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PERIODICALLY FORCED DAMPED BEAMS
RESTING ON NONLINEAR ELASTIC BEARINGS

MICHAL FEČKAN

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ABSTRACT. We show the existence of periodic solutions for certain damped linear beam equations with periodic perturbations resting on nonlinear elastic bearings.

1. Introduction

We consider the equation

\[ u_{tt} + u_{xxxx} + \delta u_t + h(x, t) = 0, \]
\[ u_{xx}(0, \cdot) = u_{xx}(\pi/4, \cdot) = 0, \]
\[ u_{xxx}(0, \cdot) = -f(u(0, \cdot)), \quad u_{xxx}(\pi/4, \cdot) = g(u(\pi/4, \cdot)), \]

where \( \delta > 0 \) is a constant, \( f, g \) are analytic and \( h \) is a forcing term \( T \)-periodic in \( t \). Equation (1) describes vibrations of a beam resting on two different bearings with purely elastic responses which are determined by \( f \) and \( g \). The length of the beam is \( \pi/4 \). We are interested in forced periodic vibrations of (1).

The existence of periodic, homoclinic and chaotic solutions is shown in the papers [1] [4] for several types of nonlinearities of (1).

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2. Setting of the problem

By a weak $T$-periodic solution of (1), we mean any $u(x, t) \in C([0, \pi/4] \times S^T)$ satisfying the identity

$$
\int_0^T \int_0^{\pi/4} \left[ u(x, t) \{ v_{tt}(x, t) + v_{xxxx}(x, t) - \delta v_t(x, t) \} + h(x, t) v(x, t) \right] \, dx \, dt \\
+ \int_0^T (f(u(0, t)) v(0, t) + g(u(\pi/4, t)) v(\pi/4, t)) \, dt = 0
$$

for any $v(x, t) \in C^\infty([0, \pi/4] \times S^T)$ such that the following boundary value conditions hold

$$
v_{xx}(0, \cdot) = v_{xx}(\pi/4, \cdot) = v_{xxx}(0, \cdot) = v_{xxx}(\pi/4, \cdot) = 0.
$$

Here $S^T = \mathbb{R}/\{T\mathbb{Z}\}$ is the circle. The eigenvalue problem

$$
w_{xxxx}(x) = \mu^4 w(x), \\
w_{xx}(0) = w_{xx}(\pi/4) = 0, \\
w_{xxx}(0) = w_{xxx}(\pi/4) = 0
$$

is known ([4]) to possesses a sequence of eigenvalues $\mu_k$, $k = -1, 0, 1, \ldots$, with $\mu_{-1} = \mu_0 = 0$

and

$$
\cos(\mu_k \pi/4) \cosh(\mu_k \pi/4) = 1, \quad k = 1, 2, \ldots.
$$

The corresponding orthonormal in $L^2(0, \pi/4)$ system of eigenvectors is

$$
w_{-1}(x) = \frac{2}{\sqrt{\pi}}, \quad w_0(x) = \frac{16}{\pi} \left( x - \frac{\pi}{8} \right) \sqrt{\frac{3}{\pi}}, \\
w_k(x) = \frac{4}{\sqrt{\pi} W_k} \left[ \cosh(\mu_k x) + \cos(\mu_k x) \\
- \frac{\cosh \xi_k - \cos \xi_k}{\sinh \xi_k - \sin \xi_k} (\sinh(\mu_k x) + \sin(\mu_k x)) \right]
$$

where the constants $W_k$ are given by the formulas

$$
W_k = \cosh \xi_k + \cos \xi_k - \frac{\cosh \xi_k - \cos \xi_k}{\sinh \xi_k - \sin \xi_k} (\sinh \xi_k + \sin \xi_k)
$$

for $\xi_k = \mu_k \pi/4$, $k \in \mathbb{N}$. From (4) we get the asymptotic formulas

$$
1 < \mu_k = 2(2k + 1) + r(k) \quad \text{for all} \quad k \geq 1
$$

along with

$$
|r(k)| \leq \hat{c}_1 e^{-\hat{c}_2 k} \quad \text{for all} \quad k \geq 1,
$$

where $\hat{c}_1$, $\hat{c}_2$ are positive constants. Moreover, the eigenfunctions $\{w_i\}_{i=-1}^\infty$ are uniformly bounded in $C([0, \pi/4])$. 

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3. Preliminary results

Let $H_1(x,t) \in C([0,\pi/4] \times S^T)$, $H_2(t), H_3(t) \in C(S^T)$ be continuous $T$-periodic functions and consider the equation

$$
\int_0^{T/4} \int_0^{\pi/4} \left[ z(x,t) \{ v_{tt}(x,t) + v_{xxxx}(x,t) - \delta v_t(x,t) \} + H_1(x,t)v(x,t) \right] \, dx \, dt 
+ \int_0^T \left\{ H_2(t)v(0,t) + H_3(t)v(\pi/4,t) \right\} \, dt = 0
$$

(5)

for any $v(x,t) \in C^\infty([0,\pi/4] \times S^T)$ satisfying the boundary conditions (3) along with

$$
\int_0^{\pi/4} v(x,t) \, dx = \int_0^{\pi/4} x v(x,t) \, dx = 0 \quad \text{for all } t \in S^T.
$$

(6)

Note that conditions (6) correspond to the orthogonality of $v(x,t)$ to $w_{-1}(x)$ and $w_0(x)$ for any $t \in S^T$. We look for $z(x,t)$ in the form

$$
z(x,t) = \sum_{i=1}^\infty z_i(t)w_i(x).
$$

(7)

We formally put (7) into (5) to get a system of ordinary differential equations

$$
\ddot{z}_i(t) + \delta \dot{z}_i(t) + \mu_i^4 z_i(t) = h_i(t),
$$

(8)

where

$$
h_i(t) = - \left( \int_0^{\pi/4} H_1(x,t)w_i(x) \, dx + H_2(t)w_i(0) + H_3(t)w_i(\pi/4) \right).
$$

(9)

Let us put

$$
M_i = \sup_{i \geq 1, \ x \in [0,\pi/4]} |w_i(x)|.
$$

Let $\omega = 2\pi/T$. We consider Banach spaces $X_\omega$ and $Y_\omega$ defined as follows

$$
X_\omega := \left\{ u(x,t) \in C([0,\pi/4] \times S^T) : u(x,t) = \sum_{k \in \mathbb{Z}} u_k(x) e^{i\omega kt}, \quad u_k \in C([0,\pi/4], \mathbb{C}), \quad \sum_{k \in \mathbb{Z}} \|u_k\|_\infty < \infty, \quad u_{-k}(x) = \overline{u_k(x)} \right\}
$$

$$
Y_\omega := \left\{ v(t) \in C(S^T) : v(t) = \sum_{k \in \mathbb{Z}} v_k e^{i\omega kt}, \quad v_k \in \mathbb{C}, \quad \sum_{k \in \mathbb{Z}} |v_k| < \infty, \quad v_{-k} = \overline{v_k} \right\}
$$

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with the norms
\[ |u| := \sum_{k \in \mathbb{Z}} \|u_k\|_\infty, \quad |v| := \sum_{k \in \mathbb{Z}} |v_k|, \]
respectively, where \( \| \cdot \|_\infty \) is the maximum norm. Clearly, \( \|u\|_\infty \leq |u| \) and \( \|v\|_\infty \leq |v| \).

