpha, u_\alpha) \in K_\alpha, \quad (y_\alpha, v_\alpha) \in K_\alpha.
\]

Define
\[
g_1(x) = \begin{cases} 
    u_\alpha & \text{if } x = x_\alpha, \ \alpha < \omega_1, \\
    f(x) - v_\alpha & \text{if } x = y_\alpha, \ \alpha < \omega_1, \\
    0 & \text{otherwise in } H,
\end{cases}
\]
\[
h_1(x) = \begin{cases} 
    f(x) - u_\alpha & \text{if } x = x_\alpha, \ \alpha < \omega_1, \\
    v_\alpha & \text{if } x = y_\alpha, \ \alpha < \omega_1, \\
    f(x) & \text{otherwise in } H,
\end{cases}
\]

and let \( g : \mathbb{R} \to \mathbb{R} \) (\( h : \mathbb{R} \to \mathbb{R} \)) be the additive extension of \( g_1 \) (of \( h_1 \)) (see, e.g., [4]).

Observe that for every \( \alpha < \omega_1 \),
\[
(x_\alpha, g(x_\alpha)) = (x_\alpha, u_\alpha) \in K_\alpha
\]
and
\[
(y_\alpha, h(y_\alpha)) = (y_\alpha, v_\alpha) \in K_\alpha.
\]

So, the functions \( g, h \) are almost continuous, and, evidently,
\[
f = g + h.
\]

This completes the proof. \( \square \)

Remark 2. The inclusion
\[
\text{Add} \cap \mathcal{C} \subset \Theta
\]
follows from [1].
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**Lemma 1.** Let \( f: \mathbb{Z} \rightarrow \mathbb{R} \) be a function with the graph \( G(f) \) of the second category in \( \mathbb{R}^2 \), and let \( g \) be an upper semi-continuous function with domain a non-degenerate interval \( J \supset \mathbb{Z} \). Then for every countable sets \( A, B \subset \mathbb{R} \) there is a point \( x \in \mathbb{Z} \setminus A \) such that \( f(x) + g(x) \) is not in \( B \).

**Proof.** Let \((b_k)_k\) be an enumeration of all points of the set \( B \). For \( k = 1, 2, \ldots \), let
\[
A_k = \{x \in \mathbb{Z} : f(x) + g(x) = b_k\}.
\]
If for every \( x \in \mathbb{Z} \setminus A \) we have
\[
f(x) + g(x) \in B,
\]
then
\[
G(f) \subset \bigcup_k G(f/A_k) \cup \{(x, y) : x \in A, \ y \in \mathbb{R}\},
\]
and for every \( k = 1, 2, \ldots \),
\[
G((f - b_k)/A_k) = G((-g)/A_k).
\]
Since the set \( G(-g) \) is nowhere dense, every set
\[
G((f - b_k)/A_k), \quad k = 1, 2, \ldots ,
\]
is also nowhere dense, and, consequently, every set \( G(f/A_k), \ k \geq 1 \), is the same. So, \( G(f) \) is of the first category, a contradiction. \( \square \)

**Remark 3.** If the graph \( G(f) \) of a function \( f \in (\text{Add} \cap \text{AC}) \setminus \mathcal{C} \) is of the second category, then for every open interval \( I = (a, b) \) the sets
\[
\{(x, f(x)) : x \in I, \ f(x) > 0\}
\]
and
\[
\{(x, f(x)) : x \in I, \ f(x) < 0\}
\]
are of the second category.

**Proof.** If \( G(f/I) \) is a subset of the first category, then for every bounded open interval \( J \) there is a function
\[
h(x) = ax + b, \quad x \in \mathbb{R}, \ a, b \in \mathbb{Q}, \ a \neq 0,
\]
such that \( J \subset h(I) \), and, consequently,
\[
G(f/J) \subset aG(f/I) + (b, f(b)) \cdot
\]
So, \( G(f/J) \) is of the first category for every open interval \( J \), and we obtain a contradiction, since \( G(f) \) is of the second category. Suppose that the set
\[
\{(x, f(x)) : x \in I, \ f(x) > 0\}
\]
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is of the first category. Since \( f \) is a discontinuous additive function with the Darboux property, the set \( \{ \lambda \in \mathbb{R} : f(x) = 0 \} \) is dense ([4]). There is an open interval \( J \subset I \) such that, at its center \( z \), we have \( f(z) = 0 \). Observe that, if \( x \in J \) and \( f(x) > 0 \), then
\[
f(z - (x - z)) = -f(x) < 0.
\]
Since the function
\[
h(x, y) = (2z - x, -y), \quad x \in I, \ y \in \mathbb{R},
\]
is a homeomorphism, and the set
\[
\{(x, f(x)) : x \in I, \ f(x) > 0\}
\]
is of the first category, the set
\[
\{(x, f(x)) : x \in I, \ f(x) < 0\}
\]
is the same. But the set
\[
\{(x, f(x)) : x \in I, \ f(x) = 0\}
\]
cannot be residual in \( I \times \mathbb{R} \), so we have a contradiction. \( \square \)

