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BASES OF CLASSES OF TREES CLOSED UNDER SUBDIVISIONS

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At the Fifth Hungarian Colloquium on Combinatorics held in Keszthely in 1976 L. Lovász has proposed the following problem [1]:

Given two finite graphs G, H, let $G < H$ mean that H contains a subdivision of G. Let us call a class $\mathcal{K}$ of finite graphs closed if $G \in \mathcal{K}$, $G < H$ implies $H \in \mathcal{K}$. (For example, the class of non-planar graphs is closed.) The (well-defined) minimal members of a closed class form its basis (e.g. K_5 and $K_{3,3}$ for the class of non-planar graphs).

Conjecture. *If two closed classes have a finite basis, so does their intersection.*

We shall not solve here this problem in general, but we restrict ourselves to trees. A closed class of trees is defined analogously as a closed class of graphs.

Let $\mathcal{T}_1$, $\mathcal{T}_2$ be two closed classes of trees, let $\mathcal{B}_1$, $\mathcal{B}_2$ be their bases, respectively. Denote $\mathcal{T} = \mathcal{T}_1 \cap \mathcal{T}_2$. A tree T belongs to $\mathcal{T}$ if and only if it contains simultaneously a subdivision of a tree from $\mathcal{B}_1$ and a subdivision of a tree from $\mathcal{B}_2$.

Suppose that $\mathcal{B}_1$ and $\mathcal{B}_2$ are finite. Then $\mathcal{B}_1 = \{X_1, \ldots, X_m\}$ $\mathcal{B}_2 = \{Y_1, \ldots, Y_n\}$. By $\mathcal{C}_{ij}$ we denote the class of all trees which contain simultaneously a subdivision of X_i and a subdivision of Y_j; $i = 1, \ldots, m$; $j = 1, \ldots, n$. Evidently $\mathcal{T} = \bigcup_{i=1}^{m} \bigcup_{j=1}^{n} \mathcal{C}_{ij}$. Let $\mathcal{D}_{ij}$ be the basis of $\mathcal{C}_{ij}$ for any i and j; then each tree from $\mathcal{T}$ contains a subdivision of a tree from some $\mathcal{D}_{ij}$ and every tree dontaining a subdivision of a tree from some $\mathcal{D}_{ij}$ belongs to $\mathcal{T}$. The basis of $\mathcal{T}$ is then $\mathcal{B} = \bigcup_{i=1}^{m} \bigcup_{j=1}^{n} \mathcal{D}_{ij}$. If we prove that each $\mathcal{D}_{ij}$ is finite, then also $\mathcal{B}$ is finite.

Before proving this, we shall define some auxiliary concepts.

To each finite tree T we define the reduced tree $R(T)$ so that the vertex set of $R(T)$ is the set of all vertices of T whose degrees are different from 2 and two vertices u and v of $R(T)$ are joined by an edge in $R(T)$ if and only if they are connected in T by a path, all of whose inner vertices (if any) have the degree 2. Evidently T is a subdivision of $R(T)$ and $R(T)$ for each T is determined uniquely.

The vertices of T whose degrees are different from 2 will be called node vertices of T.

Now let U be a subset of the vertex set V of a tree T. The set U is called linearly independent if any three pairwise distinct elements u, v, w of U have the property that none of them belongs to the path connecting the other two in T. (If U has less than 3 elements, it is always linearly independent.) It is easy to prove that the maximal number of elements of a linearly independent set of vertices of T is equal to the number of terminal vertices of T. If a subset of V is not linearly independent, it is called linearly dependent.

Now let T_1 and T_2 be two finite trees. Let T be a tree which contains simultaneously a subdivision T_1' of T_1 and a subdivision T_2' of T_2. Let T_0' be the intersection of T_1' and T_2'; it is either an empty graph, or a tree. (The graph consisting of one vertex and the graph consisting of one edge with its end vertices is also considered a tree.) By T_3' we denote the least subtree of T containing both T_1' and T_2'. If T_0' is non-empty, then T_3' is the union of T_1' and T_2'. If T_0' is empty, then T_3' consists of the trees T_1' and T_2' and of the path connecting the vertex of T_1' which is the nearest to T_2' with the vertex of T_2' which is the nearest to T_1'.

