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ON THE $b$-EQUIVALENCE OF MULTILATTICES

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The notion of the $b$-isomorphism for lattices was investigated by Kolibiar [5]; he proved the following theorem:

(A) Let $M$ and $M'$ be distributive lattices. Then the following conditions are equivalent:

- (i) $M$ and $M'$ are $b$-equivalent;
- (ii) there are lattices $M_1$ and $M_2$ such that $M$ is isomorphic with $M_1 \times M_2$ and $M'$ is isomorphic with $M_1 \times M_2$.

Klaučová [4] generalized theorem (A) for directed distributive multilattices. Jakubík [2] studied pairs of modular lattices of locally finite lengths with isomorphic unoriented graphs; he proved that two modular lattices $M$ and $M'$ of locally finite lengths have isomorphic unoriented graphs if and only if (ii) is valid. Jakubík [3] also proved that if $M$ and $M'$ are lattices of locally finite lengths such that the unoriented graphs of $M$ and $M'$ are isomorphic and if $M$ is modular, then $M'$ is modular as well.

In this note it will be shown that if $M$ and $M'$ are $b$-equivalent directed multilattices and if $M$ is distributive, then $M'$ must also be distributive. Hence in the above mentioned theorem of [4] it suffices to assume that $M$, $M'$ are directed multilattices and that $M$ is distributive.

Let us recall some basic concepts that will be used later.

A multilattice [1] is a poset $M$ in which condition (i) and its dual (ii) are satisfied:

- (i) If $a$, $b$, $h \in M$ and $a \leq h$, $b \leq h$, then there exists $v \in M$ such that (a) $v \leq h$, $v \geq a$, $v \geq b$ and (b) $z \in M$, $z \geq a$, $z \geq b$, $z \leq v$ implies $z = v$. $(a \vee b)_h$ designates the set of all elements $v \in M$ satisfying (i); the symbol $(a \wedge b)_d$ has a dual meaning.

We denote $a \vee b = \cup(a \vee b)_h$, $a \wedge b = \cup(a \wedge b)_d$.

For any multilattice $M$ we denote by $M$ the multilattice dual to $M$.

A poset $A$ is called upper (lower) directed if for every pair of the elements $a$, $b \in A$ there exists an element $h \in A$ ($d \in A$) such that $a \leq h$, $b \leq h$ ($d \leq a$, $d \leq b$). The upper and lower directed poset $A$ is called directed [5].

A multilattice $M$ is said to be distributive iff for every $a$, $b$, $b'$, $d$, $h \in M$ satisfying $d \leq a$, $b$, $b' \leq h$. $(a \vee b)_h = (a \vee b')_h = h(a \wedge b)_d = (a \wedge b')_d = d$ we have $b = b'$ [1].

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The following definitions have been introduced in [4].

Let \( M \) be a directed multilattice \( a, b, x \in M \). We say that \( x \) is between \( a \) and \( b \) and write \( axb \) if the following condition is satisfied.

\[(b) \quad [(a \wedge x) \vee (b \wedge x)]_x = x, \quad (a \wedge x) \wedge (b \wedge x) \subseteq a \wedge b.\]

Directed multilattices \( M, M' \) are said to be \( b \)-equivalent if there exists a bijection \( f \) of \( M \) onto \( M' \) such that, for each \( a, b, x \in M \), we have \( axb \iff f(a) f(x) = f(b) \).

Further we assume that \( M \) and \( M' \) are directed \( b \)-equivalent and that the multilattice \( M \) is distributive. If \( f \) is the corresponding bijection and \( x \in M \), we put \( f(x) = x' \). The partial ordering and multioperations in \( M \) and \( M' \) will be denoted by \( \leq \), \( \vee \), \( \wedge \) and \( \subseteq \), \( \cup \), \( \cap \), respectively. Let \( u, v \in M, u \leq v \). The interval \( [u, v] \) is the set \( \{x \in M: u \leq x \leq v\} \). We say that the interval \( [u, v] \) is preserved (reserved) if \( u' \leq v' \ (v' \leq u') \) in \( M' \); the interval \( [u, u] \) is simultaneously preserved and reversed.