We also consider the Banach spaces \( X_{\omega,0} \) defined as follows
\[ X_{\omega,0} = \{ v(x,t) \in X_\omega : \text{conditions of (6) hold}\} \]
with the same norm \( |\cdot| \) as for \( X_\omega \).

If \( H_1(x,t) \in X_{\omega,0} \) and \( H_2(t), H_3(t) \in Y_\omega \), then
\[ H_1(x,t) = \sum_{k \in \mathbb{Z}} h_{1,k}(x) e^{i\omega kt}, \]
\[ \int_0^{\pi/4} h_{1,k}(x) \, dx = \int_0^{\pi/4} x h_{1,k}(x) \, dx = 0, \quad h_{1,-k}(x) = \overline{h_{1,k}(x)}, \]
\[ H_2(t) = \sum_{k \in \mathbb{Z}} h_{2,k} e^{i\omega kt}, \quad h_{2,-k} = \overline{h_{2,k}}, \]
\[ H_3(t) = \sum_{k \in \mathbb{Z}} h_{3,k} e^{i\omega kt}, \quad h_{3,-k} = \overline{h_{3,k}}. \]

Hence \( h_i(t) \) from (9) has the form
\[ h_i(t) = \sum_{k \in \mathbb{Z}} h_{i,k} e^{i\omega kt} \quad (10) \]
with
\[ h_{i,k} = -\left( \int_0^{\pi/4} h_{1,k}(x)w_i(x) \, dx + h_{2,k}w_i(0) + h_{3,k}w_i(\pi/4) \right). \]

Clearly \( h_{i,-k} = \overline{h_{i,k}} \). Consequently, we get
\[ |h_i| = \sum_{k \in \mathbb{Z}} |h_{i,k}| \leq M_1 \left( \frac{\pi}{4} |H_1| + |H_2| + |H_3| \right). \quad (11) \]

Now we look for a solution \( z_i \in Y_\omega \) of (8) with \( h_i(t) \) of the form (10). Hence from \( z_i(t) = \sum_{k \in \mathbb{Z}} z_{i,k} e^{i\omega kt} \) we get
\[ z_{i,k} = \frac{h_{i,k}}{\mu_i^4 - \omega^2 k^2 + i\delta \omega k}, \]
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Clearly, \( z_{i,-k} = \overline{z_{i,k}} \). For any \( t \geq 0 \), we have

\[
(\mu_i^4 - t)^2 + \delta^2 t \geq \gamma_i^2
\]

for the constants \( \gamma_i \) defined as follows

\[
\gamma_i = \gamma(\mu_i, \delta, \omega) := \begin{cases} 
\frac{\mu_i^4}{2} & \text{for } \delta^2 \geq 2\mu_i^4, \\
\Delta_i & \text{for } 0 < \delta^2 \leq 2\mu_i^4.
\end{cases}
\]

Thus we get

\[
|z_i| = \sum_{k \in \mathbb{Z}} |z_{i,k}| \leq |h_i|/\gamma_i.
\]

Clearly such \( z_i(t) \) satisfies (8). Now the series \( \sum_{i=1}^{\infty} 1/\gamma_i \) converges, so the function (7) is well-defined and

\[
z(x, t) = \sum_{i=1}^{\infty} z_i(t)w_i(x) = \sum_{i=1}^{\infty} \sum_{k \in \mathbb{Z}} \frac{h_{i,k}}{\mu_i^4 - \omega^2 k^2 + i\delta \omega k} e^{i\omega k t} w_i(x)
\]

\[
= \sum_{k \in \mathbb{Z}} \left( \sum_{i=1}^{\infty} \frac{h_{i,k}}{\mu_i^4 - \omega^2 k^2 + i\delta \omega k} w_i(x) \right) e^{i\omega k t}.
\]

Hence \( z(x, t) \in X_{\omega,0} \) and by (11), it satisfies

\[
|z| = \sum_{k \in \mathbb{Z}} \left( \sum_{i=1}^{\infty} \frac{h_{i,k}}{\mu_i^4 - \omega^2 k^2 + i\delta \omega k} w_i(x) \right) \leq M_1 \sum_{i=1}^{\infty} \frac{1}{\gamma_i} \sum_{k \in \mathbb{Z}} |h_{i,k}|
\]

\[
= M_1 \sum_{i=1}^{\infty} \frac{1}{\gamma_i} |h_i| \leq M_2 \left( \frac{\pi}{4} |H_1| + |H_2| + |H_3| \right),
\]

where

\[
M_2 := M_1 \sum_{i=1}^{\infty} \frac{1}{\gamma_i} < \infty.
\]

Summarizing, we get the next result.

**PROPOSITION 1.** For any given functions \( H_1(x, t) \in X_{\omega,0}, H_2(t), H_3(t) \in Y_\omega \) equation (5) has a unique solution \( z(x, t) \in X_{\omega,0} \) of the form

\[
z(x, t) = \sum_{i=1}^{\infty} z_i(t)w_i(x)
\]

with \( z_i(t) \in Y_\omega \) for any \( i \geq 1 \). Such a solution satisfies:

(a) \( |z| \leq M_2 \left( \frac{\pi}{4} |H_1| + |H_2| + |H_3| \right) \).

