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# DARBOUX PROPERTY OF MEASURES AND CONTENTS

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Let us call a non-negative and finitely additive set function $v$ on an algebra $\mathfrak{A}$ of subsets of a set $S$ with a $v(\emptyset)=0$ content. Furthermore a content $v$ is said to be continuous if for any $\varepsilon>0$ there exists a partition $A_1, \ldots, A_n$ of the underlying set $S$ such that $v(A_i)<\varepsilon$, $i=1, \ldots, n$ holds. Finally a content $v$ has the Darboux property if for any $A \in \mathfrak{A}$ and for all $a \in [0, v(A)]$ there is a $B \in \mathfrak{A}$ with $v(B)=a$ and $B \subset A$.

If furthermore $\mathfrak{A}'$ is an algebra of subsets of $S$ with $\mathfrak{A} \subset \mathfrak{A}'$, and $v$ is finite, i.e. $v(S)<\infty$, the family of finite contents $v': \mathfrak{A}' \to \mathbf{R}$ with $v'|\mathfrak{A}=v$ will be denoted by $\mathscr{C}(\mathfrak{A}, v, \mathfrak{A}')$. Now the following auxiliary result can be proved:

**Lemma.** *$v$ is continuous iff every $v' \in \mathscr{C}(\mathfrak{A}, v, \mathfrak{A}')$ is continuous.*

Proof. According to [7] there exists an extreme point $v'$ of $\mathscr{C}(\mathfrak{A}, v, \mathfrak{A}')$. Furthermore an extreme point $v'$ of $\mathscr{C}(\mathfrak{A}, vm\ \mathfrak{A}')$ has the following property (see [7], theorem 1): For any $\varepsilon>0$ and $A \in \mathfrak{A}'$ there exists a $B \in \mathfrak{A}$ with $v'(A \triangle B)<\varepsilon$. If in addition $v'$ is continuous, this property implies that $v=v'|\mathfrak{A}$ is also continuous, which can be seen as follows (see [8], lemma 3.1): Let $\varepsilon>0$ and let $n$ be a natural number such that $\frac{1}{n}<\varepsilon$. There exists a partition $A_1, \ldots, A_n$ of $S$ with

$$0<v'(A_i)<\varepsilon \cdot \frac{nv'(S)}{1+nv'(S)}, \qquad i=1, \ldots, n.$$

Furthermore there exists $B_i \in \mathfrak{A}$ with

$$v'(A_i \triangle B_i)<\frac{v'(A_i)}{nv'(S)}, \qquad i=1, \ldots, n,$$

which implies

$$v(B_i) \leqq v'(A_i)\left(1+\frac{1}{nv'(S)}\right)<\varepsilon, \qquad i=1, \ldots, n.$$

Defining $C_1=B_1$, $C_i=B_i \backslash (B_1 \cup \ldots \cup B_{i-1})$, $i=2, \ldots, n+1$, where $B_{n+1} = S \backslash (B_1 \cup \ldots \cup B_n)$ yields a partition $C_1, \ldots, C_{n+1}$ of $S$ with $v(C_i)<\varepsilon$, $i=1, \ldots, n+1$, because

$$v'(B_{n+1})=v'\left(\bigcup_{i=1}^{n}A_i\setminus\bigcup_{i=1}^{n}B_i\right)\leqq v'\left(\bigcup_{i=1}^{n}(A_i\setminus B_i)\right)\leqq$$

$$\leqq\sum_{i=1}^{n}v'(A_i\setminus B_i)\leqq\sum_{i=1}^{n}v'(A_i\triangle B_i)<\frac{1}{n}<\varepsilon.$$

This implies the continuity of $v$.