We need the following results (cf. [4]):

**Lemma 1.** Let \( a, b \in M, a \leq b \). Then \( axb \iff a \leq x \leq b \).

**Lemma 2.** Let \( a, b, u, v \in M, u \leq a \leq b \leq v \) and let the interval \([u, v]\) be preserved (reversed). Then the interval \([a, b]\) is preserved (reversed).

**Lemma 3.** Let \( a, b, x \in M, x \leq a, x \leq b \ (a \leq x, b \leq x) \). Then \( axb \iff x \in a \wedge b \ (x \in a \vee b) \).

**Lemma 4.** Let \( a, b \in M, u \in a \wedge b, v \in a \vee b \). If the interval \([a, v]\) ([\(u, b]\]) is preserved and the interval \([b, v]\) ([\(u, a]\]) is reversed, then the interval \([u, b]\) ([\(u, a]\]) is reversed.

The assertions of Lemma 1, 2 were stated in [4] under the assumption that both \( M \) and \( M' \) are directed distributive multilattices. But it follows from the method of their proofs that they remain valid also without the assumption of distributivity of \( M' \).

**Lemma 5.** Let \( a, b \in M, u \in a \wedge b, v \in a \vee b \). If the intervals \([a, v], [b, v]\) or the intervals \([u, a], [u, b]\) are preserved (reversed), then the interval \([u, v]\) is preserved (reversed).

**Lemma 6.** Let \( a, b \in M \). Put \( aR,b \ (aR_2,b) \) iff there exists an element \( v \in M, v \in a \vee b \), such that the intervals \([a, v], [b, v]\) are reversed (preserved). The relations \( R, R_2 \) are equivalences on \( M \).

For \( a', b' \in M' \) set \( a'R_1b' \ (a'R_2,b') \) iff there exists an element \( v' \in M', v' \in a' \cup b' \) such that the intervals \([a', v'], [b', v']\) are reversed (preserved), i.e. \( a \geq v, b \geq v \ (a \leq v, b \leq v) \).

**Lemma 1.** Let \( a, b \in M \). The relation \( aR,b \ (aR_2,b) \) is satisfied iff \( a'R_1b' \ (a'R_2,b') \) is valid.

Proof. Let \( aR,b \) be valid. Then there exists an element \( v \in a \vee b \) such that the intervals \([a, v], [b, v]\) are reversed. Choose \( u \in a \wedge b \). By the Lemmas 1 and 2, the
intervals \([u, a], [u, b]\) are reversed. Consequently \(u' \supseteq a', u' \supseteq b'\). Moreover by Lemma I, we have \(a' \cap b'\) holds. It follows that \(u' \in a' \cap b'\) according to Lemma I. Thus the relation \(a'Rb'\) is valid.

Conversely, the assumption \(a'Rb'\) implies that there exists \(v' \in a' \cap b'\) such that the intervals \([a', v'], [b', v']\) are reversed. By Lemma I, we have \(v' \in a' \land b\). Choose \(u \in a \lor b \); then from Lemmas I, I, it follows that the intervals \([a, u], [b, u]\) are reversed and hence \(aRb\) is valid.

Analogously we can prove the assertion concerning \(RL\).

**Lemma 2.** Let \(a', b' \in M', u' \in a' \cap b', v' \in a' \cap b'\). If the intervals \([a', v'], [b', v']\) are preserved (reversed), then the interval \([u', v']\) is preserved (reversed).

**Proof.** Let the intervals \([a', v'], [b', v']\) be preserved. Choose \(r \in a \land u, s \in b \land u\). From Lemma I, it follows that \(aru, bsu\). Consequently \(a'r'\), \(b's'\). Using Lemma I, we obtain that the intervals \([r, a], [s, b]\) are preserved and the intervals \([r, u], [s, u]\) are reversed. Choose \(t \in r \lor s\). By Lemma I, we have \(a'u'b'\). Hence \(aub\). It follows that \(t \in a \land b\), \(u \in r \lor s\) according to the condition \((b)\). Using Lemma I, we infer that the interval \([t, v]\) is preserved. Consequently the intervals \([t, s], [t, r]\) are preserved by Lemma I. According to Lemma I, the interval \([t, u]\) is simultaneously preserved and reversed. Hence \(t = r = s = u\). Thus \(u \leq a \leq v\).

