Robert Šulka

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ON THREE LATTICES THAT BELONG TO EVERY SEMIGROUP

ROBERT ŠULKA

Dedicated to Academician Štefan Schwarz on the occasion of his 70th birthday

In paper [6] three kinds of nilpotency were introduced. By means of them we define three lattices which are subsets of the Boolean $\langle P(S), \subseteq \rangle$ of a semigroup S. Some properties of these lattices were found. For example it is proved that two of these lattices are complete and one of them is complemented.

We give conditions for a subset M of a semigroup S to belong to these lattices. It is proved that all ideals, (m, n)-ideals and (m, n)-quasiideals of a semigroup are elements of all these lattices.

In the case of cyclic semigroups we describe all elements of these lattices.

In the last section we are dealing with these lattices of subsemigroups of a semigroup.

Basic definitions and properties

In paper [6] the following definitions are introduced (see also [4] and [1]).

Definition 1. *Let S be a semigroup, $M \subseteq S$ and $x \in S$.*

a) If there exists a positive integer $n_0(x)$ such that $x^n \in M$ holds for all positive integers $n \geq n_0(x)$, then x will be called strongly M-potent. The set of all strongly M-potent elements of S will be denoted by $N_1(M)$.

b) If $x^n \in M$ holds for infinitely many positive integers n, then x will be called weakly M-potent. The set of all weakly M-potent elements of S will be denoted by $N_2(M)$.

c) If $x^n \in M$ holds for at least one positive integer n, then x will be called almost M-potent. The set of all almost M-potent elements of S will be denoted by $N_3(M)$.

d) Let J be a (two-sided) ideal of S such that $J \subseteq N_1(M)$ holds. Then J will be called a strong M-ideal. If $J \subseteq N_2(M)$, then J will be called a weak M-ideal. The union of all strong M-ideals will be denoted by $R_1^*(M)$ and the union of all weak M-ideals will be denoted by $R_2^*(M)$.

If $M \subseteq S$, $M_1 \subseteq S$ and $M_2 \subset S$, then the following statements are true (see [6]):

(i) $N_1(M) \subseteq N_2(M) \subseteq N_3(M)$.

(ii) *If* $M_1 \subseteq M_2$, *then* $N_i(M_1) \subseteq N_i(M_2)$ *for* $i = 1, 2, 3$.

(iii) $N_1(M_1 \cap M_2) = N_1(M_1) \cap N_1(M_2)$.

(iv) $N_2(M_1 \cup M_2) = N_2(M_1) \cup N_2(M_2)$.

(v) $N_3(\cup\{M_i \mid i \in I\}) = \cup\{N_3(M_i) \mid i \in I\}$.

(vi) $R_1^*(M_1 \cap M_2) = R_1^*(M_1) \cap R_1^*(M_2)$.

It is easy to prove the following Lemmas.

Lemma 1. *Let* $M_i \subseteq S$ *and* $N_1(M_i) = N_2(M_i)$ *for* $i = 1, 2$. *Then* $N_1(M_1 \cup M_2) = N_2(M_1 \cup M_2)$.

Proof. (i), (iv), the assumptions of Lemma 1 and (ii) imply

$$N_1(M_1 \cup M_2) \subseteq N_2(M_1 \cup M_2) = N_2(M_1) \cup N_2(M_2) = \\ = N_1(M_1) \cup N_1(M_2) \subseteq N_1(M_1 \cup M_2).$$

Hence $N_1(M_1 \cup M_2) = N_2(M_1 \cup M_2)$.

Similarly one can prove the following two Lemmas.

Lemma 2. *Let* $M_i \subseteq S$ *and* $N_1(M_i) = N_3(M_i)$ *for all* $i \in I$. *Then* $N_1(\cup\{M_i \mid i \in I\}) = N_3(\cup\{M_i \mid i \in I\})$.

Lemma 3. *Let* $M_i \subseteq S$ *and* $N_2(M_i) = N_3(M_i)$ *for all* $i \in I$. *Then* $N_2(\cup\{M_i \mid i \in I\}) = N_3(\cup\{M_i \mid i \in I\})$.

Lemma 4. *Let* $M_i \subseteq S$ *and* $N_1(M_i) = N_2(M_i)$ *for* $i = 1, 2$. *Then* $N_1(M_1 \cap M_2) = N_2(M_1 \cap M_2)$.

Proof. From the assumptions of Lemma 4, from (iii), (i) and (ii) it follows that

$$N_2(M_1) \cap N_2(M_2) = N_1(M_1) \cap N_1(M_2) = \\ = N_1(M_1 \cap M_2) \subseteq N_2(M_1 \cap M_2) \subseteq N_2(M_1) \cap N_2(M_2).$$

Hence $N_1(M_1 \cap M_2) = N_2(M_1 \cap M_2)$.

Similarly we can prove

Lemma 5. *Let* $M_i \subseteq S$ *and* $N_1(M_i) = N_3(M_i)$ *for* $i = 1, 2$. *Then* $N_1(M_1 \cap M_2) = N_3(M_1 \cap M_2)$.

Lemma 6. *Let* $M \subseteq S$, $M_1 \subseteq S$ *and* $M_2 \subseteq S$. *Then the following statements hold:*

(i) $R_1^*(M) \subseteq R_2^*(M)$, $R_1^*(M) \subseteq N_1(M)$, $R_2^*(M) \subseteq N_2(M)$.

(ii) *If* $M_1 \subseteq M_2$, *then* $R_i^*(M_1) \subseteq R_i^*(M_2)$ *for* $i = 1, 2$.

(iii) *If* $N_1(M) = N_2(M)$, *then* $R_1^*(M) = R_2^*(M)$.

Proof. (i) and (ii) follow immediately from Definition 1.

If $N_1(M) = N_2(M)$, then for every ideal J the relation $J \subseteq N_1(M)$ holds iff $J \subseteq N_2(M)$. Using Definition 1, this implies (iii).