This lemma remains true if the property of a content to be continuous is replaced by the property to be atomless, as can be seen in the following way: If $v$ is atomless and $A_0\in\mathfrak{A}'$ is a $v'$-Atom for some $v'\in\mathscr{C}(\mathfrak{A}, v, \mathfrak{A}')$, then $v'_{A_0}|\mathfrak{A}\leqq v$ holds with $v'_{A_0}$ as the concentration of $v'$ at $A_0$, where $v'_{A_0}|\mathfrak{A}$ is two valued content. According to the decomposition of Hammer—Sobczyk (see [8]) we have $v'_{A_0}=0$, which is a contradiction. If now $A_0\in\mathfrak{A}$ is a $v$-Atom, $v_{A_0}$ is two valued and therefore $v'_{A_0}$ too for every extreme point $v'$ of $\mathscr{C}(\mathfrak{A}, v, \mathfrak{A}')$ because of [7], theorem 1. Hence $A_0$ is a $v'$-Atom for every extreme point of $\mathscr{C}(\mathfrak{A}, v, \mathfrak{A}')$.

According to [3], theorem 2, a content $v$ defined on a $\sigma$-algebra $\mathfrak{A}$ of subsets of a set $S$ is continuous iff $v$ has the Darboux property. Therefore the lemma above implies

**Theorem.** *Let $\mathfrak{A}$ and $\mathfrak{A}'$ be $\sigma$-algebras with $\mathfrak{A}\subset\mathfrak{A}'$ and $v$ a finite content on $\mathfrak{A}$. Then $v$ has the Darboux property iff every $v'\in\mathscr{C}(\mathfrak{A}, v, \mathfrak{A}')$ has the Darboux property.*

This theorem is not true in general if the assumption that $\mathfrak{A}$ and $\mathfrak{A}'$ are $\sigma$-algebras is replaced by assuming that $\mathfrak{A}$ and $\mathfrak{A}'$ are algebras, as the following special case shows: Choose $S=[0, 1]$ and $\mathfrak{A}$ as the field generated by $\{[a, b]|0\leqq a\leqq b\leqq 1$, $a$ and $b$ rationals$\}$ and $\mu$ as the Lebesgue measure restricted to $\mathfrak{A}$. If furthermore $\mathfrak{A}'$ is the Borel $\sigma$-algebra of $S$, then every $\mu'\in\mathscr{C}(\mathfrak{A}, \mu, \mathfrak{A}')$ is continuous and has therefore the Darboux property (see [3], theorem 2), whereas $\mu$ has not this property.

In the following $\mathfrak{A}$ and $\mathfrak{A}'$ are defined to be $\sigma$-algebras of subsets of a set $S$. If furthermore $\mu$ is a finite measure on $\mathfrak{A}$ and $\mathscr{M}(\mathfrak{A}, \mu, \mathfrak{A}')$ denotes the family of all finite measures $\mu'$ on $\mathfrak{A}'$ with $\mu'|\mathfrak{A}=\mu$, theorem 1 is not true in general if $\mathscr{C}(\mathfrak{A}, v, \mathfrak{A}')$ is replaced by $\mathscr{M}(\mathfrak{A}, \mu, \mathfrak{A}')$. This shows the following

Example 1: Let $\mathfrak{A}=\{A\subset\mathbf{R}|A$ or $\mathbf{R}\setminus A$ is countable$\}$ be the $\sigma$-algebra of subsets of the set $\mathbf{R}$ of real numbers generated by the singletons and $\mu$ the measure defined by $\mu(A)=0$ if $A$ is countable and $\mu(A)=1$ if $\mathbf{R}\setminus A$ is countable. If furthermore $\mathfrak{A}'$ is the $\sigma$-algebra of Borel subsets of $\mathbf{R}$ the family $\mathscr{M}(\mathfrak{A}, \mu, \mathfrak{A}')$ consists of all atomless probability measures on $\mathfrak{A}'$. Hence in this case every $\mu'\in\mathscr{M}(\mathfrak{A}, \mu, \mathfrak{A}')$ has the Darboux property, but not $\mu$. Furthermore $\mathscr{M}(\mathfrak{A}, \mu, \mathfrak{A}')$ in this example has no extreme points (see [7]).

