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ON CONSEQUENCES OF THE BANACH-KURATOWSKI THEOREM FOR STONE ALGEBRA VALUED MEASURES

ZDENKA RIEČANOVÁ

1. Introduction, definitions, notation

The aim of this paper is to present for Stone algebra valued measures or submeasure an analogy of the classical *Banach problem* of the existence of a nontrivial σ-additive real-valued measure on the family of all subsets of the interval $\langle 0, 1\rangle$ which is zero in all one point subsets ([17], p. 141).

The theory of Stone algebra valued measures has been developed in [4, 5, 6, 7, 8, 9, 10 and 11]. A compact Hausdorff space S is Stonean if the closure of every open set is open. We let $C(S)$ denote the space of all continuous real-valued functions on S with the usual linear structure, norm and order. Such $C(S)$ is called a Stone algebra. M. H. Stone [3] showed that each bounded subset of the vector lattice $C(\Omega)$ of real continuous functions on a compact Hausdorff space Ω has a least upper bound in $C(\Omega)$ if and only if the closure of each open subset of Ω is open. So in this event we call $C(\Omega)$ a *Stone algebra.*

We recall that, in any topological space a set is nowhere dense if its closure has an empty interior. Also, a set is meagre if it is contained in the union of a sequence of nowhere dense sets.

A few preliminary remararks and notation. $C(S)$ always denotes a Stone algebra and $\Theta \in C(S)$ is the zero function with value 0 at each point of S. We write $\bigvee_{x\in I} a_x$ for the least upper bound and $\bigwedge_{x\in I} a_x$ for the greatest lower bound of the set $\{a_x\}_{x\in I}$, when these exist. $a_n \uparrow a$ means that $\{a_n\}_{n=1}^{\infty}$ is a monotone increasing sequence with the least upper bound a, and $b_n \downarrow b$ meaning that $\{b_n\}_{n=1}^{\infty}$ is a monotone decreasing sequence with the greatest lower bound b. Note that if in $C(S)$ $a_n \uparrow a$, then $a_n(s) \uparrow a(s)$ except on a meagre subset of S ([11], lemma 1.1). If $a, b \in C(S)$, then we write $a < b$ if $a \leqq b$ and $a \neq b$.

2. Finite $C(S)$-valued submeasure

Definition 2.1. *Let X be a nonempty set and $\mathscr{S}$ be a σ-ring of subsets of X. Let $C(S)$ be a Stone algebra. A $C(S)$-valued submeasure is defined to be a map $m: \mathscr{S} \to C(S)$ such that*

(i) $m(E) \geqq \Theta$ *for each* $E \in \mathscr{S}$,

(ii) $A \subset B$ *implies* $m(A) \leqq m(B)$ *for all* $A, B \in \mathscr{S}$,

(iii) $m(A \cup B) \leqq m(A) + m(B)$ *for all* $A, B \in \mathscr{S}$,

(iv) *if* $\{E_n\}_{n=1}^{\infty}$ *is a monotone decreasing sequence of elements of* $\mathscr{S}$ *such that* $\bigcap_{n=1}^{\infty} E_n = \emptyset$, *then* $\bigwedge_{n=1}^{\infty} m(E_n) = \Theta$.

Theorem 2.1. *Let X be a non-empty set and let $\mathscr{S}$ be a σ-ring of subsets of X. Let $C(S)$ be a Stone algebra such that each meagre subset of S is nowhere dense. Let $m: \mathscr{S} \to C(S)$ be a finite $C(S)$-valued submeasure. Let us assume the continuum hypothesis. If* $\operatorname{card}(E) = c$ *and $\{A_x\}_{x \in E}$ is a family of pairwise disjoint sets in $\mathscr{S}$ such that $\bigcup_{x \in F} A_x \in \mathscr{S}$ for all $F \subset E$, then*

$$m\left(\bigcup_{x \in E} A_x\right) = \bigvee_{\substack{I \subset E \\ I \text{ is finite}}} m\left(\bigcup_{x \in I} A\right)$$