(b) The mapping \( L_1: X_{\omega,0} \times Y_\omega \times Y_\omega \to X_{\omega,0} \) defined by \( L_1(H_1, H_2, H_3) := z(x, t) \) is compact.
Proof. It remains to prove the compactness of $L_1$. For this reason, let us put

$$
\gamma_{i,k} := \sqrt{(\mu_i^4 - \omega^2 k^2)^2 + \delta^2 \omega^2 k^2}.
$$

Clearly, $\gamma_{i,k} \geq \gamma_i$ and $\gamma_{i,k} \geq \delta \omega |k|$ for any $i \geq 1$ and $k \in \mathbb{Z}$. Now let us take a bounded sequence $\{(H_{1,n}, H_{2,n}, H_{3,n})\}_{n \in \mathbb{N}} \subset X_{\omega,0} \times Y_{\omega} \times Y_{\omega}$. Hence

$$
H_{1,n}(x,t) = \sum_{k \in \mathbb{Z}} h_{1,k,n}(x) e^{i \omega kt},
$$

$$
\int_0^{\pi/4} h_{1,k,n}(x) \, dx = \int_0^{\pi/4} x h_{1,k,n}(x) \, dx = 0, \quad h_{1,-k,n}(x) = \overline{h_{1,k,n}(x)},
$$

$$
H_{2,n}(t) = \sum_{k \in \mathbb{Z}} h_{2,k,n} e^{i \omega kt}, \quad h_{2,-k,n} = \overline{h_{2,k,n}},
$$

$$
H_{3,n}(t) = \sum_{k \in \mathbb{Z}} h_{3,k,n} e^{i \omega kt}, \quad h_{3,-k,n} = \overline{h_{3,k,n}}.
$$

Then we get

$$
z_n(x,t) = \sum_{k \in \mathbb{Z}} \left( \sum_{i=1}^{\infty} \frac{h_{i,k,n}}{\mu_i^4 - \omega^2 k^2 + i \delta \omega k} w_i(x) \right) e^{i \omega kt},
$$

where

$$
h_{i,k,n} = - \left( \int_0^{\pi/4} h_{1,k,n}(x) w_i(x) \, dx + h_{2,k,n} w_i(0) + h_{3,k,n} w_i(\pi/4) \right).
$$

We note that there is a constant $\tilde{K}_1 > 0$ such that

$$
\sum_{k \in \mathbb{Z}} |h_{i,k,n}| \leq \tilde{K}_1
$$

for any $i,n \in \mathbb{N}$. By using the Cantor diagonal procedure, we can suppose that $h_{i,k,n} \to h_{i,k,0}$ as $n \to \infty$ for any $i \in \mathbb{N}$ and $k \in \mathbb{Z}$. Then clearly

$$
\sum_{k \in \mathbb{Z}} |h_{i,k,0}| \leq \tilde{K}_1
$$

for any $i \in \mathbb{N}$. So the function

$$
z_0(x,t) = \sum_{k \in \mathbb{Z}} \left( \sum_{i=1}^{\infty} \frac{h_{i,k,0}}{\mu_i^4 - \omega^2 k^2 + i \delta \omega k} w_i(x) \right) e^{i \omega kt}
$$

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belongs to $X_{\omega,0}$. We prove that $z_n(x,t) \to z_0(x,t)$ in $X_{\omega,0}$. So let $\varepsilon > 0$ be given. We take so large $i_0, k_0 \in \mathbb{N}$ that

$$\sum_{i=i_0+1}^{\infty} \frac{1}{\gamma_i} \leq \frac{\varepsilon}{6K_1 M_1}, \quad 6K_1 M_1 i_0 \leq \varepsilon \delta \omega (k_0 + 1).$$

Then

$$\sum_{k \in \mathbb{Z}} \sum_{i=i_0+1}^{\infty} \frac{|h_{i,k,n} - h_{i,k,0}|}{\gamma_i, k} \leq \sum_{k \in \mathbb{Z}} \sum_{i=i_0+1}^{\infty} \frac{|h_{i,k,n} - h_{i,k,0}|}{\gamma_i}$$

$$= \sum_{i=i_0+1}^{\infty} \frac{1}{\gamma_i} \sum_{k \in \mathbb{Z}} |h_{i,k,n} - h_{i,k,0}|$$

$$\leq 2K_1 \sum_{i=i_0+1}^{\infty} \frac{1}{\gamma_i} \leq \frac{\varepsilon}{3M_1},$$

and

$$\sum_{i=1}^{i_0} \sum_{|k| \geq k_0 + 1} \frac{|h_{i,k,n} - h_{i,k,0}|}{\gamma_i, k} \leq \sum_{i=1}^{i_0} \sum_{|k| \geq k_0 + 1} \frac{|h_{i,k,n} - h_{i,k,0}|}{\delta \omega |k|}$$

$$\leq \frac{1}{\delta \omega (k_0 + 1)} \sum_{i=1}^{i_0} \sum_{|k| \geq k_0 + 1} |h_{i,k,n} - h_{i,k,0}|$$

$$\leq \frac{2K_1 i_0}{\delta \omega (k_0 + 1)} \leq \frac{\varepsilon}{3M_1}.$$

By using the above estimates, we obtain

$$|z_0 - z_n| \leq M_1 \sum_{k \in \mathbb{Z}} \sum_{i=1}^{\infty} \frac{|h_{i,k,n} - h_{i,k,0}|}{\gamma_i, k}$$

$$= M_1 \sum_{|k| \leq k_0} \sum_{i=1}^{i_0} \frac{|h_{i,k,n} - h_{i,k,0}|}{\gamma_i, k} + M_1 \sum_{|k| \geq k_0 + 1} \sum_{i=1}^{i_0} \frac{|h_{i,k,n} - h_{i,k,0}|}{\gamma_i, k}$$

$$+ M_1 \sum_{k \in \mathbb{Z}} \sum_{i=i_0+1}^{\infty} \frac{|h_{i,k,n} - h_{i,k,0}|}{\gamma_i, k}$$

$$\leq M_1 \sum_{|k| \leq k_0} \sum_{i=1}^{i_0} \frac{|h_{i,k,n} - h_{i,k,0}|}{\gamma_i} + \frac{2\varepsilon}{3}.$$

Since $h_{i,k,n} \to h_{i,k,0}$ as $n \to \infty$ for any $i \in \mathbb{N}$ and $k \in \mathbb{Z}$, there is an $n_0 \in \mathbb{N}$ such that for any $n \geq n_0$, it holds that

$$M_1 \sum_{|k| \leq k_0} \sum_{i=1}^{i_0} \frac{|h_{i,k,n} - h_{i,k,0}|}{\gamma_i} \leq \frac{\varepsilon}{3}.$$
Summarizing, we get \(|z_0 - z_n| \leq \varepsilon\) for any \(n \geq n_0\). Since \(\varepsilon > 0\) is arbitrary, the compactness of \(L_1\) is proved. The proof is finished.

Let us put

\[ Y_{\omega,0} := \left\{ h = \sum_{k \in \mathbb{Z}} h_k e^{i\omega k t} \in Y_{\omega} : h_0 = 0 \right\} \]

with the same norm \(|\cdot|\) as for \(Y_{\omega}\). We introduce the projection \(Q: Y_{\omega} \to Y_{\omega}\) given by

\[ Qy = \frac{1}{T} \int_0^T y(s) \, ds , \]

and the projection \(P: Y_{\omega} \to Y_{\omega,0}, P = I - Q\). Note that \(T = 2\pi/\omega\). Clearly \(|P| = |Q| = 1\).

Now we consider the equation

\[ \ddot{y} + \delta \dot{y} = h(t) \quad (12) \]

for \(y, w, h \in Y_{\omega}\). We need the following simple result.