If, however, one assumes that the set of extreme points of $\mathscr{M}(\mathfrak{A}, \mu, \mathfrak{A}')$ is not empty the method of proof for the theorem above yields the

**Corollary.** *Let $\mathfrak{A}$ and $\mathfrak{A}'$ be $\sigma$-algebras with $\mathfrak{A} \subset \mathfrak{A}'$ and $\mu$ a finite measure on $\mathfrak{A}$. If $\mathcal{M}(\mathfrak{A}, \mu, \mathfrak{A}')$ has at least one extreme point, then $\mu$ has the Darboux property iff every $\mu' \in \mathcal{M}(\mathfrak{A}, \mu, \mathfrak{A}')$ has the Darboux property.*

Here are some examples, where the set of extreme points of $\mathcal{M}(\mathfrak{A}, \mu, \mathfrak{A}')$ is not empty:

Example 2. Let $\mathfrak{A}$ be a $\sigma$-algebra of subsets of an arbitrary set $S$ and $\mathfrak{A}'$ the $\sigma$-algebra generated by $\mathfrak{A} \cup \{A\}$, where $A$ is a fixed subset of $S$. Then $\mathcal{M}(\mathfrak{A}, \mu, \mathfrak{A}')$ has extreme points, where $\mu$ is a finite measure on $\mathfrak{A}$ (see [7]).

Example 3. Let $\mathfrak{A}$ be a $\sigma$-algebra of subsets of an arbitrary set $S$ and $\mu$ a finite measure on $\mathfrak{A}$. If $\mathfrak{A}'$ is the $\sigma$-algebra generated by $\mathfrak{A}$ and a family of internal negligible sets closed under countable unions, then the proof of theorem 31 in [5] shows that there exists an extension $\mu' \in \mathcal{M}(\mathfrak{A}, \mu, \mathfrak{A}')$ of $\mu$ with the property: For $A \in \mathfrak{A}'$ there is a $B \in \mathfrak{A}$ with $\mu'(A \triangle B) = 0$. Hence $\mu'$ is an extreme point of $\mathcal{M}(\mathfrak{A}, \mu, \mathfrak{A}')$ (see [7]).

Example 4. Let $S$ be a locally compact space. If in addition it is assumed that $S$ is $\sigma$-compact, then the Baire, resp. Borel, $\sigma$-algebra coincides with the Baire, resp. Borel $\sigma$-ring. It is well known (see [1]) that every Baire measure $\mu$ can be extended uniquely to a regular Borel measure $\mu'$, from which it follows that $\mu'$ is an extreme point of $\mathcal{M}(\mathfrak{A}, \mu, \mathfrak{A}')$. For a generalization to completely regular spaces $S$ compare [4].

Example 5. Let $S$ be a compact space and $\mathfrak{A}'$ the baire $\sigma$-algebra, resp. $\mathfrak{A} = f^{-1}(\mathfrak{A}')$, where $f: S \to S$ is assumed to be continuous, which implies $\mathfrak{A} \subset \mathfrak{A}'$; $\mu$ is defined to be a finite measure on $\mathfrak{A}$. If furthermore $C(S)$ denotes the family of all real valued, continuous functions, then $\mathcal{M}(\mathfrak{A}, \mu, \mathfrak{A}')$ is compact with respect to the weak* topology of the dual space $C^*(S)$ of $C(S)$, which can be seen as follows: A net $\mu'_\alpha \in \mathcal{M}(\mathfrak{A}, \mu, \mathfrak{A}')$ converges to $\mu'$ with respect to the weak* topology iff $\int g \, d\mu'_\alpha \to \int g \, d\mu'$ holds for every $g \in C(S)$. This implies $\int g \circ f \, d\mu'_\alpha \to \int g \circ f \, d\mu'$, $g \in C(S)$, from which $\mu' \in \mathcal{M}(\mathfrak{A}, \mu, \mathfrak{A}')$ follows, since $\mu'_\alpha \in \mathcal{M}(\mathfrak{A}, \mu, \mathfrak{A}')$ implies $\int g \circ f \, d\mu'_\alpha = \int g \circ f \, d\mu = \int g \, d\mu^f$ and $C(S)$ is dense in the space $L_1(S, \mathfrak{A}', \mu^f)$ of $\mu^f$-integrable functions with respect to the $L_1$-norm (see [1]). The theorem of Krein—Milman now implies that there exist extreme points of $\mathcal{M}(\mathfrak{A}, \mu, \mathfrak{A}')$ if $\mathcal{M}(\mathfrak{A}, \mu, \mathfrak{A}')$ is non empty. This is true if in addition it is assumed that $S$ is metrizable (see [2]).

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