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A NOTE TO THE TRANSCENDENCE OF SPECIAL INFINITE SERIES

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ABSTRACT. The main result of this paper is a criterion for the sums of infinite series to be transcendental. The terms of these series are positive rational numbers which converge rapidly to zero. The speed of the convergence oscillates.

1. Introduction

Many recent results of the theory of transcendence can be found in the book of Parshkin and Shafarevich in [9]. The book of Nishioka [7] on Mahler theory is also interesting.

Some new recent results for the transcendence of infinite series which rapidly converge can also be found in Adhikari, Saradha, Shorey and Tijdeman [1], Hančl [4] and Nyblom [8].

Duverney in [3] proved a theorem which gives a criterion for the sums of infinite series to be transcendental. The terms of these series consist of the rational numbers and converge regulary and very quickly to zero.

Recently Hančl [5] introduced the concept of transcendental sequences in the following way.

DEFINITION 1. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of positive real numbers. If for every sequence $\{c_n\}_{n=1}^{\infty}$ of positive integers the number $\sum\limits_{n=1}^{\infty} 1/(a_n c_n)$ is transcendental, then the sequence $\{a_n\}_{n=1}^{\infty}$ is called *transcendental*.

He also proved in this paper the criterium for sequences to be transcendental. Criteria for the sums of the series to be Liouville numbers, which are special kinds of the transcendental numbers, can be found in [6].

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2. Main results

The main result of this paper is the theorem which shows that sums of certain infinite series are transcendental.

THEOREM 2.1. *Let δ and ε be positive real numbers and $s \in \mathbb{N}$. Let $\{L_i(x)\}_{i=0}^{\infty}$ be the sequence of logarithmic functions defined in the following way:*

$$L_0(x) = x$$
$$\text{and} \qquad L_i(x) = \underbrace{\log \dots \log}_{i\text{-times}} x \,, \qquad 0 < i \le s \,.$$

Assume that $\{a_k\}_{k=1}^{\infty}$ and $\{b_k\}_{k=1}^{\infty}$ are two sequences of positive integers such that

$$\limsup_{k\to\infty} \frac{L_s^{\varepsilon}\left(\frac{a_{k+1}}{b_{k+1}\left(\prod_{i=1}^{s} L_i\left(\frac{a_{k+1}}{b_{k+1}}\right)\right) L_s^{\varepsilon}\left(\frac{a_{k+1}}{b_{k+1}}\right)}\right)}{(a_1 a_2 \cdots a_k)^{2+\delta}} = \infty \,, \tag{1}$$

and for every sufficiently large k

$$\frac{a_{k+1}}{b_{k+1}\left(\prod_{i=1}^{s} L_i\left(\frac{a_{k+1}}{b_{k+1}}\right)\right) L_s^{\varepsilon}\left(\frac{a_{k+1}}{b_{k+1}}\right)} \ge \frac{a_k}{b_k\left(\prod_{i=1}^{s} L_i\left(\frac{a_k}{b_k}\right)\right) L_s^{\varepsilon}\left(\frac{a_k}{b_k}\right)} + 1 \,. \tag{2}$$

Then the number

$$\xi = \sum_{k=1}^{\infty} \frac{b_k}{a_k} < \infty$$

is transcendental.

EXAMPLE 1. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of positive integers such that $a_1 = 7$ and for every $n = 1, 2, \dots$

$$a_{n+1} = \begin{cases} 2^{n^{(a_1 a_2 \dots a_n)^3}} & \text{if } n = 3^{3^{3^{3^m}}} \text{ where } m \in \mathbb{N}, \\ a_n + \left[2L_1(a_n) \cdot L_2^2(a_n)\right] & \text{otherwise.} \end{cases}$$

Let us put $\delta = \varepsilon = 1$ and $s = 2$ in Theorem 2.1. Then we obtain that the number

$$\sum_{n=1}^{\infty} \frac{1}{a_n}$$

is transcendental.

EXAMPLE 2. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of positive integers such that $a_1 = 14$ and for every $n = 1, 2, \dots$

$$a_{n+1} = \begin{cases} n^{(a_1 a_2 \dots a_n)^3} & \text{if } n = 2^{3^m} \text{ where } m \in \mathbb{N}, \\ a_n + [2 \log^2 a_n] & \text{otherwise.} \end{cases}$$

Let us put $s = 1$ and $\delta = \varepsilon = 1$ in Theorem 2.1. Then we obtain that the number

$$\sum_{n=1}^{\infty} \frac{1}{a_n}$$

is transcendental.

OPEN PROBLEM 1. *Let* $\operatorname{lcm}(1,2,3,\dots,n)$ *be the least common multiply of the numbers* $1,2,3,\dots,n$. *Is there any sequence* $\{c_n\}_{n=1}^{\infty}$ *of positive integers such that the number* $\sum_{n=1}^{\infty} 1/\big(2^{\operatorname{lcm}(1,2,3,\dots,n)} c_n\big)$ *is algebraic?*

3. Proofs

Proof of Theorem 2.1. Let $x = F(y)$ be the inverse function of the function

$$y = f(x) = \frac{x}{\Big(\prod_{i=1}^{s} L_i(x)\Big) L_s^{\varepsilon}(x)} \,. \tag{3}$$

It follows that $y \le x$ for every sufficiently large x. From this we obtain the fact that

$$y = \frac{x}{\Big(\prod_{i=1}^{s} L_i(x)\Big) L_s^{\varepsilon}(x)} \le \frac{x}{\Big(\prod_{i=1}^{s} L_i(y)\Big) L_s^{\varepsilon}(y)} = \frac{F(y)}{\Big(\prod_{i=1}^{s} L_i(y)\Big) L_s^{\varepsilon}(y)} \,.$$