First consider the case when T_0' is non-empty. A vertex u of T_1' which is not a node vertex of this tree is a node vertex of T_3' if either u is a node vertex of T_2', or u is not a node vertex of T_2', but u belongs to T_2' and is incident with an edge of T_1' not belonging to T_2' and with an edge of T_2' not belonging to T_1'. Analogously for vertices of T_2'. Let K_1 (or K_2) be the set of all vertices of T_1' (or T_2') which are not node vertices of T_1' (or T_2'), but are node vertices of T_2' (or T_1' respectively). Let L be the set of vertices which belong to both T_1' and T_2', are node vertices in none of them and are adjacent with an edge of T_1' not belonging to T_2' and with an edge of T_2' not belonging to T_1'.

The number of vertices of K_1 is less than or equal to the number of node vertices of T_2'; this number is equal to the number n_2 of node vertices of T_2, because at the forming of subdivisions no new node vertex arises. Analogously the number of elements of K_2 is less than or equal to the number n_1 of node vertices of T_1.

Consider the set L. Suppose that it is linearly dependent in T_3'. This means that there exist three pairwise distinct vertices u, v, w of L such that v is an inner vertex of the path connecting u and w in T_3'. But u and w belong simultaneously to T_1' and T_2' therefore this path is contained in both T_1' and T_2'. Let h_1, h_2 be the edges of this path incident with v. As v is not a node vertex of T_1', it cannot be incident with an edge of T_1' other than h_1 and h_2 and therefore it is incident with no edge of T_1' not belonging to T_2' (the edges h_1, h_2 belong to T_2'). This is a contradiction with the assumption that $v \in L$. We have proved that L is linearly independent in T_3'.

As L is linearly independent in T_3', it is linearly independent in anyone of its subtrees, therefore also in T_1' and T_2'. Its number of vertices is less than or equal to the minimum q of the numbers of terminal vertices of T_1' and T_2'. This number q is also the minimum of the numbers of terminal vertices of T_1 and T_2, because at the forming of subdivisions no new terminal vertex arises.

Take the reduced tree $R(T_3')$. Let T_1'', T_2'', T_3'' be the subtrees of $R(T_3')$ corresponding to the trees T_1', T_2', T_3' respectively. The tree T_1'' (or T_2'') is a subdivision of $R(T_1)$ (or $R(T_2)$) obtained by inserting at most n_2+q (or n_1+q respectively) vertices onto its edges. Such subdivisions are finitely many. The tree T_3'' can be constructed from T_1'' and T_2'' so that we choose isomorphic subtrees of T_1'' and T_2'' and an isomorphism which maps one of them onto the other and we identify pairs of vertices corresponding to each other in this isomorphism. This can be done in finitely many ways, therefore to the given trees T_1 and T_2 we can construct only finitely many trees $R(T_3')$. A subdivision of some of these trees $R(T_3')$ is contained in each tree from the class of all trees which contain non-disjoint subdivisions of T_1 and T_2 as their subtrees.

It remains to consider the case when the intersection T_0' of T_1' and T_2' is empty. Then evidently $R(T_3')$ consists of the disjoint copies of $R(T_1)$ and $R(T_2)$ and an edge joining a vertex of $R(T_1)$ with a vertex of $R(T_2)$; this edge is formed from the path joining these vertices at the transition to the reduced tree. The number of such trees $R(T_3')$ is again finite.

If $T_1 = X_i$, $T_2 = Y_j$, then we have the result that $\mathcal{D}_{ij}$ is finite and, as mentioned above, also $\mathcal{B}$ is finite. We have a theorem.

Theorem. *If two classes of trees closed under subdivisions have a finite basis, so has their intersection.*

REFERENCE

[1] Proceedings of the Fifth Hungarian Colloquium on Combinatorics held in Keszthely in 1976. Akadémiai Kiadó, Budapest (to appear).

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БАЗИСЫ КЛАССОВ ДЕРЕВЬЕВ ЗАМКНУТЫХ ОТНОСИТЕЛЬНО ПОДРАЗДЕЛЕНИЙ

Богдан Зелинка

Резюме

Класс $\mathcal{K}$ деревьев называется замкнутым, если он с каждым деревом T содержит все деревья, которые содержат подразделение дерева T. Минимальное семейство $\mathcal{B}$ деревьев обладающее тем свойством, что всякое дерево из $\mathcal{K}$ содержит подразделение некоторого дерева из $\mathcal{B}$, называется базисом класса $\mathcal{K}$. Доказано, что если два замкнутых класса $\mathcal{T}_1$, $\mathcal{T}_2$ деревьев имеют конечные базисы, то $\mathcal{T}_1 \cap \mathcal{T}_2$ тоже имеет конечный базис. Это является частичным решением проблемы Л. Ловаса.