Corollary. *Let $M_i \subseteq S$ and $N_1(M_i) = N_2(M_i)$ for $i = 1, 2$. Then $R_1^*(M_i) = R_2^*(M_i)$ for $i = 1, 2$, $R_1^*(M_1 \cap M_2) = R_2^*(M_1 \cap M_2)$ and $R_1^*(M_1 \cup M_2) = R_2^*(M_1 \cup M_2)$.*

The proof follows from Lemmas 6, 4 and 1.

Lemma 7. *Let $M_i \subseteq S$ and $R_1^*(M_i) = R_2^*(M_i)$ for $i = 1, 2$. Then $R_1^*(M_1 \cap M_2) = R_2^*(M_1 \cap M_2)$.*

The proof of Lemma 7 is similar to the proof of Lemma 4.

Let $\langle P(S), \subseteq \rangle$ be the Boolean of S.

Lemmas 1—7 imply

Theorem 1. *Let S be a semigroup,*

$$\mathcal{N}_{12} = \{M \subseteq S \mid N_1(M) = N_2(M)\}, \quad \mathcal{N}_{13} = \{M \subseteq S \mid N_1(M) = N_3(M)\},$$
$$\mathcal{N}_{23} = \{M \subseteq S \mid N_2(M) = N_3(M)\} \text{ and } \mathcal{R} = \{M \subseteq S \mid R_1^*(M) = R_2^*(M)\}.$$

Then the following statements hold:

a) *$\emptyset$ and S are contained in $\mathcal{N}_{12}$, $\mathcal{N}_{13}$, $\mathcal{N}_{23}$ and $\mathcal{R}$.*

b) *$\mathcal{N}_{13} \subseteq \mathcal{N}_{12} \subseteq \mathcal{R}$ and $\mathcal{N}_{13} \subseteq \mathcal{N}_{23}$.*

c) *$\langle \mathcal{N}_{12}, \subseteq \rangle$ is a lattice.*

d) *$\langle \mathcal{N}_{13}, \subseteq \rangle$ is a complete lattice.*

e) *$\langle \mathcal{N}_{23}, \subseteq \rangle$ is a complete lattice.*

f) *$\langle \mathcal{R}, \subseteq \rangle$ is a lower semilattice.*

g) *$\langle \mathcal{N}_{12}, \cap, \cup \rangle$ is a sublattice of $\langle P(S), \cap, \cup \rangle$.*

h) *$\langle \mathcal{N}_{13}, \cap, \cup \rangle$ is a sublattice of $\langle \mathcal{N}_{12}, \cap, \cup \rangle$.*

i) *$\langle \mathcal{N}_{13}, \cap, \cup \rangle$ is a sublattice of $\langle \mathcal{N}_{23}, \wedge, \cup \rangle$.*

j) *$\langle \mathcal{N}_{13}, \subseteq \rangle$ is a complete upper subsemilattice of $\langle P(S), \subseteq \rangle$.*

k) *$\langle \mathcal{N}_{23}, \subseteq \rangle$ is a complete upper subsemilattice of $\langle P(S), \subseteq \rangle$.*

l) *$\langle \mathcal{N}_{12}, \cap, \cup \rangle$ is a distributive lattice.*

m) *$\langle \mathcal{N}_{13}, \cap, \cup \rangle$ is a distributive lattice.*

n) *$\langle \mathcal{R}, \cap \rangle$ is a lower subsemilattice of $\langle P(S), \cap \rangle$.*

Lemmas 1—7 and (iii)—(vi) imply

Theorem 2. *Let S be a semigroup. Then the following statements are true:*

a) *The mapping N_{12}: $\langle \mathcal{N}_{12}, \cap, \cup \rangle \to \langle P(S), \cap, \cup \rangle$, $N_{12}(M) = N_1(M) = N_2(M)$ is a homomorphism.*

b) *The mapping N_{13}: $\langle \mathcal{N}_{13}, \cap, \cup \rangle \to \langle P(S), \cap, \cup \rangle$, $N_{13}(M) = N_1(M) = N_3(M)$ is a homomorphism. It preserves infinite joins (set-theoretical unions).*

c) *The mapping N_{23}: $\langle \mathcal{N}_{23}, \cup \rangle \to \langle P(S), \cup \rangle$, $N_{23}(M) = N_2(M) = N_3(M)$ is a homomorphism. It preserves infinite joins (set-theoretical unions).*

d) *The mapping R_{12}^*: $\langle \mathcal{R}, \cap \rangle \to \langle P(S), \cap \rangle$, $R_{12}^*(M) = R_1^*(M) = R_2^*(M)$ is a homomorphism.*

Some examples

Let N be the set of all positive integers.

If $S=\langle a\rangle$ is a cyclic semigroup generated by the element a and $J(x)$ is the principal two-sided ideal generated by an element $x\in S$, then $x=a^{n_0}$ for some $n_0\in N$ and $J(a^{n_0})=\{a^n \mid n\geqq n_0\}$.

Theorem 3. *Let $\emptyset\neq M\subseteq S$ and $S=\langle a\rangle$ be a cyclic semigroup, generated by the element a. Then $M\in\mathcal{N}_{13}$ iff there exists an element $x\in S$ such that $J(x)\subseteq M$ holds.*

Proof. a) Let $N_1(M)=N_3(M)$ hold. Since $M\neq\emptyset$, there exists a positive integer k such that $a^k\in M$ is true. Hence we have $a\in N_3(M)=N_1(M)$. This implies the existence of a positive integer n_0 such that for all positive integers $n\geqq n_0$ the relation $a^n\in M$ holds. This means that $J(a^{n_0})=\{a^n \mid n\geqq n_0\}\subseteq M$.

b) Let the relation $J(a^{n_0})=\{a^n \mid n\geqq n_0\}\subseteq M$ hold. Let z be an arbitrary element of $S=\langle a\rangle$. Then $z=a^k$ holds for some positive integer k. Since $J(a^{n_0})\subseteq M$, we have $a^n\in M$ for all positive integers $n\geqq n_0$. Hence we have also $z^n=(a^k)^n\in M$ for all positive integers $n\geqq n_0$. This means that $z\in N_1(M)$. In this way we get that $N_1(M)=S$. Hence by (i) $N_1(M)=N_3(M)$.