Proof. It follows from the property (ii) of the submeasure that for every finite set $I \subset E$

$$m\left(\bigcup_{x \in I} A_x\right) \leqq m\left(\bigcup_{x \in E} A_x\right)$$

and hence

$$\bigvee_{\substack{I \subset E \\ I \text{ is finite}}} m\left(\bigcup_{x \in I} A_x\right) \leqq m\left(\bigcup_{x \in E} A_x\right).$$

We now define a map $\lambda: 2^E \to C(S)$ by $\lambda(F) = m\left(\bigcup_{x \in F} A_x\right)$ for all $F \subset E$. It is easy to see that λ is a $C(S)$-valued submeasure defined on 2^E.

Since $\operatorname{card}(E) = c$, it follows from the Banach–Kuratowski theorem ([1], p. 27) that there exist the subsets $F_{kr} \subset E$ $(k = 1, 2, \ldots; r = 1, 2, \ldots)$

(a): $\forall r \in N: \bigcup_{k \in N} F_{kr} = E$,

(b): $\forall k \in N \forall r \in N: F_{kr} \subset F_{k+1,r}$,

(c): $\forall f: N \to N: \bigcap_{n=1}^{\infty} F_{f(n),n}$ is a countable set.

Choose the sequences

$$F_{kl} \uparrow E$$

$$\ldots$$

$$F_{11} \cap F_{k2} \uparrow F_{11}$$

$$F_{21} \cap F_{k2} \uparrow F_{21}$$
$$\dots$$
$$F_{11} \cap F_{12} \cap F_{k3} \uparrow F_{11} \cap F_{12}$$
$$F_{11} \cap F_{22} \cap F_{k3} \uparrow F_{11} \cap F_{22}$$
$$\dots$$
$$F_{21} \cap F_{12} \cap F_{k3} \uparrow F_{21} \cap F_{12}$$
$$F_{21} \cap F_{22} \cap F_{k3} \uparrow F_{21} \cap F_{22}$$
$$\dots$$

We get countable many sequences. The submeasure λ of the members of each sequence converges pointwise except on a nowhere dense subset of S. Let $A \subset S$ be the union of these countable many nowhere dense sets. Then A is a meagre set and so, by the assumption A and also $\bar{A}$ are nowhere dense subsets of S. Thus for each point $s \in S \backslash \bar{A}$ there exists a non-empty clopen set $K_s \subset S \backslash \bar{A}$ such that $s \in K_s$. Choose $\varepsilon > 0$. Since K_s is compact and $\lambda(F_{kr})$ $(k = 1, 2, \dots, r = 1, 2, \dots)$ are continuous functions, we can find (by Dini's theorem) positive integers $k_1, k_2, \dots$ such that

$$\lambda(F_{k_1 1} \cap F_{k_2 2} \cap \dots \cap F_{k_n n})(\bar{s}) > \lambda(E)(\bar{s}) - \varepsilon$$

for all $\bar{s} \in K_s$ and all positive integers n. Since

$$F_{k_1 1} \cap F_{k_2 2} \cap \dots \cap F_{k_n n} \downarrow \bigcap_{n=1}^{\infty} F_{k_n n}$$

it follows from the properties of the submeasure that

$$\lambda(F_{k_1 1} \cap F_{k_2 2} \cap \dots \cap F_{k_n n}) \downarrow \lambda\left(\bigcap_{n=1}^{\infty} F_{k_n n}\right).$$

So

$$\lim_{n \to \infty} \lambda(F_{k_1 1} \cap \dots \cap F_{k_n n})(\bar{s}) = \lambda\left(\bigcap_{n=1}^{\infty} F_{k_n n}\right)(\bar{s})$$

except a nowhere dense set B.