Now, let
\[
\Omega = \{ f \in \text{Add} \cap AC : G(f) \text{ is of the second category} \}
\]
and
\[
\Delta = \{ f : \text{for every } g \in \Omega, \ f + g \in \Omega \}.
\]

**PROBLEM 1.** Does exist a function \( f \in (\text{Add} \cap AC) \setminus C \) with the graph of the first category?

**Remark 4.** The inclusion
\[
\Delta \supset \text{Add} \cap AC
\]
is true.

**Proof.** If \( f \in \text{Add} \cap C \), then for every function \( g \in \Omega \) we have
\[
f + g \in \text{Add} \cap AC
\]
([1]). There is \( a \in \mathbb{R} \) such that
\[
f(x) = ax, \quad x \in \mathbb{R}.
\]
Since the function
\[
F(x, y) = (x, y + ax), \quad (x, y) \in \mathbb{R}^2
\]
is a homeomorphism, the graph of the function
\[
h(x) = g(x) + f(x) = g(x) + ax, \quad x \in \mathbb{R},
\]
is of the second category. So, \( f + g \in \Omega \), and the proof is completed. \( \square \)
THEOREM 2. If the function $f \in \Omega$, then there is a function $h \in \Omega$ such that $f + h$ is not in $\mathcal{AC}$.

Proof. Let

$$H = \{ t_0, \ldots, t_\alpha, \ldots \}$$

be a Hamel basis, let

$$M_0, \ldots, M_\alpha, \ldots, \quad \alpha < \omega_1,$$

be a transfinite sequence of all $G_\delta$-sets of the second category in $\mathbb{R}^2$, and let

$$G(u_0), \ldots, G(u_\alpha), \ldots, \quad \alpha < \omega_1,$$

be a transfinite sequence of the graphs of all upper semi-continuous functions with domains being non-degenerate intervals such that the domains $I_0$ of $u_0$ and $I_1$ of $u_1$ have positive endpoints.

There exists a point $(x_0, u_0(x_0)) \in G(u_0)$ such that

$$f(x_0) + u_0(x_0) > 0.$$ 

Let $(s_0, w_0) \in M_0$ be a point such that the sets $\{ x_0, s_0 \}$ and $\{ f(x_0) + u_0(x_0), f(s_0) + w_0 \}$ are linearly independent over $\mathbb{Q}$. Moreover, if $x_0, s_0, t_0$ are linearly independent over $\mathbb{Q}$, we find $v_0$ such that $f(t_0) + v_0, f(x_0) + u_0(x_0), f(s_0) + w_0$ are linearly independent over $\mathbb{Q}$.

Denote by $E_1$ ($F_1$, resp.) the linear subspace over $\mathbb{Q}$ generated by $\{ x_0, t_0, s_0 \}$ ($\{ f(t_0) + v_0, f(x_0) + u_0(x_0), f(s_0) + w_0 \}$).

By Lemma 1 and Remark 3, there is a point $x_1 \in I \setminus E_1$ such that $0 > f(x_1) + u_1(x_1)$ is not in $F_1$.

Let $(s_1, w_1) \in M_1$ be a point such that the set $\{ s_1, x_1 \} \cup E_1$ is linearly independent over $\mathbb{Q}$, and the set $\{ f(s_1) + w_1, f(s_1) + u_1(x_1) \} \cup F_1$ is also linearly independent over $\mathbb{Q}$. If the points

$$x_0, x_1, t_0, t_1, s_0, s_1$$

are linearly independent over $\mathbb{Q}$, we find a point $v_1$ such that the set $\{ f(t_1) + v_1, f(x_1) + u_1(x_1), f(s_1) + w_1 \} \cup F_1$ is linearly independent over $\mathbb{Q}$.