Multiplying both sides of this inequality by $\Big(\prod_{i=1}^{s} L_i(y)\Big) L_s^{\varepsilon}(y)$ we get for every large y

$$F(y) \ge \Big(\prod_{i=0}^{s} L_i(y)\Big) L_s^{\varepsilon}(y) \,. \tag{4}$$

Assumption (2) can be rewritten in the form

$$f\big(\tfrac{a_{k+1}}{b_{k+1}}\big) \ge f\big(\tfrac{a_k}{b_k}\big) + 1 \,.$$

From this and by using mathematical induction we obtain for every sufficiently large k and every positive integer t

$$f\big(\tfrac{a_{k+t}}{b_{k+t}}\big) \ge f\big(\tfrac{a_k}{b_k}\big) + t \,. \tag{5}$$

It follows that $\lim_{k\to\infty} f(a_k/b_k) = \infty$. Recall that the function $f(x)$ is increasing on (a,∞) for any sufficiently large a. This implies that the function $F(y)$ is increasing on (b,∞) for any sufficiently large number b. This fact together with the fact that $\lim_{k\to\infty} f(a_k/b_k) = \infty$ and inequality (5) imply that for every sufficiently large k and every positive integer t

$$\frac{a_{k+t}}{b_{k+t}} = F\Big(f\big(\tfrac{a_{k+t}}{b_{k+t}}\big)\Big) \geq F\Big(f\big(\tfrac{a_k}{b_k}\big) + t\Big)\,.$$

From this together with (4) we obtain

$$\frac{a_{k+t}}{b_{k+t}} \geq F\Big(f\big(\tfrac{a_k}{b_k}\big) + t\Big) \geq \left(\prod_{i=0}^{s} L_i\Big(f\big(\tfrac{a_k}{b_k}\big) + t\Big)\right) L_s^{\varepsilon}\Big(f\big(\tfrac{a_k}{b_k}\big) + t\Big)\,. \tag{6}$$

We have for every sufficiently large real number z

$$\sum_{r=0}^{\infty} \frac{1}{\Big(\prod\limits_{i=0}^{s} L_i(z+r)\Big) L_s^{\varepsilon}(z+r)} < \int_{z-1}^{\infty} \frac{\mathrm{d}x}{\Big(\prod\limits_{i=0}^{s} L_i(x)\Big) L_s^{\varepsilon}(x)} = \frac{1}{\varepsilon L_s^{\varepsilon}(z-1)}\,. \tag{7}$$

Let M be a positive real number. Then (1) implies that there exist infinitely many k such that

$$\frac{1}{L_s^{\varepsilon}\Big(f\big(\frac{a_{k+1}}{b_{k+1}}\big)\Big)} < \frac{1}{M(a_1 a_2 \dots a_k)^{2+\delta}}\,. \tag{8}$$

Now, we use (6) to get that for infinitely many k and for every positive integer t

$$\begin{aligned}
\left|\xi - \sum_{i=1}^{k} \frac{b_i}{a_i}\right| = \left|\sum_{i=k+1}^{\infty} \frac{b_i}{a_i}\right| &\leq \left|\sum_{t=0}^{\infty} \frac{1}{F\Big(f\big(\frac{a_{k+1}}{b_{k+1}}\big) + t\Big)}\right| \\
&\leq \left|\sum_{t=0}^{\infty} \frac{1}{\Big(\prod\limits_{i=0}^{s} L_i\Big(f\big(\frac{a_{k+1}}{b_{k+1}}\big) + t\Big)\Big) L_s^{\varepsilon}\Big(f\big(\frac{a_{k+1}}{b_{k+1}}\big) + t\Big)}\right|.
\end{aligned}$$

This and (7) imply that

$$\begin{aligned}
\left|\xi - \sum_{i=1}^{k} \frac{b_i}{a_i}\right| &\leq \left|\sum_{t=0}^{\infty} \frac{1}{\Big(\prod\limits_{i=0}^{s} L_i\Big(f\big(\frac{a_{k+1}}{b_{k+1}}\big) + t\Big)\Big) L_s^{\varepsilon}\Big(f\big(\frac{a_{k+1}}{b_{k+1}}\big) + t\Big)}\right| \\
&\leq \frac{1}{\varepsilon L_s^{\varepsilon}\Big(f\big(\frac{a_{k+1}}{b_{k+1}}\big) - 1\Big)} < \frac{c}{\varepsilon L_s^{\varepsilon}\Big(f\big(\frac{a_{k+1}}{b_{k+1}}\big)\Big)}\,,
\end{aligned}$$

where the positive real constant c depends on ε and s only. From this, (8) and choosing $M > c/\varepsilon$ we get for infinitely many positive integers k

$$\left|\xi - \sum_{i=1}^{k} \frac{b_i}{a_i}\right| \le \frac{c}{\varepsilon L_s^{\varepsilon}\left(f\left(\frac{a_{k+1}}{b_{k+1}}\right)\right)} \le \frac{c}{\varepsilon M} \cdot \frac{1}{(a_1 a_2 \cdots a_k)^{2+\delta}} < \frac{1}{(a_1 a_2 \cdots a_k)^{2+\delta}} . \tag{9}$$

Also if we write

$$\left|\xi - \sum_{i=1}^{k} \frac{b_i}{a_i}\right| = \left|\xi - \frac{B_k}{a_1 a_2 \cdots a_k}\right|, \qquad B_k \in \mathbb{N},$$

then from this together with (9) and Roth's theorem we obtain that the number ξ is transcendental. □

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