Since $\bigcap_{n=1}^{\infty} F_{k_n n}$ is a countable set, $\bigcap_{n=1}^{\infty} F_{k_n n} = \{\varkappa_1, \varkappa_2, \dots, \varkappa_r, \dots\}$ and thus for all $\bar{s} \in K_s \backslash B$ there holds

$$\lambda(E)(\bar{s}) - \varepsilon \leqq \lambda\left(\bigcap_{n=1}^{\infty} F_{k_n n}\right)(\bar{s}) = \lambda(\{\varkappa_1, \varkappa_2, \dots\})(\bar{s}) =$$
$$= m\left(\bigcup_{n=1}^{\infty} A_{\varkappa_n}\right)(\bar{s}) = \left(\bigvee_{n=1}^{\infty} m\left(\bigcup_{r=1}^{n} A_{\varkappa_r}\right)\right)(\bar{s}) \leqq \left(\bigvee_{\substack{I \subset E \\ I \text{ is finite}}} m\left(\bigcup_{\varkappa \in I} A_{\varkappa}\right)\right)(\bar{s}).$$

Since B is a nowhere dense set

$$\lambda(E)(\bar{s}) - \varepsilon \leqq \left(\bigvee_{\substack{I \subset E \\ I \text{ is finite}}} m\left(\bigcup_{x \in I} A_x \right) \right)(\bar{s})$$

for all $\bar{s} \in K_s$ and each $\varepsilon > 0$. Hence

$$\lambda(E)(s) \leqq \left(\bigvee_{\substack{I \subset E \\ I \text{ is finite}}} m\left(\bigcup_{x \in I} A_x \right) \right)(s)$$

for all $s \in S \backslash \bar{A}$ and thus for all $s \in S$.

Theorem 2.2. *Let X be a non-empty set and let $\mathscr{S}$ be a σ-ring of subsets of X. Let $C(S)$ be a Stone algebra such that each meagre subset of S is nowhere dense. Let us assume the continuum hypothesis. Let* $\operatorname{card}(X) = c$. *Let $m: \mathscr{S} \to C(S)$ be a $C(S)$-valued submeasure such that $m(\{x\}) = \Theta$ for all $x \in X$. If there exists a set $E \in \mathscr{S}$ such that $m(E) > \Theta$, then there exists $F \subset X$ such that $F \notin \mathscr{S}$.*

Proof. Let us assume that for every set $E \subset X$ there holds $E \in \mathscr{S}$. If $E \in \mathscr{S}$, then $\operatorname{card}(E) \leqq c$ and hence by theorem 2.1

$$m(E) = m\left(\bigcup_{x \in E} \{x\} \right) = \bigvee_{\substack{A \subset E \\ A \text{ is finite}}} m\left(\bigcup_{x \in A} \{x\} \right) = \Theta.$$

3. σ-finite $C(S)$-valued submeasure

It is convenient to adjoin an object $+\infty$, not in $C(S)$ and extend the partial ordering and addition operation of $C(S)$ to $C(S) \cup \{+\infty\}$ in the obvious way by defining $a < +\infty$ for all $a \in C(S)$. Further, when $\{a_x\}_{x \in E}$ is an unbounded set in $C(S)$, we define $\bigvee_{x \in E} a_x$ be $+\infty$.

Definition 3.1. *Let $C(S)$ be a Stone algebra. Let X be a non-empty set and let $\mathscr{S}$ be a σ-ring of subsets of X. A σ-finite $C(S)$-valued submeasure is a map $m: \mathscr{S} \to C(S) \cup \{+\infty\}$ such that*

(i) *$m(E) \geqq \Theta$ for each $E \in \mathscr{S}$,*

(ii) *$A \subset B$ implies $m(A) \leqq m(B)$ for all $A, B \in \mathscr{S}$,*

(iii) *$m(A \cup B) \leqq m(A) + m(B)$ for all $A, B \in \mathscr{S}$,*

(iv) *if $\{E_n\}_{n=1}^{\infty}$ is a monotone decreasing sequence of elements of $\mathscr{S}$ such that $m(E_1) < +\infty$ and $\bigcap_{n=1}^{\infty} E_n = \emptyset$, then $\bigwedge_{n=1}^{\infty} m(E_n) = \Theta$,*