Fix an countable ordinal $\alpha > 1$ and suppose that we have defined points $x_\beta$, $(s_\beta, w_\beta) \in M_\beta$ and, if necessary, $v_\beta$, $1 < \beta < \alpha$, such that

$$f(x_\beta) + u_\beta(x_\beta), \quad f(t_\beta) + v_\beta, \quad f(s_\beta) + w_\beta, \quad \beta < \alpha,$$

are linearly independent over $\mathbb{Q}$, and $x_\beta, s_\beta$ and such $t_\beta$ for which $v_\beta$ exist are also linearly independent over $\mathbb{Q}$. Denote by $E_\alpha$ ($F_\alpha$) the linear subspace over $\mathbb{Q}$ generated by $\{ x_\beta : \beta < \alpha \} \cup \{ s_\beta : \beta < \alpha \}$ and such $t_\beta$ for which $v_\beta$
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exist \( \{ f(x_\beta) + u_\beta(x_\beta) : \beta < \alpha \} \cup \{ f(t_\beta) + v_\beta : \beta < \alpha \text{ and } v_\beta \text{ is chosen} \} \cup \{ f(s_\beta) + w_\beta : \beta < \alpha \} \). By Lemma 1, there is a point

\[ x_\alpha \in I_\alpha \setminus E_\alpha, \]

where \( I_\alpha \) is the domain of the function \( u_\alpha \), such that \( f(x_\alpha) + u_\alpha(x_\alpha) \) is not in \( E_\alpha \).

Let \((s_\alpha, w_\alpha) \in M_\alpha\) be a point such that the set \( \{ s_\alpha, x_\alpha \} \cup E_\alpha \) is linearly independent over \( \mathbb{Q} \), and the set \( \{ f(s_\alpha) + w_\alpha, f(x_\alpha) + u_\alpha(x_\alpha) \} \cup F_\alpha \) is also linearly independent over \( \mathbb{Q} \). If the set \( \{ t_\alpha, x_\alpha, s_\alpha \} \cup E_\alpha \) is linearly independent over \( \mathbb{Q} \), then we find a real \( v_\alpha \) such that the set \( \{ f(t_\alpha) + v_\alpha, f(s_\alpha) + w_\alpha, f(x_\alpha) + u_\alpha(x_\alpha) \} \cup F_\alpha \) is linearly independent over \( \mathbb{Q} \). All points \( x_\alpha, s_\alpha, \alpha < \omega_1 \), and such points \( t_\alpha \) for which \( v_\alpha \) exist form a Hamel basis \( H_1 \). Let \( h \) be the additive extension on \( \mathbb{R} \) of the function

\[ h_1(x) = \begin{cases} u_\alpha(x_\alpha) & \text{if } x = x_\alpha, \alpha < \omega_1, \\ v_\alpha & \text{if } x = t_\alpha \in H_1 \setminus \{ x_\beta, s_\beta : \beta < \omega_1 \}, \alpha < \omega_1, \\ w_\alpha & \text{if } x = s_\alpha, \alpha < \omega_1. \end{cases} \]

Since

\[ G(h) \cap G(u_\alpha) \neq \emptyset \]

and

\[ G(h) \cap M_\alpha \neq \emptyset \]

for every \( \alpha < \omega_1 \), the function \( h \) is almost continuous ([3]), and its graph \( G(h) \) is of the second category. Suppose that

\[ f(x) + h(x) = 0 \]

for some \( x > 0 \). Then

\[ x = r_1 z_1 + \ldots + r_k z_k, \]

where \( r_i \in \mathbb{Q} \setminus \{0\} \) and \( z_i \in H_1 \) for \( i \leq k \). Thus

\[ r_1(f + h)(z_1) + \ldots + r_k(f + h)(z_k) = 0, \]

which is a contradiction with the linear independence of

\( (f + h)(z_1), \ldots, (f + h)(z_k) \).