**Proposition 2.** Equation (12) has a solution \(y \in Y_{\omega}\) if and only if \(h \in Y_{\omega,0}\) and this solution is unique for \(y := L_2 h \in Y_{\omega,0}\) satisfying

\[ |y| \leq \frac{1}{\omega \sqrt{\delta^2 + \omega^2}} |h| . \]

Moreover, the linear mapping \(L_2: Y_{\omega,0} \to Y_{\omega,0}\) is compact.

**Proof.** If equation (12) has a solution \(y \in Y_{\omega}\) then clearly \(\int_0^T h(t) \, dt = 0\), so \(h \in Y_{\omega,0}\). On the other hand, if \(h \in Y_{\omega,0}\), then

\[ h(t) = \sum_{k \in \mathbb{Z} \setminus \{0\}} h_k e^{i\omega k t} . \]

Let

\[ y(t) := \sum_{k \in \mathbb{Z} \setminus \{0\}} \frac{h_k}{-\omega^2 k^2 + i\delta \omega k} e^{i\omega k t} . \]

Hence \(y \in Y_{\omega,0}\) and

\[ |y| \leq \sum_{k \in \mathbb{Z} \setminus \{0\}} \frac{|h_k|}{\sqrt{\omega^4 k^4 + \delta^2 \omega^2 k^2}} \leq \sum_{k \in \mathbb{Z} \setminus \{0\}} \frac{|h_k|}{\sqrt{\omega^4 + \delta^2 \omega^2}} = \frac{|h|}{\omega \sqrt{\omega^2 + \delta^2}} . \]

Similarly we can show that \(\dot{y}, \ddot{y} \in Y_{\omega,0}\) and thus \(y(t)\) solves (12). This proves the first part of Proposition 2.
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To show the compactness of $L_2$: $Y_{\omega,0} \to Y_{\omega,0}$, we take a bounded sequence 
\( \{h_n(t)\}_{n \in \mathbb{N}} \subset Y_{\omega,0} \) with 
\[
h_n(t) = \sum_{k \in \mathbb{Z} \setminus \{0\}} h_{k,n} e^{\omega kt}, \quad h_{k,n} = \overline{h_{-k,n}},
\]
and there is a constant $\tilde{K}_2 > 0$ such that 
\[
\sum_{k \in \mathbb{Z} \setminus \{0\}} |h_{k,n}| \leq \tilde{K}_2
\]
for any $n \in \mathbb{N}$. Again by using the Cantor diagonal procedure, we can suppose 
that $h_{k,n} \to h_{k,0}$ as $n \to \infty$ for any $k \in \mathbb{Z} \setminus \{0\}$. Then 
\[
\sum_{k \in \mathbb{Z} \setminus \{0\}} |h_{k,0}| \leq \tilde{K}_2
\]
and the function 
\[
y_0(t) := \sum_{k \in \mathbb{Z} \setminus \{0\}} \frac{h_{k,0}}{-\omega^2 k^2 + i\delta \omega k} e^{\omega kt}
\]
bilds to $Y_{\omega,0}$. Let $\varepsilon > 0$ be given. We take $k_0 \in \mathbb{N}$ so large that 
\[
\sqrt{\omega^4 k_0^4 + \delta^2 \omega^2 k_0^2} \geq \frac{4\tilde{K}_2}{\varepsilon}
\]
and put $y_n = L_2 h_n$. Then 
\[
|y_0 - y_n| = \sum_{k \in \mathbb{Z} \setminus \{0\}} \frac{|h_{k,n} - h_{k,0}|}{\sqrt{\omega^4 k^4 + \delta^2 \omega^2 k^2}}
\]
\[
= \sum_{0 < |k| \leq k_0} \frac{|h_{k,n} - h_{k,0}|}{\sqrt{\omega^4 k^4 + \delta^2 \omega^2 k^2}} + \sum_{|k| \geq k_0 + 1} \frac{|h_{k,n} - h_{k,0}|}{\sqrt{\omega^4 k^4 + \delta^2 \omega^2 k^2}}
\]
\[
\leq \sum_{0 < |k| \leq k_0} \frac{|h_{k,n} - h_{k,0}|}{\sqrt{\omega^4 k^4 + \delta^2 \omega^2 k^2}} + \frac{\varepsilon}{4\tilde{K}_2} \sum_{|k| \geq k_0 + 1} |h_{k,n} - h_{k,0}|
\]
\[
\leq \sum_{0 < |k| \leq k_0} \frac{|h_{k,n} - h_{k,0}|}{\sqrt{\omega^4 k^4 + \delta^2 \omega^2 k^2}} + \frac{\varepsilon}{2}.
\]
Since $h_{k,n} \to h_{k,0}$ as $n \to \infty$ for any $k \in \mathbb{Z} \setminus \{0\}$, there is an $n_0 \in \mathbb{N}$ such that 
for any $n \geq n_0$, it holds that 
\[
\sum_{0 < |k| \leq k_0} \frac{|h_{k,n} - h_{k,0}|}{\sqrt{\omega^4 k^4 + \delta^2 \omega^2 k^2}} \leq \frac{\varepsilon}{2}.
\]
Summarizing, we get $|y_0 - y_n| \leq \varepsilon$ for any $n \geq n_0$. Since $\varepsilon > 0$ is arbitrary, the 
compactness of $L_2$ is proved. The proof is finished. \(\square\)
4. Periodic solutions

In this section, we present the main results concerning equation (1). We seek a solution $u(x,t)$ of (2) in the form

$$u(x,t) = y_1(t)w_{-1}(x) + y_2(t)w_0(x) + z(x,t)$$

where $y_1(t), y_2(t) \in Y_\omega$ and $z(x,t) \in X_{\omega,0}$ belongs to the infinite dimensional space spanned by $\{w_i\}_{i=1}^\infty$. To get the equations for $y_1(t)$, $y_2(t)$, and $z(x,t)$ we take $v(x,t) = \phi_1(t)w_{-1}(x) + \phi_2(t)w_0(x) + v_0(x,t)$ in (2) with $\phi_i \in C^\infty(S^T)$, $v_0(x,t) \in C^\infty([0,\pi/4] \times S^T)$ satisfying besides (3) also (6). Plugging the above expression for $v(x,t)$ into (2) and using the orthonormality, we arrive at the system of equations

$$\ddot{y}_1(t) + \delta y_1(t) + \frac{2}{\sqrt{\pi}} \int_0^{\pi/4} h(x,t) \, dx$$

$$+ \frac{2}{\sqrt{\pi}} f \left( \frac{2}{\sqrt{\pi}} y_1(t) - 2 \sqrt{\frac{3}{\pi}} y_2(t) + z(0,t) \right)$$

$$+ \frac{2}{\sqrt{\pi}} g \left( \frac{2}{\sqrt{\pi}} y_1(t) + 2 \sqrt{\frac{3}{\pi}} y_2(t) + z(\pi/4,t) \right) = 0,$$