Remark. The condition $J(x)\subseteq M$ is equivalent to the condition that M contains an ideal.

Theorem 4. *Let $M\subseteq S$ and $S=\langle a\rangle$ be a cyclic semigroup. Then $M\in\mathcal{N}_{12}$ iff either the relation $a^n\in M$ holds only for a finite number of positive integers n or $J(x)\subseteq M$ for some $x\in S$.*

Proof. If the relation $a^n\in M$ holds only for a finite number of positive integers n, then the condition $N_1(M)=N_2(M)$ is satisfied. Therefore it is sufficient to consider subsets M of S satisfying the relation $a^n\in M$ for infinitely many positive integers n.

a) Let $N_1(M)=N_2(M)$ and let $a^n\in M$ hold for infinitely many positive integers n. Then $a\in N_2(M)=N_1(M)$ and we get that $J(x)\subseteq M$ as in the first part of the proof of Theorem 3.

b) Let $J(a^{n_0})=\{a^n \mid n\geqq n_0\}\subseteq M$ (then clearly $a^n\in M$ holds for infinitely many positive integers n). From the second part of the proof of Theorem 3 we know that $N_1(M)=S$. From this and (i) we get that $N_1(M)=N_2(M)$.

If S is a cyclic semigroup of infinite order, we can formulate Theorem 4 as follows:

Theorem 4a. *Let $M\subseteq S$ and $S=\langle a\rangle$ be a cyclic semigroup of infinite order. Then $M\in\mathcal{N}_{12}$ iff either M is a finite subset of S or $J(x)\subseteq M$ for an element $x\in S$.*

Let $S=\{a, a^2, \dots, a^r, a^{r+1}, \dots, a^{r+m-1}\}$ be a cyclic semigroup of finite order with index r and with period m (see [1] and [3]). Denote $G=\{a^r, a^{r+1}, \dots, a^{r+m-1}\}$ the maximal subgroup of S and $P=\{a, a^2, \dots, a^{r-1}\}$. In such semigroup the condition

$J(x) \subseteq M$ is equivalent to the condition $G \subseteq M$. This follows from the fact that for all positive integers n_0 the inclusion $G \subseteq J(a^{n_0})$ holds and $G = J(a^t)$.

Hence we have

Theorem 3a. *Let $\emptyset \neq M \subseteq S$ and $S = \langle a \rangle$ be a cyclic semigroup of finite order. Then $M \in \mathcal{N}_{13}$ iff $G \subseteq M$.*

If $S = \langle a \rangle$ is a cyclic semigroup of finite order, then the condition that the relation $a^n \in M$ holds only for a finite number of positive integers is equivalent to the condition $M \subseteq P$.

This implies

Theorem 4b. *Let $M \subseteq S$ and $S = \langle a \rangle$ be a cyclic semigroup of finite order. Then $M \in \mathcal{N}_{12}$ iff either $M \subseteq P$ or $G \subseteq M$.*

Theorem 5. *Let S be a semigroup and $M \subseteq S$. Then $M \in \mathcal{N}_{23}$ iff $M \subseteq N_2(M)$.*

Proof. a) Let $M \in \mathcal{N}_{23}$ i.e. $N_2(M) = N_3(M)$. Since $M \subseteq N_3(M)$ we have $M \subseteq N_2(M)$.

b) Let $M \subseteq N_2(M)$. If $x \in N_3(M)$, then for a positive integer m we have $x^m = y \in M$. Since $M \subseteq N_2(M)$, there exists a strictly increasing sequence $(k_n)_{n=1}^{\infty}$ of positive integers k_n such that $y^{k_n} \in M$ i.e. $x^{mk_n} \in M$ for all $n \in N$. This means that $x \in N_2(M)$ and we have $N_3(M) \subseteq N_2(M)$. Since by (i) $N_2(M) \subseteq N_3(M)$, we get $N_2(M) = N_3(M)$ i.e. $M \in \mathcal{N}_{23}$.

Theorem 6. *Let S be a semigroup and $M \subseteq S$. Then $M \subseteq N_2(M)$ iff for every $x \in M$ there exists a positive integer $n > 1$ such that $x^n \in M$.*

Proof. a) Let $M \subseteq N_2(M)$. If $x \in M$, then $x \in N_2(M)$ and there exists a positive integer $n > 1$ such that $x^n \in M$.

b) If for every $x \in M$ there exists a positive integer $n > 1$ such that $x^n \in M$, then for every $x \in M$ there exists a sequence of positive integers $k_n > 1$ such that $x, x^{k_1k_2}$, $x^{k_1k_2k_3}, \ldots, x^{k_1k_2\cdots k_n}, \ldots$ belong to M i.e. $x \in N_2(M)$. Hence $M \subseteq N_2(M)$.

Theorems 5 and 6 imply the following

Corollary. *Let S be a semigroup and $x \in S$. Let $(k_n)_{n=1}^{\infty}$ be a sequence of positive integers $k_n > 1$ and $M = \{x, x^{k_1}, x^{k_1k_2}, \ldots, x^{k_1k_2\cdots k_n}, \ldots\}$. Then $M \in \mathcal{N}_{23}$.*

Lemma 8. *Let S be a semigroup and x an element of S of infinite order. Let $(k_n)_{n=1}^{\infty}$ and $(r_n)_{n=1}^{\infty}$ be two distinct sequences of positive integers $k_n > 1$ and $r_n > 1$. Let*

$$M_1 = \{x, x^{k_1}, x^{k_1k_2}, \ldots, x^{k_1k_2\cdots k_n}, \ldots\}$$

and

$$M_2 = \{x, x^{r_1}, x^{r_1r_2}, \ldots, x^{r_1r_2\cdots r_n}, \ldots\}.$$

Then $M_1 \neq M_2$.