(v) *for every $E \in \mathscr{S}$ there exist $E_n \in \mathscr{S}$, $m(E_n) < +\infty$, $n = 1, 2, \ldots$ such that $E = \bigcup_{n=1}^{\infty} E_n$ and $m(E) = \bigvee_{n=1}^{\infty} m\left(\bigcup_{k=1}^{n} E_k \right)$.*

Theorem 3.1. *Let X be a non-empty set and let $\mathscr{S}$ be a σ-ring of subsets of X. Let $C(S)$ be a Stone algebra such that each meagre subset of S is nowhere dense.*

Let $m: \mathscr{S} \to C(S) \cup \{+\infty\}$ *be a* σ*-finite* $C(S)$*-valued submeasure. Let us assume the continuum hypothesis. Let* $\{A_x\}_{x\in E}$ *be a family of parwise disjoint sets in* $\mathscr{S}$ *such that* $\operatorname{card}(E) = c$ *and* $\bigcup_{x\in F} A_x \in \mathscr{S}$ *for all* $F \subset E$. *Then*

$$m\left(\bigcup_{x\in E} A_x\right) = \bigvee_{\substack{I\subset E\\ I \text{ is finite}}} m\left(\bigcup_{x\in I} A_x\right).$$

Proof. This is a trivial consequence of theorem 2.1 if $m\left(\bigcup_{x\in E} A_x\right) < +\infty$. If $m\left(\bigcup_{x\in E} A_x\right) = +\infty$, then there exist the sets $E_n \in \mathscr{S}$, $m(E_n) < +\infty$, $n = 1, 2, \ldots$, such that

$$\bigcup_{x\in E} A_x = \bigcup_{n=1}^{\infty} E_n \quad \text{and} \quad m\left(\bigcup_{x\in E} A_x\right) = \bigvee_{n=1}^{\infty} m\left(\bigcup_{k=1}^{n} E_k\right).$$

Denote $B_n = \bigcup_{k=1}^{n} E_k$, $n = 1, 2, \ldots$. Then for each positive integer n there is $m(B_n) < +\infty$ and by Theorem 2.1 we have

$$m(B_n) = m\left(\bigcup_{x\in E} (A_x \cap B_n)\right) = \bigvee_{\substack{I\subset E\\ I \text{ is finite}}} m\left(\bigcup_{x\in I} (A_x \cap B_n)\right) \leqq \bigvee_{\substack{I\subset E\\ I \text{ is finite}}} m\left(\bigcup_{x\in I} A_x\right).$$

Hence

$$m\left(\bigcup_{x\in E} A_x\right) = \bigvee_{n=1}^{\infty} m(B_n) \leqq \bigvee_{\substack{I\subset E\\ I \text{ is finite}}} m\left(\bigcup_{x\in I} A_x\right).$$

Evidently also

$$\bigvee_{\substack{I\subset E\\ I \text{ is finite}}} m\left(\bigcup_{x\in I} A_x\right) \leqq m\left(\bigcup_{x\in E} A_x\right).$$

Theorem 3.2. *Let* X *be a non-empty set and let* $\mathscr{S}$ *be a* σ*-ring of subsets of* X. *Let* $C(S)$ *be a Stone algebra such that each meagre subset of* S *is nowhere dense. Let* $m: \mathscr{S} \to C(S) \cup \{+\infty\}$ *be a* σ*-finite* $C(S)$*-valued submeasure. Let us assume the continuum hypothesis and let* $\operatorname{card}(X) = c$. *Let* $m(\{x\}) = \Theta$ *for all* $x \in X$. *If there exists a set* $E \in \mathscr{S}$ *such that* $m(E) > \Theta$, *then there exits* $F \subset X$ *such that* $F \notin \mathscr{S}$.