(13)

$$\ddot{y}_2(t) + \delta y_2(t) + \frac{16}{\pi} \sqrt{\frac{3}{\pi}} \int_0^{\pi/4} h(x,t) \left( x - \frac{\pi}{8} \right) \, dx$$

$$- 2 \sqrt{\frac{3}{\pi}} f \left( \frac{2}{\sqrt{\pi}} y_1(t) - 2 \sqrt{\frac{3}{\pi}} y_2(t) + z(0,t) \right)$$

$$+ 2 \sqrt{\frac{3}{\pi}} g \left( \frac{2}{\sqrt{\pi}} y_1(t) + 2 \sqrt{\frac{3}{\pi}} y_2(t) + z(\pi/4,t) \right) = 0,$$

(14)

$$\int_0^T \int_0^{\pi/4} \left[ z(x,t) \left\{ v_{tt}(x,t) + v_{xxxx}(x,t) - \delta v_1(x,t) \right\} + h(x,t)v(x,t) \right] \, dx \, dt$$

$$+ \int_0^T \left\{ f(u(0,t))v(0,t) + g(u(\pi/4,t))v(\pi/4,t) \right\} \, dt = 0$$

(15)

where we wrote $v(x,t)$ instead $v_0(x,t)$. Thus, in equation (15), $v(x,t)$ is any function in $C^\infty([0,\pi/4] \times S^T)$ such that the conditions (3) and (6) hold. We remark that in this way we have split the original equation in two parts: to the resonant finite-dimensional part represented by (13)–(14) and to the non-resonant infinite-dimensional part represented by (15).

Now we take in (13)–(15) the decomposition $y_i(t) \leftrightarrow c_i + y_i(t)$ for $i = 1, 2$ and $c_i \in \mathbb{R}$, $y_i(t) \in Y_{\omega,0}$, and then we also plug this system to a homotopy with
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a parameter $\lambda \in [0, 1]$. So we get the system

$$
j\dot{y}_1(t) + \delta y_1(t) + P \left\{ \frac{2}{\sqrt{\pi}} \int_0^{\pi/4} h(x, t) \, dx \right\} + \frac{2}{\sqrt{\pi}} f \left( \frac{2}{\sqrt{\pi}} c_1 + \lambda \frac{2}{\sqrt{\pi}} y_1(t) - 2 \sqrt{\frac{3}{\pi}} c_2 - \lambda 2 \sqrt{\frac{3}{\pi}} y_2(t) + \lambda z(0, t) \right) + \frac{2}{\sqrt{\pi}} g \left( \frac{2}{\sqrt{\pi}} c_1 + \lambda \frac{2}{\sqrt{\pi}} y_1(t) + 2 \sqrt{\frac{3}{\pi}} c_2 + \lambda 2 \sqrt{\frac{3}{\pi}} y_2(t) + \lambda z(\pi/4, t) \right) \right\} = 0,
$$

(16)

$$
j\dot{y}_2(t) + \delta y_2(t) + P \left\{ \frac{16}{\pi} \int_0^{\pi/4} h(x, t) \left( x - \frac{\pi}{8} \right) \, dx \right\} - \frac{2}{\sqrt{\pi}} \int_0^{\pi/4} h(x, t) \left( x - \frac{\pi}{8} \right) \, dx dt
$$

$$
\int_0^T \int_0^{\pi/4} \left[ z(x, t) \left\{ v_{tt}(x, t) + v_{xxxx}(x, t) - \delta v_t(x, t) \right\} + \lambda h(x, t) v(x, t) \right] \, dx \, dt
$$

$$
+ \frac{\lambda}{\pi} \int_0^T \left\{ \int_0^{\pi/4} f \left( \frac{2}{\sqrt{\pi}} c_1 + \lambda \frac{2}{\sqrt{\pi}} y_1(t) - 2 \sqrt{\frac{3}{\pi}} c_2 - \lambda 2 \sqrt{\frac{3}{\pi}} y_2(t) + \lambda z(0, t) \right) v(0, t) + g \left( \frac{2}{\sqrt{\pi}} c_1 + \lambda \frac{2}{\sqrt{\pi}} y_1(t) + 2 \sqrt{\frac{3}{\pi}} c_2 + \lambda 2 \sqrt{\frac{3}{\pi}} y_2(t) + \lambda z(\pi/4, t) \right) v(\pi/4, t) \right\} \, dt = 0
$$

(18)

$$
\int_0^T f \left( \frac{2}{\sqrt{\pi}} c_1 + \lambda \frac{2}{\sqrt{\pi}} y_1(t) - 2 \sqrt{\frac{3}{\pi}} c_2 - \lambda 2 \sqrt{\frac{3}{\pi}} y_2(t) + \lambda z(0, t) \right) \, dt = \theta_1,
$$

$$
\int_0^T g \left( \frac{2}{\sqrt{\pi}} c_1 + \lambda \frac{2}{\sqrt{\pi}} y_1(t) + 2 \sqrt{\frac{3}{\pi}} c_2 + \lambda 2 \sqrt{\frac{3}{\pi}} y_2(t) + \lambda z(\pi/4, t) \right) \, dt = \theta_2,
$$

(19)

$$\theta_1 = \frac{4}{\pi} \int_0^{\pi/4} \int_0^T h(x, t) \left( x - \frac{\pi}{4} \right) \, dx \, dt, \quad \theta_2 = -\frac{4}{\pi} \int_0^{\pi/4} \int_0^T h(x, t) x \, dx \, dt.$$
We note that system (19) is derived from the system

\[
(I - P) \left\{ \int_0^{\pi/4} h(x, t) \, dx 
+ f \left( \frac{2}{\sqrt{\pi}} c_1 + \lambda \frac{2}{\sqrt{\pi}} y_1(t) - 2\sqrt{\frac{3}{\pi}} c_2 - \lambda 2\sqrt{\frac{3}{\pi}} y_2(t) + \lambda z(0, t) \right) 
+ g \left( \frac{2}{\sqrt{\pi}} c_1 + \lambda \frac{2}{\sqrt{\pi}} y_1(t) + 2\sqrt{\frac{3}{\pi}} c_2 + \lambda 2\sqrt{\frac{3}{\pi}} y_2(t) + \lambda z(\pi/4, t) \right) \right\} = 0,
\]

\[
(I - P) \left\{ \int_0^{\pi/4} h(x, t) \left( x - \frac{\pi}{8} \right) \, dx 
- f \left( \frac{2}{\sqrt{\pi}} c_1 + \lambda \frac{2}{\sqrt{\pi}} y_1(t) - 2\sqrt{\frac{3}{\pi}} c_2 - \lambda 2\sqrt{\frac{3}{\pi}} y_2(t) + \lambda z(0, t) \right) 
+ g \left( \frac{2}{\sqrt{\pi}} c_1 + \lambda \frac{2}{\sqrt{\pi}} y_1(t) + 2\sqrt{\frac{3}{\pi}} c_2 + \lambda 2\sqrt{\frac{3}{\pi}} y_2(t) + \lambda z(\pi/4, t) \right) \right\} = 0.
\]