Proof. Let i be the least index such that $k_i \neq r_i$ holds. Suppose that $k_i < r_i$. Then $x^{k_1 k_2 \cdots k_i} \in M_1$ but $x^{k_1 k_2 \cdots k_i} \notin M_2$.

Theorem 7. *If the semigroup S contains at least one element of infinite order, the* card $\mathcal{N}_{23} \geqslant 2^{\aleph_0}$.

Proof. Let $\mathcal{M}$ be the system of all sets $M = \{x, x^{k_1}, x^{k_1 k_2}, \ldots, x^{k_1 k_2 \cdots k_n}, \ldots\}$ where $(k_n)_{n=1}^{\infty}$ is a sequence of positive integers $k_n > 1$. By Lemma 8 we have card $\mathcal{M} = 2^{\aleph_0}$. Corollary of Theorems 5 and 6 implies $\mathcal{M} \subseteq \mathcal{N}_{23}$, therefore card $\mathcal{N}_{23} \geqslant 2^{\aleph_0}$.

Corollary. *Let $S = \langle a \rangle$ be a cyclic semigroup of infinite order. Then* card $\mathcal{N}_{23} = 2^{\aleph_0}$.

The proof follows from Theorem 7 and from the fact that card $P(S) = 2^{\aleph_0}$.

Now we can prove that the sets $\mathcal{N}_{12}$, $\mathcal{N}_{13}$, $\mathcal{N}_{23}$, $\mathcal{R}$ and $P(S)$ may be distinct. This follows from the foregoing Theorems and their Corollaries.

Example. Let $S = \langle a \rangle$ be the cyclic semigroup of infinite order. Then $A = \{a\} \in \mathcal{N}_{12}$ but $A \notin \mathcal{N}_{23}$ and $A \notin \mathcal{N}_{13}$, because $N_1(A) = N_2(A) = \emptyset$ but $N_3(A) = A \neq \emptyset$. Hence $\mathcal{N}_{12} \neq \mathcal{N}_{23}$ and $\mathcal{N}_{12} \neq \mathcal{N}_{13}$. Moreover $N_3(A) = \{a\}$ is neither an ideal in S nor it contains an ideal. Therefore $R_1^*(A) = R_2^*(A) = \emptyset$ i.e. $A \in \mathcal{R}$. Hence we have $A \in \mathcal{R}$ but $A \notin \mathcal{N}_{23}$ i.e. $\mathcal{R} \neq \mathcal{N}_{23}$. Moreover $A \notin \mathcal{N}_{23}$ implies that $\mathcal{N}_{23} \neq P(S)$.

Let $B = S \setminus \{a^p \mid p \in N,\ p \text{ is prime}\}$. Then $R_1^*(B) = S \setminus \{a\} \neq S = R_2^*(B)$. Hence $B \notin \mathcal{R}$ and we have $\mathcal{R} \neq P(S)$.

By Corollary of Theorems 5 and 6 $M = \{a^{2^k} \mid k \in N\} \in \mathcal{N}_{23}$ but $M \notin \mathcal{N}_{13}$ since $a \notin N_1(M)$ but $a \in N_3(M)$. Therefore $\mathcal{N}_{23} \neq \mathcal{N}_{13}$. Now we shall prove that $M \in \mathcal{R}$ but $M \notin \mathcal{N}_{12}$. Clearly $a \in N_2(M)$ but $a \notin N_1(M)$, therefore $M \notin \mathcal{N}_{12}$. On the other hand $M \in \mathcal{N}_{23}$ and $N_2(M) = N_3(M) = \{a\} \cup M$, but this is neither an ideal of S nor it contains an ideal of S. Hence $R_1^*(M) = R_2^*(M) = \emptyset$ i.e. $M \in \mathcal{R}$. We have $\mathcal{R} \neq \mathcal{N}_{12}$. Moreover $M \in \mathcal{R}$, $M \notin \mathcal{N}_{12}$ imply $M \in \mathcal{R}$, $M \notin \mathcal{N}_{13}$. Hence $\mathcal{R} \neq \mathcal{N}_{13}$.

Some other properties

In the following example it will be shown that $\langle \mathcal{N}_{12}, \subseteq \rangle$ need not be a complete lattice.

Example. Let $S = \langle a \rangle$ be the cyclic semigroup of infinite order generated by a. Let $M_i = \{a^{2^i}\}$ for all $i \in N$. Denote $M = \cup\{M_i \mid i \in N\}$. Then clearly $N_1(M_i) = N_2(M_i) = \emptyset$ for all $i \in N$. On the other hand $a \in N_2(M)$ but $a \notin N_1(M)$, hence $N_1(M) \neq N_2(M)$. This means that the union of infinitely many elements of $\mathcal{N}_{12}$ does not belong to $\mathcal{N}_{12}$. Moreover we shall prove that in $\langle \mathcal{N}_{12}, \subseteq \rangle$ the sup $\{M_i \mid i \in N\} = \vee\{M_i \mid i \in N\}$ does not exist.

Let $a \in \mathcal{N}_{12}$ and $A \supseteq M$. Since $a \in N_1(M)$, we have $a \in N_1(A)$ and A contains a set $\{a^k \mid k \geqslant n_0\}$, where $n_0 \in N$. Now it is easy to see that every upper bound A of M in $\langle \mathcal{N}_{12}, \subseteq \rangle$ is of the form $A = M \cup \{a^k \mid k \geqslant n_0\}$, where $n_0 \in N$. But the system of all these upper bounds of M has no minimal element. This implies that $\langle \mathcal{N}_{12}, \subseteq \rangle$ is not a complete lattice.

