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A REMARK ON ALMOST CONTINUOUS MULTIFUNCTIONS

LUBICA HOLÁ

The term "almost continuity" is used here in the sense of Husain. The notion of almost continuity of a function was studied by Blumberg, Banach, Pták and by several other authors ([1], [6], [10]). We investigate "upper almost continuity" of multifunctions. In this paper we give a characterization of upper almost continuity. We show that under some assumptions on spaces for each compact-valued multifunction \( F \) there is a dense set \( A \) in domain, such that \( F/A \) is upper semicontinuous.

We introduce some definitions which we shall use. By a multifunction \( F \) of \( X \) to \( Y \) (\( F: X \to Y \)) we mean a function which to every point \( x \in X \) assigns a nonempty subset \( F(x) \) of \( Y \). For any \( A \subset Y \) we denote \( F^{-}(A) = \{ x \in X : F(x) \cap A \neq \emptyset \} \) and \( F^{+}(A) = \{ x \in X : F(x) \subset A \} \).

All topological spaces considered in this paper are supposed to be Hausdorff. For a subset \( A \) of a topological space \( X \), \( \overline{A} \) and \( \text{Int} \, A \) denote the closure or the interior of \( A \) respectively.

A multifunction \( F: X \to Y \) is called upper (lower) semicontinuous at a point \( x \) if for any open set \( V \subset Y \) such that \( x \in F^{-}(V)(x \in F^{+}(V)) \) there exists a neighbourhood \( U \) of \( x \) such that \( U \subset F^{-}(V)(U \subset F^{+}(V)) \).

A multifunction \( F: X \to Y \) is upper (lower) almost continuous at a point \( x \in X \) if for every open set \( V \subset Y \), \( x \in F^{+}(V)(\overline{F^{-}(V)}) \) implies \( x \in \text{Int} \overline{F^{+}(V)}(x \in \text{Int} F^{-}(V)) \).

By a graph of a multifunction \( F: X \to Y \) we mean the set \( \text{Gr} \, F = \{(x, y) : x \in X, y \in F(x)\} \).

If a single-valued function \( f: X \to Y \) is given, then it is considered as a multifunction which associates \( \{f(x)\} \) to any \( x \in X \). Thus \( f \) is upper (lower) almost continuous exactly if it is almost continuous in the sense as introduced in [1].

A subset \( A \) of a topological space \( X \) is called almost open (or nearly open [8]) if \( A \subset \text{Int} \, A \) and almost closed if \( X \setminus A \) is almost open. If for some \( x \in X \) and an almost open set \( A \subset X \) we have \( x \in A \), we say that \( A \) is an almost-neighbourhood of \( x \).

Remark 1. The following properties of almost open sets are evident:
(a) A set $A \subset X$ is almost open if and only if there is an open set $U$ such that $A \subset U$ and $A$ is dense in $U$.

(b) The intersection of an open set and an almost open set is almost open.

(c) If $A \subset X$ is almost open in $X$ and $B \subset A$ is almost open in $A$ (with the induced topology), then $B$ is almost open in $X$.

(d) The union of almost open sets is almost open.

The following remark is a trivial exercise. We will frequently use it without a specific reference.

Remark 2. The following conditions on a multifunction $F: X \to Y$ are equivalent:

(a) $F$ is upper (lower) almost continuous at $x \in X$;

(b) for any open set $V \subset Y$ such that $x \in F^+(V)(x \in F^-(V))$ there exists an almost neighbourhood $G$ of $x$ such that $G \subset F^+(V)(G \subset F^-(V))$;

(c) for any open set $V$ such that $x \in F^+(V)(x \in F^-(V))$ there exists an open neighbourhood $U$ of $x$ such that $F^+(V)(F^-(V))$ is dense in $U$.

The proofs of the following two propositions are based only on the topological properties of the domain of multifunctions. We give proofs only for single-valued functions since their generalization for multifunctions is evident.

Proposition 1. Let $f: X \to Y$ be a function. Let $A$ be an almost open set and $f/A$ be almost continuous. Then $f$ is almost continuous at every $x \in A$.

Proof. The proof is clear from Remark 1 (c).

For $A$ dense in $X$ and $f$ such that $f/A$ is continuous, Proposition 1 is proved in [1].

If $f: X \to Y$ is almost continuous and $A$ is an almost open set, then $f/A$ need not be almost continuous. (See Example 3 in [1]) But the following proposition is true.

Proposition 2. Let $f: X \to Y$ be a function. Let $M = G \setminus R$, where $G$ is a nonempty open set in $X$ and $R$ is a nowhere dense set in $X$. Then $f$ is almost continuous at $x \in M$ if and only if $f/M$ is almost continuous at $x$.