If the intervals \([a', v'], [b', v']\) are reversed, then choose \(w \in a \lor b\). Consider \(r, s, t\) as above. By Lemma I, the interval \([v, w]\) is reversed, hence the intervals \([a, w], [b, w]\) are reversed according to Lemma I. Again from Lemma I, it follows that the interval \([t, w]\) is reversed. Consequently the intervals \([r, a], [s, b]\) are reversed. Hence \(r = a, s = b\), thus \(u \geq b \geq v\).

**Lemma 2'.** Let \(a', b' \in M', u' \in a' \cap b', v' \in a' \cap b'\). If the intervals \([u', a'], [u', b']\) are preserved (reversed), then the interval \([u', v']\) is preserved (reversed).

**Proof.** Let the intervals \([u', a']\), \([u', b']\) be preserved. Choose \(r \in a \lor v, s \in b \land v\). Similarly as in the proof of Lemma 2 (by using Lemma I, and Lemma I,) we obtain that the intervals \([r, a], [s, b]\) are preserved and the intervals \([s, v], [r, v]\) are reversed. Choose \(w \in a \lor b\), \(t \in r \lor s\). Since \(avb\), we have \(t \in a \land b\) according to the condition \((b)\). By Lemma I, the interval \([u, w]\) is preserved. Therefore the intervals \([a, w], [b, w]\) are preserved by Lemma I. Again by Lemma I, the interval \([t, w]\) is preserved. Hence the intervals \([r, a], [s, b]\) are preserved. Consequently \(r = a, s = b\). Thus \(v \geq a \geq u\).

Let the intervals \([u', a'], [u', b']\) be reversed and let \(r, s, t\) be as above. The interval \([t, u]\) is reversed by Lemma I. Then the intervals \([t, s], [t, r]\) are reversed according to Lemma I. Hence \(r = u \leq a \leq u\).

**Lemma 3.** Let \(a', b' \in M', a'R b'\). If \(w' \in a' \cap b'\), then the intervals \([a', w'], [b', w']\) are reversed.

**Proof.** Let \(a'Rb'\). Then there exists \(v' \in a' \cap b'\) such that the intervals \([a', v'], [b', v']\),
[b', v'] are reversed. Choose u' e a' \cap b'. The interval [u', v'] is reversed by Lemma 2. Hence the intervals [u', a'], [u', b'] are reversed according to Lemma I_2. If w' e a' \cup b', then again by Lemma 2 the interval [u', w'] is reversed. Therefore the intervals [a', w'], [b', w'] are reversed by Lemma I_2.

Analogously we can prove:

**Lemma 3'.** Let a', b' e M', a'R I b'. If w' e a' \cup b', then the intervals [a', w'], [b', w'] are preserved.

**Lemma 4.** Let a', b' e M', a'R_2 b' (a'R I b'). If u' e a' \cap b', then the intervals [u', a'], [u', b'] are reversed (preserved).

**Proof.** Let a'R_2 b', u' e a' \cap b', v' e a' \cup b'. By Lemma 3 the intervals [a', v'], [b', v'] are reversed. Hence the interval [u', v'] is reversed by Lemma 2. Therefore the intervals [u', a'], [u', b'] are reversed according to Lemma I_2. Similarly we can prove the analogous assertion concerning R_2.

**Lemma 5.** The relations R_1, R_2 are equivalence relations on M' and they satisfy the following conditions

(i) \( R_1 \cdot R_2 = R_2 \cdot R_1 \)

(ii) \( R_1 \cup R_2 = I', R_1 \cap R_2 = 0' \) (where 0'(I') is the least (greatest) element of the lattice of all equivalence relations on the set M').

(iii) If a', b', c' e M', a' \subseteq c', a'R_1 b', b'R_2 c', then a' \subseteq b' \subseteq c'.

(iv) Let a', b', c', d' e M', a'R_1 b', c'R_2 d', a'R_2 c', b'R_1 d'. Then from a' \subseteq b' it follows that c' \subseteq d' and from a' \subseteq c' it follows that b' \subseteq d'.

The Lemma can be proved in the same way as [4, Lemma 9].

The following assertions K_1, K_2 were proved by Kolibiar.