Theorem 8. *Let S be a semigroup. Then* $\langle \mathcal{N}_{12}, \cap, \cup \rangle$ *is a complemented lattice.*

Proof. We prove it indirectly. Let $M \in \mathcal{N}_{12}$ i.e. $N_1(M) = N_2(M)$ and $S \setminus M \notin \mathcal{N}_{12}$ i.e. $N_1(S \setminus M) \neq N_2(S \setminus M)$. Since $N_1(S \setminus M) \subseteq N_2(S \setminus M)$ and $N_1(S \setminus M) \neq N_2(S \setminus M)$ there exists an x such that $x \in N_2(S \setminus M)$ but $x \notin N_1(S \setminus M)$. Now $x \notin N_1(S \setminus M)$ implies that $x \in N_2(M) = N_1(M)$. However $x \in N_1(M)$ and $x \in N_2(S \setminus M)$ cannot hold. We have got a contradiction. Hence $S \setminus M \in \mathcal{N}_{12}$.

Corollary. *Let S be a semigroup. Then* $\langle N_{12}(\mathcal{N}_{12}), \cap, \cup \rangle$ *is a complemented lattice.*

The proof follows from the fact that N_{12}: $\langle \mathcal{N}_{12}, \cap, \cup \rangle \rightarrow \langle P(S), \cap, \cup \rangle$ is a homomorphism.

In the next example it will be shown that $\langle \mathcal{N}_{13}, \cap, \cup \rangle$ need not be a complemented lattice.

Example. Let $S = \langle a \rangle$ be the cyclic semigroup of infinite order. Let $M = \{a^n \mid n > n_0\}$, where $n_0 \in N$. Then clearly $M \in \mathcal{N}_{13}$ but $S \setminus M = \{a^k \mid k \leq n_0\} \notin \mathcal{N}_{13}$ by Theorem 3.

In the following example it will be shown that $\langle \mathcal{N}_{13}, \subseteq \rangle$ need not be a complete sublattice of $\langle P(S), \subseteq \rangle$.

Example. Let $S = \langle a \rangle$ be the cyclic semigroup of infinite order. Let $M_k = \{a\} \cup \{a^n \mid n \geq k\}$ for all $k \in N$. Then clearly $M_k \in \mathcal{N}_{13}$ for all $k \in N$ but $\cap \{M_k \mid k \in N\} = \{a\} \notin \mathcal{N}_{13}$.

Theorem 9. *Let S be a semigroup. Then* $\langle N_{13}(\mathcal{N}_{13}), \cap, \cup \rangle$ *is a complemented lattice.*

Proof. First we prove that $A = S \setminus N_{13}(M)$ is a union of cyclic semigroups. If $x^n \in N_{13}(M) = N_3(M)$, then $x \in N_3(M) = N_{13}(M)$. Hence $x \in A$ implies $\langle x \rangle \subseteq A$.

Next we show that $N_3(A) = A$. Clearly $N_3(A) \supseteq A$. If $x \in N_3(A)$, then infinitely many powers x^n belong to A, because A is a union of cyclic semigroups. From this follows that $x \notin N_{13}(M)$ (i.e. $x \in A$) since $x \in N_{13}(M) = N_1(M)$ implies that almost all powers x^n are contained in $M \subseteq N_{13}(M)$. Hence $x \in N_3(A)$ implies that $x \in A$ i.e. $N_3(A) \subseteq A$. We got $N_3(A) = A$.

Now we prove that $N_1(A) = A$. Clearly $N_1(A) \subseteq N_3(A) = A$, therefore $N_1(A) \subseteq A$. Since A is a union of cyclic semigroups, $N_1(A) \supseteq A$ holds. We have $N_1(A) = A = N_3(A)$, therefore $A = N_{13}(A) \in N_{13}(\mathcal{N}_{13})$.

We got the following results: $N_{13}(M) \in N_{13}(\mathcal{N}_{13})$, $N_{13}(A) = A \in N_{13}(\mathcal{N}_{13})$, $N_{13}(M) \cup N_{13}(A) = S$ and $N_{13}(M) \cap N_{13}(A) = \emptyset$. This means that $\langle N_{13}(\mathcal{N}_{13}), \cap, \cup \rangle$ is a complemented lattice.

In the following example it will be shown that in the complete lattice $\langle \mathcal{N}_{23}, \subseteq \rangle$ even the finite meets need not be set-theoretical intersections.

E x a m p l e. Let $S=\langle a\rangle$ be the cyclic semigroup of infinite order. If $M_1=\{a\}\cup\{a^{2^i}\mid i\in N\}$ and $M_2=\{a^{2^i-1}\mid i\in N\}$, then $M_1\in\mathcal{N}_{23}$ and $M_2\in\mathcal{N}_{23}$ by Theorem 5 and 6. But $M_1\cap M_2=\{a\}\notin\mathcal{N}_{23}$. Hence $M_1\wedge M_2=\emptyset$.

From the following example we shall see that the complete lattice $\langle\mathcal{N}_{23},\subseteq\rangle$ need not be complemented.

E x a m p l e. Let $S=\langle a\rangle$ be the cyclic semigroup of infinite order. Let $M_1=\{a^p\mid p\in N,\ p=1$ or p is a prime$\}$ and $M_2=\{a^n\mid n\in N,\ n\neq 1,\ n$ is not a prime$\}$. Then $M_1\notin\mathcal{N}_{23}$ since $N_3(M_1)=M_1$ but $N_2(M_1)=\{a\}$. By Theorem 5 and 6, $M_2\in\mathcal{N}_{23}$ because M_2 contains with every element $a^n\in M_2$ also its power $(a^n)^2=a^{2n}$.