But

\[ (f + h)(x_0) > 0 \]

and

\[ (f + h)(x_1) < 0, \]

so \( f + h \) has not the Darboux property. Thus \( f + h \) is not almost continuous ([3]), and the function \( f \) is not in the collection \( \Omega \). This completes the proof. \( \square \)
PROBLEM 2. Are the following equalities true:

\[ \text{Add} \cap \mathcal{C} = \Theta = \Delta ? \]

THEOREM 3. If \( f \in \text{Add} \), then there is a sequence of functions \( f_n \in \text{Add} \cap \mathcal{AC} \), \( n \geq 1 \), such that \( f = \lim_{n \to \infty} f_n \).

Proof. Let \( H \subset \mathbb{R} \) be a Hamel basis satisfying the condition from Remark 1. For every \( \alpha < \omega_1 \) there is a sequence of points

\[ x_{\alpha,n} \in H \cap \text{pr}_x(K_\alpha), \quad n = 1, 2, \ldots, \]

such that

\[ x_{\alpha,n} \neq x_{\beta,k} \]

if

\[ (\alpha, n) \neq (\beta, k), \quad \beta < \alpha, \quad k, n = 1, 2, \ldots. \]

For each point \( x_{\alpha,n} \) there is a point \( y_{\alpha,n} \) such that

\[ (x_{\alpha,n}, y_{\alpha,n}) \in K_\alpha, \quad \alpha < \omega_1, \quad n \geq 1. \]

Define, for \( n = 1, 2, \ldots, \)

\[ g_n(x) = \begin{cases} y_{\alpha,k} & \text{if } x = x_{\alpha,k}, \quad \alpha < \omega_1, \quad k \geq n, \\ f(x) & \text{otherwise in } H, \end{cases} \]

and let \( f_n \) be the additive extension of \( g_n \) on \( \mathbb{R} \). Since

\[ (x_{\alpha,n}, y_{\alpha,n}) \in K_\alpha \cap G(f_n) \]

for \( \alpha < \omega_1 \) and \( n \geq 1 \), all functions \( f_n \) are almost continuous. Moreover, if \( x = x_{\alpha,k}, \quad \alpha < \omega_1, \quad k \geq 1 \), then \( f_n(x) = f(x) \) for \( n > k \), and if \( x \in H \), and \( x \neq x_{\alpha,k} \) for all \( \alpha < \omega_1 \) and \( k \geq 1 \), then \( f_n(x) = f(x) \) for all \( n \geq 1 \). So, \( f = \lim_{n \to \infty} f_n \) on \( H \) and, consequently, on \( \mathbb{R} \). Thus the proof is completed. \( \square \)

THEOREM 4. If \( f \in \text{Add} \), then there is a transfinite sequence of functions \( f_\alpha \in \text{Add} \cap \mathcal{AC} \), \( \alpha < \omega_1 \), such that \( \lim_{\alpha} f_\alpha = f \), i.e.,

\[ \forall x \exists \beta < \omega_1 \forall \omega_1 > \alpha > \beta \quad f_\alpha(x) = f(x). \]

Proof. Let a Hamel basis \( H \) be the same as in the proof of Theorem 3. There are pairwise disjoint sets \( T_\alpha \), \( \alpha < \omega_1 \), such that every set

\[ H \cap \text{pr}_x(K_\alpha) \cap T_\alpha, \quad \alpha < \omega_1, \]

is uncountable. For each \( \alpha < \omega_1 \) let

\[ (x_{\alpha,\beta})_{\beta < \omega_1} \]
be a transfinite sequence of all points of the set
\[ H \cap \text{pr}_x(K_\alpha) \cap T_\alpha, \]
and let
\[ g_\alpha(x) = \begin{cases} y_{\alpha,\beta} & \text{if } x = x_{\alpha,\beta}, \; \omega_1 > \beta \geq \alpha, \\ f(x) & \text{otherwise in } H, \end{cases} \]
where \( y_{\alpha,\beta} \) are points such that
\[ (x_{\alpha,\beta}, y_{\alpha,\beta}) \in K_\beta, \quad \alpha, \beta < \omega_1, \]
and let \( f_\alpha \) be the additive extension \( g_\alpha \) on \( \mathbb{R} \). Analogously as in the proof of Theorem 3, we can observe that all functions \( f_\alpha \) are almost continuous and
\[ \lim_{\alpha} f_\alpha = f. \]
This completes the proof. \( \square \)

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