Since \((I - P)y = Qy = \frac{1}{T} \int_0^T y(s) \, ds\), system (20) is equivalent to the system

\[
\int_0^T f \left( \frac{2}{\sqrt{\pi}} c_1 + \lambda \frac{2}{\sqrt{\pi}} y_1(t) - 2\sqrt{\frac{3}{\pi}} c_2 - \lambda 2\sqrt{\frac{3}{\pi}} y_2(t) + \lambda z(0, t) \right) \, dt 
+ \int_0^T g \left( \frac{2}{\sqrt{\pi}} c_1 + \lambda \frac{2}{\sqrt{\pi}} y_1(t) + 2\sqrt{\frac{3}{\pi}} c_2 + \lambda 2\sqrt{\frac{3}{\pi}} y_2(t) + \lambda z(\pi/4, t) \right) \, dt 
= - \int_0^T \int_0^{\pi/4} h(x, t) \, dx \, dt,
\]

\[
\int_0^T f \left( \frac{2}{\sqrt{\pi}} c_1 + \lambda \frac{2}{\sqrt{\pi}} y_1(t) - 2\sqrt{\frac{3}{\pi}} c_2 - \lambda 2\sqrt{\frac{3}{\pi}} y_2(t) + \lambda z(0, t) \right) \, dt 
+ \int_0^T g \left( \frac{2}{\sqrt{\pi}} c_1 + \lambda \frac{2}{\sqrt{\pi}} y_1(t) + 2\sqrt{\frac{3}{\pi}} c_2 + \lambda 2\sqrt{\frac{3}{\pi}} y_2(t) + \lambda z(\pi/4, t) \right) \, dt 
= \frac{8}{\pi} \int_0^T \int_0^{\pi/4} h(x, t) \left( \frac{\pi}{8} - x \right) \, dx \, dt.
\]
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It is clear that system (21) implies system (19). Now, we take

\[
d_1 = \frac{2}{\sqrt{\pi}} c_1 - 2\sqrt{\frac{3}{\pi}} c_2, \quad \zeta_1(t) = \frac{2}{\sqrt{\pi}} y_1(t),
\]

\[
d_2 = \frac{2}{\sqrt{\pi}} c_1 + 2\sqrt{\frac{3}{\pi}} c_2, \quad \zeta_2(t) = 2\sqrt{\frac{3}{\pi}} y_2(t),
\]

and we split

\[
h(x, t) = 8 \frac{\theta_2 - 2\theta_1}{T\pi} + 96 \frac{\theta_1 - \theta_2}{T^2} x + p(x, t)
\]

such that

\[
\int_0^T \int_0^{\pi/4} p(x, t) \, dx \, dt = \int_0^T \int_0^{\pi/4} xp(x, t) \, dx \, dt = 0.
\]

By using these notations along with Propositions 1 and 2 we can rewrite system (16) (19) as the following semi-fixed point problem

\[
\zeta_1(t) = -\frac{4}{\pi} L_2 \left\{ \int_0^{\pi/4} p(x, t) \, dx + f\left( d_1 + \lambda \zeta_1(t) - \lambda \zeta_2(t) + \lambda z(0, t) \right) \right.
\]

\[
\left. + g\left( d_2 + \lambda \zeta_1(t) + \lambda \zeta_2(t) + \lambda z(\pi/4, t) \right) \right\},
\]

\[
\zeta_2(t) = -\frac{12}{\pi} L_2 \left\{ \int_0^{\pi/4} \frac{8}{\pi} p(x, t) \left( x - \frac{\pi}{8} \right) \, dx - f\left( d_1 + \lambda \zeta_1(t) - \lambda \zeta_2(t) + \lambda z(0, t) \right) \right.
\]

\[
\left. + g\left( d_2 + \lambda \zeta_1(t) + \lambda \zeta_2(t) + \lambda z(\pi/4, t) \right) \right\},
\]

\[
z(x, t) = \lambda L_1 \left( p(x, t), f\left( d_1 + \lambda \zeta_1(t) - \lambda \zeta_2(t) + \lambda z(0, t) \right), \right.
\]

\[
\left. g\left( d_2 + \lambda \zeta_1(t) + \lambda \zeta_2(t) + \lambda z(\pi/4, t) \right) \right),
\]

\[
\int_0^T f\left( d_1 + \lambda \zeta_1(t) - \lambda \zeta_2(t) + \lambda z(0, t) \right) \, dt = \theta_1,
\]

\[
\int_0^T g\left( d_2 + \lambda \zeta_1(t) + \lambda \zeta_2(t) + \lambda z(\pi/4, t) \right) \, dt = \theta_2.
\]
Since \( f \) and \( g \) are analytic, we have expansions 
\[
f(x) = \sum_{n=0}^{\infty} a_n x^n \quad \text{and} \quad g(x) = \sum_{n=0}^{\infty} b_n x^n.
\]
Then we put 
\[
F(x) = \sum_{n=0}^{\infty} |a_n| x^n, \quad G(x) = \sum_{n=0}^{\infty} |b_n| x^n.
\]
We note that \( X_{\omega}, X_{\omega,0}, Y_{\omega} \), and \( Y_{\omega,0} \) are all Banach algebras [6]. Now again by using Propositions 1 and 2, from (24)–(26) we get 
\[
|\zeta_1| \leq \frac{4}{\pi \omega \sqrt{\omega^2 + \delta^2}} \left\{ \frac{\pi}{4} |p| + \left( F(A + |d_1|) + G(A + |d_2|) \right) \right\},
\]
\[
|\zeta_2| \leq \frac{12}{\pi \omega \sqrt{\omega^2 + \delta^2}} \left\{ \frac{\pi}{8} |p| + \left( F(A + |d_1|) + G(A + |d_2|) \right) \right\},
\]
\[
|z| \leq M_2 \left( \frac{\pi}{4} |p| + \left( F(A + |d_1|) + G(A + |d_2|) \right) \right),
\]
where \( A = |\zeta_1| + |\zeta_2| + |z| \). By summing up the above inequalities, we obtain the following:

**Proposition 3.** Let \( h \in X_{\omega} \). If system (24)–(27) has a solution \( \zeta_1(t), \zeta_2(t) \in Y_{\omega,0} \) and \( z(x,t) \in X_{\omega,0} \), then it holds that 
\[
A \leq \left( \frac{5}{2 \omega \sqrt{\omega^2 + \delta^2}} + M_2 \frac{\pi}{4} \right) |p| + \left( F(A + |d_1|) + G(A + |d_2|) \right) \left( \frac{16}{\pi \omega \sqrt{\omega^2 + \delta^2}} + M_2 \right),
\]
where \( A = |\zeta_1| + |\zeta_2| + |z| \).