Now we have $M_1\cup M_2=S$ and $M_1\cap M_2=\emptyset$. We want to find an $K\in\mathcal{N}_{23}$ such that $M_2\cup K=S$ and $M_2\wedge K=\emptyset$.

The condition $M_2\cup K=S$ implies $K\supseteq M_1$. By Theorem 6, the conditions $K\in\mathcal{N}_{23}$ and $K\supseteq M_1$ imply that for the element $a\in M_1$ there has to exist a sequence $(n_k)_{k=1}^{\infty}$, $n_k\in N$ such that $T=\{a^{n_1 n_2},\ \dots,\ a^{n_1 n_2\cdots n_k},\ \dots\}\subseteq K$ holds. But then $M_2\cap K\supseteq T\neq\emptyset$ and by Theorem 6 $T\in\mathcal{N}_{23}$. Hence $M_2\wedge K\neq\emptyset$. We see that $M_2\in\mathcal{N}_{23}$ has no complement in $\langle\mathcal{N}_{23},\subseteq\rangle$, therefore $\langle\mathcal{N}_{23},\subseteq\rangle$ is not complemented.

In the following example it is shown that the mapping $N_{12}\colon\langle\mathcal{N}_{12},\cap,\cup\rangle\to\langle P(S),\cap,\cup\rangle$ need not preserve infinite joins and infinite meets.

E x a m p l e. Let S be the free semigroup generated by the set $\{0,a\}\cup\{b_k\mid k\in N\}$ and by the relations $0\cdot 0=0\cdot a=a\cdot 0=0\cdot b_k=b_k\cdot 0=a\cdot b_k=b_k\cdot a=0$ for all $k\in N$ and $b_k\cdot b_l=b_l\cdot b_k=0$ for all $k,l\in N$, $k\neq l$.

a) Let $M_k=\{a^k\}\cup\langle b_k\rangle$ for all $k\in N$. Clearly $N_1(M_k)=N_2(M_k)=\langle b_k\rangle=N_{12}(M_k)$, hence $M_k\in\mathcal{N}_{12}$.

We have $\cup\{M_k\mid k\in N\}=\langle a\rangle\cup(\cup\{\langle b_k\rangle\mid k\in N\})=S\setminus\{0\}$. Moreover

$$N_1(\cup\{M_k\mid k\in N\})=N_2(\cup\{M_k\mid k\in N\})=$$
$$=\langle a\rangle\cup(\cup\{\langle b_k\rangle\mid k\in N\})=S\setminus\{0\}=N_{12}(\cup\{M_k\mid k\in N\}),$$

hence $\cup\{M_k\mid k\in N\}\in\mathcal{N}_{12}$.

On the other hand we get $M=\cup\{N_{12}(M_k)\mid k\in N\}=\cup\{\langle b_k\rangle\mid k\in N\}\neq S\setminus\{0\}$, $N_1(M)=N_2(M)=M=N_{12}(M)$ i.e. $\cup\{N_{12}(M_k)\mid k\in N\}=M\in N_{12}(\mathcal{N}_{12})$.

Hence $N_{12}(\cup\{M_k\mid k\in N\})=S\setminus\{0\}\neq\cup\{N_{12}(M_k)\mid k\in N\}$.

b) Let $L_k=\{a^i\mid i\geqslant k\}\cup\langle b_k\rangle$ for all $k\in N$. Then $N_1(L_k)=N_2(L_k)=\langle a\rangle\cup\langle b_k\rangle=N_{12}(L_k)$, hence $L_k\in\mathcal{N}_{12}$.

Moreover $\cap\{L_k\mid k\in N\}=\emptyset$ and $N_1(\emptyset)=N_2(\emptyset)=\emptyset=N_{12}(\emptyset)$ imply that $\cap\{L_k\mid k\in N\}\in\mathcal{N}_{12}$ and $N_{12}(\cap\{L_k\mid k\in N\})=\emptyset$.

On the other hand $\cap\{N_{12}(L_k)\mid k\in N\}=\langle a\rangle$. But $N_1(\langle a\rangle)=N_2(\langle a\rangle)=\langle a\rangle=N_{12}(\langle a\rangle)$ means that $\cap\{N_{12}(L_k)\mid k\in N\}\in N_{12}(\mathcal{N}_{12})$.

Hence $N_{12}(\cap\{L_k\mid k\in N\})=\emptyset\neq\langle a\rangle=\cap\{N_{12}(L_k)\mid k\in N\}$.

Subsets that belong to $\mathcal{N}_{13}$ or to $\mathcal{N}_{23}$

We shall prove that if a subset M of a semigroup S satisfies some conditions, then M belongs to $\mathcal{N}_{13}$ or to $\mathcal{N}_{23}$.

Lemma 1 of [6] implies that every subsemigroup of a semigroup S belongs to $\mathcal{N}_{23}$.

From Lemma 2 of [6] it follows that every left ideal, right ideal and two-sided ideal of a semigroup S belongs to $\mathcal{N}_{13}$ (hence it belongs also to $\mathcal{N}_{12}$ and $\mathcal{N}_{23}$).

Let S be a semigroup, $M \subseteq S$ and $M \neq \emptyset$. We consider the following conditions:

(1) $M^m\, S\, M^n \subseteq M$,

(2) $M^m\, S \cap S\, M^n \subseteq M$,

where m, n are fixed nonnegative integers, not both equal 0 and for $m=0$ or $n=0$, M^0 be the empty symbol. We say that M satisfies condition (1) or (2), respectively, for the pair (m, n).

Remark. If M satisfies condition (1) or (2) for the pair (m, n), it also satisfies this condition for the pair (p, q), $p \geqslant m$, $q \geqslant n$ (see [2]).

Lemma 9. *If M satisfies condition* (1), *then* $N_1(M) = N_3(M)$.