Proof. Let $f/M$ be almost continuous at $x \in M$. Since $R$ is nowhere dense in $X$, $G \setminus R$ is almost open. By Proposition 1 $f$ is almost continuous at $x$.

Now let $f$ be almost continuous at $x \in M$. Let $V$ be an open set in $Y$ such that $f(x) \in V$. There is an open set $U$ in $X$ such that $x \in U$ and $f^{-1}(V)$ is dense in $U$. Put $H = U \cap M$. $H$ is open in $M$. We show that $(f/M)^{-1}(V)$ is dense in $H$. Let $H_1$ be a nonempty open set in $M$ such that $H_1 \subset H$. Then $H_1 = V_i \cap M$ for some open set $V_i$ in $X$. $V_i \cap U \cap G$ is a nonempty open set in $X$. Since $R$ is nowhere dense in $X$ there exists a nonempty open set $G_i$ in $X$ such that $G_i \subset V_i \cap U \cap G$ and $G_i \subset X \setminus R$. The density $f^{-1}(V)$ in $U$ implies that $f^{-1}(V) \cap G_i \neq \emptyset$, i.e. $f^{-1}(V) \cap H_1 \neq \emptyset$.

Remark 3. Let $F: X \to Y$ be a multifunction. Denote the set of points of upper (lower) almost continuity by $A_t(F)(A_l(F))$. In the paper [2] it is proved

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that if \( Y \) is a second countable space, then for any multifunction \( F: X \to Y \), 
\( \mathcal{AL}(F) \) is a complement of a set of the first category. If \( F \) is a compact-valued multifunction of \( X \) to a second countable space \( Y \), then the same is true for the set \( \mathcal{AL}(F) \). Thus if \( X \) is a second category space and \( Y \) a second countable space, the sets \( \mathcal{AL}(F), \mathcal{AV}(F) \) are nonempty and in spaces in which any set of the first category is nowhere dense the restrictions \( F/\mathcal{AL}(F) \) and \( F/\mathcal{AV}(F) \) are lower or upper almost continuous respectively. (See Proposition 2) But in general the restriction \( F/\mathcal{AL}(F) (F/\mathcal{AV}(F)) \) need not be lower (upper) almost continuous.

**Example 1.** Let \( X \) be the unit interval with the usual topology and \( Y \) be the set of real numbers with the usual topology. Let \( \{x_n\} \) be a sequence of different real numbers convergent to 2. For any \( n \in \mathbb{N} \) let \( A_n \) be the set of rational numbers in the open interval \((1/(n + 1), 1/n)\) and \( f_n \) be a bijection from \( A_n \) onto the set \( \{x_m: m \geq n\} \). Define the function \( f \) as follows: \( f(0) = 2, f(x) = f_n(x) \) for \( x \in A_n \) and \( f(x) = x \) otherwise. It is easy to verify that \( \mathcal{AL}(f) = X \setminus \bigcup_{n=1}^{\infty} A_n \) and \( f/\mathcal{AL}(f) \) is not almost continuous at 0.

**Proposition 3.** Let \( X \) be a Baire space and \( Y \) be a second countable space. Let \( F: X \to Y \) be a multifunction. There is a dense set \( D \) in \( \mathcal{AL}(F) \) such that \( F/D \) is lower almost continuous. If \( F \) is a compact-valued multifunction, then there exists a dense set \( T \) in \( \mathcal{AV}(F) \) such that \( F/T \) is upper almost continuous.

Proposition 3 is stated here for reference. The case of lower almost continuity is proved in [10] and the proof of upper almost continuity is similar.

The following theorem gives a characterization of upper almost continuity.

**Theorem 1.** Let \( X, Y \) be topological spaces, \( F: X \to Y, x \in X \). Let there exist a countable base of neighbourhoods of \( F(x) \) and a countable family of closed neighbourhoods of \( x \) the intersection of which is the set \( \{x\} \). Then \( F \) is upper almost continuous at \( x \) if and only if there exists an almost neighbourhood \( A \) of \( x \) such that \( F/A \) is upper semicontinuous at \( x \).

**Proof.** Let \( A \) be an almost-neighbourhood of \( x \) such that \( F/A \) is upper semicontinuous at \( x \). By Proposition 1, \( F \) is upper almost continuous at \( x \).

Now let \( F \) be upper almost continuous at \( x \). If \( \{x\} \) is open, then the theorem is proved. Suppose \( \{x\} \) is not open. Let \( \{G_n\} \) be a non-increasing base of open neighbourhoods of \( F(x) \) and \( \{V_n\} \) be a sequence of closed neighbourhoods of \( x \) such that \( \bigcap_{n=1}^{\infty} V_n = \{x\} \).