(K_1) [5]. Let M be a Cartesian product of two posets M_1, M_2. M is a multilattice iff M_1 and M_2 are multilattices. For x e M we denote by x_i the components of x(x_i e M_i). Let a, b, h, v e M. Then v e (a \lor b)_n, (v e (a \land b)_n) iff v e (a \lor b)_n, (v e (a \land b)_n) for a_i, b_i, h_i, v_i e M_i (i = 1, 2).

(K_2) [6]. Let A be a quasiordered set. There exists a one-one correspondence between the non trivial direct decompositions of the quasiordered set A into two factors and pairs (R_1, R_2) of non trivial congruence relations R_1, R_2 on A satisfying the properties (i), (ii), (iii), (iv) from Lemma 5. To each couple (R_1, R_2) with the mentioned properties there corresponds the decomposition A \sim A/R_1 \times A/R_2 and to each element a e A there corresponds the element (a_1, a_2), where a_i is the equivalence class under R_i (i = 1, 2) containing a.

Denote M/R_1 = M_1, M/R_2 = M_2, M'/R_1 = M'_1, M'/R_2 = M'_2. From the assertion K_2 and from Lemma I_6 it follows that there exists an isomorphism \( \psi : M \sim M'_1 \times M'_2 \). According to K_2 and Lemma 5 there exists an isomorphism \( \psi' : M' \sim M'_1 \times M'_2 \). Since M, M' are multilattices, we infer that M_1 \times M_2, M'_1 \times M'_2 are multilattices and by K_1, M_1, M_2, M'_1, M'_2 are multilattices as well. Let \( \varphi \) be a b-equivalence of M
onto $M'$; then it is obvious that $x = \psi' \varphi \psi^{-1}$ is a $b$-equivalence of $M_1 \times M_2$ onto $M' \times \bar{M}_2$. In the same way as in [4] we can now prove that $M_1$ and $M'$ are isomorphic, $M_2$ and $\bar{M}_2$ are anti-isomorphic. Thus the following assertion holds.

**Theorem 1.** Let $M, M'$ be directed $b$-equivalent multilattices, $\varphi$ be an $b$-equivalence of $M$ onto $M'$ and let $M$ be distributive. Then there exist multilattices $M_1, M_2$ such that $M \sim M_1 \times M_2$, $M' \sim M_1 \times \bar{M}_2$, whereby the elements $x \in M$, $x' \in M'$, $x' = \varphi(x)$ are mapped on the same pair $(x_1, x_2)$, $x_1 \in M_1$, $x_2 \in \bar{M}_2$.

**Theorem 2.** Let $M$ and $M'$ be directed $b$-equivalent multilattices. If $M$ is distributive, then $M'$ is distributive as well.

**Proof.** Let $M$, $M'$ be directed $b$-equivalent multilattices and let $M$ be distributive. Then by Theorem 1 there exist multilattices $M_1, M_2$ such that $M \sim M_1 \times M_2$, $M' \sim M_1 \times \bar{M}_2$. Since $M$ is distributive, then by the assertion $K_1, M_1$ and $M_2$ are distributive also. Consequently $\bar{M}_2$ is distributive. Thus by the assertion $K_1, M'$ is distributive.

The following assertion has been proved in [4].

(C) Let $M, M'$ be directed distributive multilattices. $M, M'$ are $b$-equivalent if and only if there exist multilattices $M_1, M_2$ such that $M \sim M_1 \times M_2$ and $M' \sim M_1 \times \bar{M}_2$.

The following result is a direct corollary of Theorem 1, Theorem 2 and the assertion (C).

**Theorem 3.** Let $M, M'$ be direct multilattices. If $M$ is distributive, then the following conditions are equivalent.

(a) $M$ and $M'$ are $b$-equivalent multilattices.

(b) There exist multilattices $M_1, M_2$ such that $M \sim M_1 \times M_2$ and $M' \sim M_1 \times \bar{M}_2$.

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О $b$-ЭКВИВАЛЕНТНЫХ МУЛТИСТРУКТУРАХ

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Резюме

В данной статье обобщена одна теорема О. Клаучовой касающаяся пар дистрибутивных мультиструктур. Затем доказано, что если $M$ и $M'$ — $b$-эквивалентные направленные мультиструктуры и если $M$ — дистрибутивна, тогда $M'$ — также должна быть дистрибутивна.