Proof. A trivial consequence of Theorem 3.1 is that if $E \in \mathscr{S}$ for all $E \subset X$, then $m(E) = \Theta$ for all $E \in \mathscr{S}$. This proves the theorem.

4. Examples and remarks

We can obtain two special cases of a $C(S)$-valued submeasure when the properties (ii) and (iii) are replaced by one of the following stronger properties:

(1) $m(A \cup B) = m(A) + m(B)$ for all disjoint sets $A, B \in \mathscr{S}$ or

(2) $m(A \cup B) = m(A) \vee m(B)$ for all sets $A, B \in \mathscr{S}$.

In this way we obtain in the case (1) a *σ-additive $C(S)$-valued measure*, which is defined to be a map $m: \mathscr{S} \to C(S) \cup \{+\infty\}$ and such that $m(E) \geqq \Theta$ for each $E \in \mathscr{S}$, $m(\emptyset) = \Theta$ and $m\left(\bigcup_{n=1}^{\infty} E_n\right) = \bigvee_{n=1}^{\infty} \sum_{k=1}^{n} m(E_k)$ whenever $\{E_n\}_{n=1}^{\infty}$ is a sequence of pairwise disjoint elements of $\mathscr{S}$ (see [4]—[11]). In the case (2) we obtain the *continuous from above σ-maxitive* $C(S)$-valued measure which is defined as a map $m: \mathscr{S} \to C(S) \cup \{+\infty\}$ such that $m(E) \geqq \Theta$ for each $E \in \mathscr{S}$, $m(\emptyset) = \Theta$, $m\left(\bigcup_{n=1}^{\infty} A_n\right) = \bigvee_{n=1}^{\infty} m(A_n)$ whenever $\{A_n\}_{n=1}^{\infty}$ is a sequence of elements of $\mathscr{S}$, and m has the property (iv). (See [12], [13].)

A positive linear functional ψ on $C(S)$ is normal if, whenever $a_n \downarrow \Theta$, we have $\psi(a_n) \downarrow 0$. S is *Hyperstonean* if the normal positive linear functionals on $C(S)$ separate points. See [11]. If S is Hyperstonean, then $C(S)$ is weakly (σ, ∞)-distributive and $C(S)$ is weakly (σ, ∞)-distributive iff every meagre subset of S is nowhere dense. See [10], p. 281. J. Dixmier [14] gives an example of a Stone algebra $C(S)$ such that each meagre subset of S is nowhere dense but $C(S)$ has not a separating family of normal functionals and hence S is not Hyperstonean. Kelley [15] gives another examples.

A vector lattice V is boundedly complete if each subset of V, which is bounded, has a least upper bound. For each positive element e of V let

$$V[e] = \{b \in V \mid -re \leqq b \leqq re \text{ for some positive } r \in R\}.$$

By the fundamental Stone—Krein—Kakutani—Yosida vector lattice representation theorem (see Theorem 4.1 of Kadison [16]), there exists a compact Hausdorff space S such that $V[e]$ is isometrically and vector lattice isomorfic to $C(S)$. Since V is boundedly complete then also is $V[e]$ and thus $C(S)$ is a Stone algebra. Further, V is weakly (σ, ∞)-distributive iff $V[e]$ is weakly (σ, ∞)-distributive for each $e > 0$ and this implies that $C(S)$ is such that every meagre subset of S is nowhere dense ([10], p. 279 and 281).

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О СЛЕДСТВИЯХ ТЕОРЕМЫ БАНАХА–КУРАТОВСКОГО ДЛЯ МЕР ПРИНИМАЮЩИХ ЗНАЧЕНИЯ В АЛГЕБРЕ СТОУНА

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Резюме

Доказывается теорема о несуществовании нетривиальной меры, заданной на всех подмножествах множества мощности континуума и принимающей значения в алгебре Стоуна, равные нулю во всех одноэлементных подмножествах.