Proof. a) $N_1(M) \subseteq N_3(M)$.

b) If $x \in N_3(M)$, then there exists a positive integer k, such that $x^k \in M$ holds. In view of condition (1) we have $x^{k(m+n)+p} \in M$ for all $p \in N$. Hence $N_3(M) \subseteq N_1(M)$.

Lemma 10. *If M satisfies condition* (2) *it satisfies also condition* (1) (see [2]).

Proof. Evidently $M^m S M^n \subseteq M^m S$ and $M^m S M^n \subseteq S M^n$. Hence $M^m S M^n \subseteq M^m S \cap S M^n \subseteq M$. Hence (2) implies (1).

Corollary. *If M satisfies condition* (2), *then* $N_1(M) = N_3(M)$.

Further let us consider the condition

(3) $S^m M \cap M S^n \subseteq M$, where m, n are fixed positive integers. We say that M satisfies condition (3) for the pair (m, n).

Lemma 11. *If M satisfies condition* (3) *for some pair* (m, n), *then it satisfies condition* (3) *for every pair* (p, q), $p \geqslant m$, $q \geqslant n$.

Proof. We have $S^p M \cap M S^q \subseteq S^m M \cap M S^n \subseteq M$.

Lemma 12. *If M satisfies condition* (3) *for some pair* (m, n), *then* $N_1(M) = N_3(M)$.

Proof. a) $N_1(M) \subseteq N_3(M)$.

b) If $m \geqslant n$ and M satisfies condition (3) for the pair (m, n), then M also satisfies this condition for the pair (m, m) and for every pair $(m+t, m+t)$, $t \in N$.

Let $x \in N_3(M)$ i.e. $x^k \in M$ for some positive integer k. Then the relation $S^{m+t} M \cap M S^{m+t} \subseteq M$, $t \in N$ implies that $x^{k+m+t} \in M$ for every $t \in N$ i.e. $x \in N_1(M)$. We have obtained that $N_3(M) \subseteq N_1(M)$. Hence $N_1(M) = N_3(M)$.

From the foregoing Lemmas we have

Theorem 10. *Let S be a semigroup. All subsets $M \subseteq S$ that satisfy some of the conditions (1), (2), (3) are elements of $\mathcal{N}_{13}$.*

Corollary. *Let S be a semigroup. Then all (m, n)-ideals and all (m, n)-quasiideals of S are elements of $\mathcal{N}_{13}$.*

The Corollary follows immediatelly from definitions (see [2] and [5]).

Lemma 13. *Let S be a semigroup, $A \subseteq S$, $B \subseteq S$ and let $AB \subseteq A \cap B$. Then $N_2(AB) = N_2(A \cap B) = N_3(A \cap B) = (N_3(AB)$.*

Proof. a) First we prove that $N_2(AB) = N_2(A \cap B)$. Evidently $N_2(AB) \subseteq N_2(A \cap B)$ since $AB \subseteq A \cap B$. Now let $x \in N_2(A \cap B)$. Then for infinitely many $k \in N$ we have $x^k \in A \cap B$ i.e. for infinitely many $k \in N$ there is $x^k \in A$ and $x^k \in B$. This implies that for infinitely many k we have $x^{2k} \in AB$ i.e. $x \in N_2(AB)$. Hence $N_2(A \cap B) \subseteq N_2(AB)$.

b) Similarly it can be proved that $N_3(AB) = N_3(A \cap B)$.

c) It remains to prove that $N_3(AB) = N_2(AB)$. It is sufficient to prove that $N_3(AB) \subseteq N_2(AB)$ because we know that $N_2(AB) \subseteq N_3(AB)$.

Let $x \in N_3(AB)$. Then there exists a $k \in N$ such that $x^k \in AB$. We shall prove that $x^{nk} \in AB$ for all $n \in N$. This statement is true for $n = 1$. Suppose that $x^{nk} \in AB \subseteq A \cap B$. Then $x^{nk} \in A$. But $x^k \in AB \subseteq A \cap B$ implies $x^k \in B$. Therefore $x^{(n+1)k} = x^{nk} \cdot x^k \in AB$. We have $x^{nk} \in AB$ for all $n \in N$ i.e. $x \in N_2(AB)$. We have proved that $N_3(AB) \subseteq N_2(AB)$. This together with $N_2(AB) \subseteq N_3(AB)$ gives $N_2(AB) = N_3(AB)$.

Theorem 11. *Let S be a semigroup, $A \subseteq S$, $B \subseteq S$ and $AB \subseteq A \cap B$. Then $AB \in \mathcal{N}_{23}$, $A \cap B \in \mathcal{N}_{23}$ and $N_{23}(AB) = N_{23}(A \cap B)$.*

Subsemigroups

Let S be a semigroup, S' a subsemigroup of S and $M' \subseteq S'$. Then $N_1'(M')$ will be the set of all strongly M'-potent elements of S', $N_2'(M')$ will be the set of all weakly M'-potent elements of S' and $N_3'(M')$ will be the set of all almost M'-potent elements of S'.

We have the following

Lemma 14. *Let S be a semigroup, S' a subsemigroup of S and $M \subseteq S$. Then $N_i'(S' \cap M) = N_i(M) \cap S'$ holds for $i = 1, 2, 3$.*

Proof. a) If $x \in N_1'(S' \cap M)$, then $x^n \in S' \cap M \subseteq M$ for almost all $n \in N$ and $x \in S'$ hold. This means that $x \in N_1(M)$ and $x \in S'$, hence $x \in N_1(M) \cap S'$. Therefore $N_1'(S' \cap M) \subseteq N_1(M) \cap S'$ is valid.

b) If $x \in N_1(M) \cap S'$, then $x \in N_1(M)$ and $x \in S'$. This implies that $x^n \in M$ for almost all $n \in N$ and $x \in S'$. Since S' is a subsemigroup, $x^n \in S'$ holds for all $n \in N$. Hence we get $x^n \in M \cap S'$ for almost all $n \in N$ and $x \in S'$ i.e. $x \in N_1'(S' \cap M)$.