Now we can prove the main results of this note.

**Theorem 1.** Let \( h \in X_{\omega} \) and \( \theta_1, \theta_2 \in \mathbb{R} \). Let \( \tilde{c}_1, \tilde{c}_2 \) be simple roots of the equations \( f(\tilde{c}_1) = \theta_1 / T \) and \( g(\tilde{c}_2) = \theta_2 / T \), respectively. We assume the existence of positive constants \( r_1, r_2, k_1, k_2, K_1, K_2 \) and \( A \) such that 
\[
k_1 \leq |f'(s_1)| \leq K_1, \quad k_2 \leq |g'(s_2)| \leq K_2
\]
for any \( |s_i - \tilde{c}_i| \leq r_i + A, \; i = 1, 2 \), and 
\[
A > \left( \frac{5}{2 \omega \sqrt{\omega^2 + \delta^2}} + M_2 \frac{\pi}{4} \right) |p| + \left( F(A + |\tilde{c}_1| + r_1) + G(A + |\tilde{c}_2| + r_2) \right) \left( \frac{16}{\pi \omega \sqrt{\omega^2 + \delta^2}} + M_2 \right).
\]
If 

\[ k_i r_i > AK_i, \quad i = 1, 2, \]

then equation (1) has a solution \( u(x, t) \in X^X. \)

Proof. We solve the system (24)–(27) on the ball \( B \) in the Banach space \( X = \mathbb{R}^2 \times Y_{\omega,0} \times X_{\omega,0} \) given by

\[ B := \{(d_1, d_2, \zeta_1, \zeta_2, z) \in X : |d_1 - \bar{c}_1| \leq r_1, \quad |d_2 - \bar{c}_2| \leq r_2, \quad |\zeta_1| + |\zeta_2| + |z| \leq A\}. \]

The norm on \( X \) is given by

\[ |(d_1, d_2, \zeta_1, \zeta_2, z)| = |d_1| + |d_2| + |\zeta_1| + |\zeta_2| + |z|. \]

We show that the system (24)–(27) has no solutions on the border (the sphere) \( \partial B \) of the ball \( B \). For \( |\zeta_1| + |\zeta_2| + |z| = A \), this follows from Proposition 3. For \( |d_1 - \bar{c}_1| = r_1 \) and \( |\zeta_1| + |\zeta_2| + |z| \leq A \), we have \( A \geq \|\zeta_1\|_\infty + \|\zeta_2\|_\infty + \|z\|_\infty \) and

\[ \left| \int_0^T f(d_1 + \lambda \zeta_1(t) - \lambda \zeta_2(t) + \lambda z(0, t)) \, dt - \theta_1 \right| \geq T |f(d_1) - f(\bar{c}_1)| - K_1 TA \geq (k_1 r_1 - K_1 A)T > 0. \]

Similarly for \( |d_2 - \bar{c}_2| = r_2 \) and \( |\zeta_1| + |\zeta_2| + |z| \leq A \). Consequently, by using Propositions 1 and 2, we can apply the Leray-Schauder degree theory to the system (24)–(27). Indeed, let us denote by \( \Psi_1(d_1, d_2, \zeta_1, \zeta_2, z, \lambda) \) the left-hand side of (27) and by \( \Psi_2(d_1, d_2, \zeta_1, \zeta_2, z, \lambda) \) the right-hand side of (24)–(26), respectively. We put

\[ \Psi_1 := \Psi_1(d_1, d_2, \zeta_1, \zeta_2, z, \lambda) + (d_1 - \theta_1, d_2 - \theta_2). \]

Then by using Propositions 1 and 2, the operators

\[ \Psi_1 : X \times [0, 1] \to \mathbb{R}^2, \quad \Psi_2 : X \times [0, 1] \to Y_{\omega,0} \times X_{\omega,0} \]

are compact and continuous. Moreover, by putting

\[ \psi := (d_1, d_2, \zeta_1, \zeta_2, z), \quad \Psi(\psi, \lambda) := (\Psi_1(\psi, \lambda), \Psi_2(\psi, \lambda)), \]

system (24)–(27) has the form

\[ \psi - \Psi(\psi, \lambda) = 0. \]

We already know that \( \psi - \Psi(\psi, \lambda) \neq 0 \) on \( \psi \in \partial B \) for any \( \lambda \in [0, 1] \). Hence we can define the Leray-Schauder degree \( \deg(I - \Psi(\cdot, \lambda), B, 0) \). Now from system
(24) (27) for $\lambda = 1$, we get (2), while for $\lambda = 0$, we get

$$
\zeta_1(t) = -\frac{4}{\pi} L_2 P \left\{ \int_0^{\pi/4} p(x, t) \, dx + f(d_1) + g(d_2) \right\},
$$

$$
\zeta_2(t) = -\frac{12}{\pi} L_2 P \left\{ \int_0^{\pi/4} p(x, t) \left( x - \frac{\pi}{8} \right) \, dx - f(d_1) + g(d_2) \right\},
$$

(28)

$$
z(x, t) = 0,
$$

$$
f(d_1) = \theta_1 / T,
$$

$$
g(d_2) = \theta_2 / T.
$$

Since $\tilde{c}_1$, $\tilde{c}_2$ are simple roots of the equations $f(\tilde{c}_1) = \theta_1 / T$ and $g(\tilde{c}_2) = \theta_2 / T$, we see that the system (28) is solvable for $d_i = \tilde{c}_i$, $i = 1, 2$, and also the corresponding Leray-Schauder degree or the coincidence topological degree $\text{deg}(I-\Psi(\cdot, 0), B, 0)$ is nonzero (see [5]). Since

$$
\text{deg}(I-\Psi(\cdot, 1), B, 0) = \text{deg}(I-\Psi(\cdot, 0), B, 0) \neq 0,
$$

system (24)-(27) is solvable in the ball $B$. The proof is finished. \Box

For instance, if $f(x) = g(x) = Kx + \varepsilon x^3$ for constants $K > 0$ and $\varepsilon$, then Theorem 1 is applicable when

$$
\eta := 4K \left( \frac{16}{\pi \omega \sqrt{\omega^2 + \delta^2}} + M_2 \right) < 1
$$

and $\varepsilon$ is sufficiently small. Indeed, we first take $\varepsilon = 0$. Hence $f(x) = g(x) = Kx$.