For the cases $i=2$ and 3 the proofs are similar.

Corollary. *Let S be a semigroup, S' a subsemigroup of S and $M \subseteq S'$. Then $N'_i(M) = N_i(M) \cap S'$ for $i=1, 2, 3$.*

Using the notations $\mathcal{N}'_{12} = \{M \subseteq S' | N'_1(M) = N'_2(M)\}$, $\mathcal{N}'_{13} = \{M \subseteq S' | N'_1(M) = N'_3(M)\}$ and $\mathcal{N}'_{23} = \{M \subseteq S' | N'_2(M) = N'_3(M)\}$, we get

Lemma 15. *Let S be a semigroup, S' be a subsemigroup of S and $M \subseteq S'$. If $M \in \mathcal{N}'_{23}$, then $M \in \mathcal{N}_{23}$.*

Proof. Let $M \subseteq S'$ and $M \in \mathcal{N}'_{23}$ i.e. $N'_2(M) = N'_3(M)$. If $x \in N_3(M)$, then there exists $n \in N$ such that $x^n \in M \subseteq S'$. This means that $x^n \in N'_3(M) = N'_2(M)$. Therefore $x^{nm} = (x^n)^m \in M$ holds for infinitely many $m \in N$ i.e. $x \in N_2(M)$. We have proved that $N_3(M) \subseteq N_2(M)$. This, together with $N_2(M) \subseteq N_3(M)$ gives $N_2(M) = N_3(M)$ i.e. $M \in \mathcal{N}_{23}$.

A subsemigroup S' of a semigroup S is called *isolated* if $x^n \in S'$ implies $x \in S'$ for all $x \in S$.

Lemma 16. *Let S be a semigroup, S' an isolated subsemigroup of S and $M \subseteq S'$. Then the following statements hold:*

a) *If $M \in \mathcal{N}'_{13}$, then $M \in \mathcal{N}_{13}$.*

b) *If $M \in \mathcal{N}'_{12}$, then $M \in \mathcal{N}_{12}$.*

Proof. a) Let $M \subseteq S'$ and $M \in \mathcal{N}'_{13}$ i.e. $N'_1(M) = N'_3(M)$. If $x \in N_3(M)$, then there exists $n \in N$ such that $x^n \in M \subseteq S'$. But since S' is an isolated subsemigroup, we have $x \in S'$ and $x^n \in M$, hence $x \in N'_3(M) = N'_1(M)$. Therefore $x^m \in M$ holds for almost all $m \in N$ i.e. $x \in N_1(M)$. We have obtained $N_3(M) \subseteq N_1(M)$ what together with $N_1(M) \subseteq N_3(M)$ gives $N_1(M) = N_3(M)$. This means that $M \in \mathcal{N}_{13}$.

The proof of b) is similar.

Theorem 12. *Let S be a semigroup and S' a subsemigroup of S. Then the following statements hold:*

a) *$\mathcal{N}'_{23} \subseteq \mathcal{N}_{23}$.*

b) *If S' is an isolated subsemigroup, then $\mathcal{N}'_{12} \subseteq \mathcal{N}_{12}$ and $\mathcal{N}'_{13} \subseteq \mathcal{N}_{13}$.*

c) *The complete lattice $\langle \mathcal{N}'_{23}, \subseteq \rangle$ is a complete sublattice of the complete lattice $\langle \mathcal{N}_{23}, \subseteq \rangle$.*

d) *If S' is an isolated subsemigroup, then the lattice $\langle \mathcal{N}'_{12}, \cap, \cup \rangle$ is a sublattice of the lattice $\langle \mathcal{N}_{12}, \cap, \cup \rangle$.*

e) *If S' is an isolated subsemigroup, then the complete lattice $\langle \mathcal{N}'_{13}, \subseteq \rangle$ is a complete sublattice of the complete lattice $\langle \mathcal{N}_{13}, \subseteq \rangle$.*

The proof follows from the foregoing Lemmas and Theorem 1.

The following example illustrates that if S' is not an isolated subsemigroup of S, then neither $\mathcal{N}'_{12} \subseteq \mathcal{N}_{12}$ nor $\mathcal{N}'_{13} \subseteq \mathcal{N}_{13}$ need be true.

Example. Let $S = \langle a \rangle$ be the cyclic semigroup of infinite order. Let $S' = M = \{a^{2k} | k \in N\}$. Then $N'_1(M) = N'_2(M) = N'_3(M) = M = S'$ i.e. $M \in \mathcal{N}'_{12}$ and $M \in \mathcal{N}'_{13}$.

On the other hand $a \notin N_1(M)$ but $a \in N_2(M)$ and $a \in N_3(M)$. Hence $N_1(M) \neq N_2(M)$ and $N_1(M) \neq N_3(M)$ i.e. $M \notin \mathcal{N}_{12}$ and $M \notin \mathcal{N}_{13}$.

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Katedra matematiky
Elektrotechnickej fakulty SVŠT
Gottwaldovo nám. 19
812 19 Bratislava

О ТРЕХ СТРУКТУРАХ, ПРИНАДЛЕЖАЩИХ ВСЯКОИ ПОЛУГРУППЕ

Robert Šulka

Резюме

С помощью понятия нильпотентности определены три структуры, элементы которых принадлежат булеану $\langle P(S), \subseteq \rangle$ полугруппы S. Доказываются некоторые свойства этих структур. Так две из этих структур являются полными и одна из них является структурой с дополнениями. Показывается, что все (m, n) – идеалы и все (m, n) – квазиидеалы содержатся во всех трех этих структурах. Изучаются тоже эти структуры в случае циклических полугрупп и в случае подполугрупп полугруппы.