Then $k_1 = k_2 = K_1 = K_2 = K$ in Theorem 1 for any $s_1$, $s_2$. We take $r_1 = r_2 = A/\eta$ to satisfy $k_i r_i > AK_i$, $i = 1, 2$. We note $\tilde{c}_i = \frac{\theta_i}{K_i}$, $i = 1, 2$. The condition of Theorem 1 for constants $\theta_1$, $\theta_2$ and function $p$ now reads as follows

$$
\left( \frac{5}{2\omega \sqrt{\omega^2 + \delta^2}} + M_2 \frac{\pi}{4} \right) |p| + \frac{16}{T} \left( \frac{16}{\pi \omega \sqrt{\omega^2 + \delta^2}} + M_2 \right) < A \frac{1 - \eta}{2}.
$$

Since we can always find $A > 0$ satisfying the above inequality, we see that when $\varepsilon = 0$, then (1) is solvable for any $h \in X_\omega$. Clearly the above inequalities remain also for $\varepsilon$ small. This gives the solvability of (1) for any fixed $h \in X_\omega$ and $\varepsilon$ small depending on $h$.

Now we present more constructive method than Theorem 1. We consider system (24)-(26) for $\lambda = 1$. Let $N(d_1, d_2, \zeta_1, \zeta_2, z)$ denote the right-hand side of (24)-(26) with $\lambda = 1$. Hence (24)-(26) with $\lambda = 1$ has the form

$$
\tau = N(d_1, d_2, \tau)
$$

(29)
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for \( \tau = (\zeta_1, \zeta_2, z) \) and \( d_1, d_2 \) are parameters. We intend to apply the Banach fixed point theorem to solve (29). For this reason, we consider on the Banach space \( Y = Y^2_{\omega, 0} \times X_{\omega, 0} \) the ball
\[
B_A = \{ \tau = (\zeta_1(t), \zeta_2(t), z(x, t)) \in Y : |\zeta_1| + |\zeta_2| + |z| \leq A \}.
\]
The norm on \( Y \) is given by \( |\tau| = |\zeta_1| + |\zeta_2| + |z| \). We suppose positive constants \( D_i, i = 1, 2 \), such that it holds that
\[
\left( \frac{5}{2\omega \sqrt{\omega^2 + \delta^2}} + M_2 \frac{\pi}{4} \right) |p| + \left( F(A + D_1) + G(A + D_2) \right) \left( \frac{16}{\pi \omega \sqrt{\omega^2 + \delta^2}} + M_2 \right) \leq A
\]
and
\[
(F'(A + D_1) + G'(A + D_2)) \left( \frac{16}{\pi \omega \sqrt{\omega^2 + \delta^2}} + M_2 \right) < 1.
\]

The conditions (30) and (31) imply that for any \((d_1, d_2) \in B_D\) with
\[
B_D := \{(d_1, d_2) \in \mathbb{R}^2 : |d_i| \leq D_i, \ i = 1, 2\},
\]
the mapping \( N(d_1, d_2, \cdot) \) maps \( B_A \) to itself with the Lipschitz contraction constant
\[
\Gamma := \left( F'(A + D_1) + G'(A + D_2) \right) \left( \frac{16}{\pi \omega \sqrt{\omega^2 + \delta^2}} + M_2 \right).
\]

Hence (29) has a unique fixed point \( \tau = \tau(d_1, d_2) \) in \( B_A \) for any \((d_1, d_2) \in B_D\). Moreover, mapping \( \tau(d_1, d_2) \) is Lipschitz continuous with the constant \( \Gamma/(1 - \Gamma) \), i.e. it holds that
\[
|\tau(d_1, d_2) - \tau(d_1', d_2')| \leq \frac{\Gamma}{1 - \Gamma} (|d_1 - d_1'| + |d_2 - d_2'|)
\]
for any \((d_1, d_2), (d_1', d_2') \in B_D\). We consider in (30) and (31) the function \( p \) as a parameter for fixed \( A, D_1, D_2 \). Hence \( \tau(d_1, d_2) = \tau(d_1, d_2, p) \). We plug this \( \tau(d_1, d_2, p) \) into (27) with \( \lambda = 1 \) to get
\[
\theta_1 = \int_0^\tau \phi(d_1 + \tau(d_1, d_2, p)(0, t)) \, dt,
\]
\[
\theta_2 = \int_0^\tau \phi(d_2 + \tau(d_1, d_2, p)(\pi/4, t)) \, dt,
\]
where
\[
\tau(d_1, d_2, p)(0, t) = \xi_1(t) - \xi_2(t) + z(0, t),
\]
\[
\tau(d_1, d_2, p)(\pi/4, t) = \xi_1(t) + \xi_2(t) + z(\pi/4, t)
\]
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for \( \tau(d_1, d_2, p) = (\xi_1(t), \xi_2(t), z(x, t)) \).

By using the above formulas for \( \theta_1, \theta_2 \), from the splitting (22) for the function \( h(x, t) \), we obtain

\[
\begin{align*}
\Phi(x) := (F(x) + G(x)) & \left( \frac{16}{\pi \omega \sqrt{x^2 + \delta^2} + M_2} \right).
\end{align*}
\]

Now, if there is an \( x_0 > 0 \) such that \( \Phi(x_0) < x_0 \) and \( \Phi'(x_0) < 1 \), then for the validity of Theorem 2, we can take

\[
\begin{align*}
B & = D_1 = D_2 = D = \frac{4(x_0 - \Phi(x_0))\omega \sqrt{x^2 + \delta^2}}{10 + \omega \sqrt{x^2 + \delta^2}(M_2 \pi + 4)}, \\
A & = x_0 - D.
\end{align*}
\]

For instance, if \( f(x) = g(x) = \Omega x^3, \Omega > 0 \), then we get

\[
\Phi(x) = 2\Omega x^3 \left( \frac{16}{\pi \omega \sqrt{x^2 + \delta^2} + M_2} \right).
\]
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and we can take

\[ x_0 = \sqrt{\frac{\pi \omega \sqrt{\omega^2 + \delta^2}}{12 \Omega (16 + \pi \omega \sqrt{\omega^2 + \delta^2} M^2)}}. \]

Then we obtain

\[ x_0 - \Phi(x_0) = \frac{5}{6} \sqrt{\frac{\pi \omega \sqrt{\omega^2 + \delta^2}}{12 \Omega (16 + \pi \omega \sqrt{\omega^2 + \delta^2} M^2)}}, \]

which according to (33) implies

\[ B = \frac{5 \omega \sqrt{\omega^2 + \delta^2}}{10 + \omega \sqrt{\omega^2 + \delta^2} (M^2 \pi + 4)} \sqrt{\frac{\pi \omega \sqrt{\omega^2 + \delta^2}}{27 \Omega (16 + \pi \omega \sqrt{\omega^2 + \delta^2} M^2)}}. \]

This gives a relationship between the magnitude of the constant \( B \) and the parameter \( \Omega \).

REFERENCES