\( \{F^+(G_n)\} \) is a non-increasing sequence such that \( x \in F^+(G_n) \) and \( F^+(G_n) \) is a neighbourhood of \( x \) for \( n = 1, 2, \ldots \). There exist open neighbourhoods \( U_1, H_1 \) of \( x \) such that \( U_1 \subset F^+(G_1) \cap V_1, H_1 \subset U_1 \) and \( U_1 \setminus H_1 \neq 0 \). By induction we can construct sequences \( \{U_n\}, \{H_n\} \) of open neighbourhoods of \( x \) such that for any \( n \in \mathbb{N} \) \( H_n \subset U_n, U_n \subset F^+(G_n) \cap V_n, U_{n+1} \subset H_n \), and \( U_n \setminus H_n \neq 0 \).
Put $A = \bigcup_{n=1}^{\infty} F^+(G_n) \cap (U_n \setminus \bar{U}_{n+1}) \cup \{x\}$. Then $A$ is the searched set.

Notice that if $F$ in Theorem 1 is upper almost continuous at every $x \in X$, then $F/A$ is upper almost continuous.

Let $z \in A \setminus \{x\}$ and $U$ be an open set in $Y$ such that $z \leq F^+(U)$. There is $n \in N$ such that $z \leq F^+(G_n) \cap (U_n \setminus \bar{U}_{n+1})$. The upper almost continuity of $F$ at $z$ implies that there is an almost-neighbourhood $H$ of $z$ such that $H \subset F^+(G_n \cap U)$, i.e. $H \cap (U_n \setminus \bar{U}_{n+1})$ is a subset of $A$. By Remark 1 the set $H \cap (U_n \setminus \bar{U}_{n+1})$ is almost open in $X$ and thus in $A$.

For a single-valued function and $X$, $Y$ metric spaces, Theorem 1 is proved in [8].

The following examples show that the assumptions in Theorem 1 are essential.

Example 2. Let $X$ be the set of all ordinal numbers less than or equal to $\omega_1$ with the topology $\{\{x \in X \colon x > \gamma\} \cup \{x, 0\} \cup \{\{x \in X \colon x \neq \omega_1, x > \gamma\}\colon \gamma \in X\}$ and $Y = \mathbb{R}$ with the usual topology. Then for any sequence $\{V_n\}$ of neighbourhoods of $\omega_1$, $\bigcap_{n=1}^{\infty} V_n \neq \{\omega_1\}$. If $\lambda$ is an ordinal number, there are a unique non-negative integer $n$ and a limit number $\beta$ such that $\lambda = \beta + n$. Define the single-valued function $f \colon X \to Y$ by $f(\lambda) = 1/n$ if $\lambda$ is a non-limit ordinal number, $f(\lambda) = 1$ if $\lambda < \omega_1$ is a limit ordinal number and $f(\omega_1) = 0$.

It is easy to verify that $f$ is almost continuous at $\omega_1$.

Suppose that $A$ is an almost-neighbourhood of $\omega_1$ and $f/A$ is continuous at $\omega_1$. For any $n \in N$ there is a neighbourhood $U_n$ of $\omega_1$ such that $f(U_n \cap A) \subset \{y \in Y \colon y < 1/n\}$. Put $U = \bigcap_{n=1}^{\infty} U_n$. Then $U$ is a neighbourhood of $\omega_1$ and $f(U \cap A) = \{0\}$, hence $U \cap A = \{\omega_1\}$, thus $\omega_1 \notin \text{Int} A$, which is a contradiction.

Example 3. Let $\{B_n\}$ be a sequence of mutually disjoint countable dense sets in $[0, 1] \setminus \{1, 1/2, \ldots, 1/n, \ldots\}$. Put $X = \left(\bigcup_{n=1}^{\infty} B_n\right) \cup \{0\}$ with the induced topology. Let $Y$ be the set of real numbers with the usual topology.

Let for every $k \in N$, $\{x_{n,k}\}$ be a sequence of different real numbers in the open interval $(k, k + 1)$ convergent to $k$ and $\{f^k\}_{j=1}^{k}$ be a sequence of bijections from $B_j \cap (1/(k + 1), 1/k)$ to the set $\{x_{n,k} \colon n \geq j\}$. Define $F$ by $F(0) = \{1, 2, \ldots, n, \ldots\}$ and $F(x) = \{1, 2, \ldots, (k - 1), f^k_j(x), (k + 1), \ldots\}$ for $x \in B_j \cap (1/(k + 1), 1/k)$.

It is easy to verify that $F$ is upper almost continuous at 0.

Suppose $A$ is an almost-neighbourhood of 0 and $F/A$ is upper semicontinuous at 0. There exists $r \in N$ such that $A$ is dense in $X \cap (0, 1/r)$. For any $l \geq r$ choose $x_l \in A \cap (1/(l + 1), 1/l)$. Let $j_l$ be such that $x_l \in B_{j_l}$. Put $V = Y \setminus \{f^k_{j_l}(x_l) \colon j \geq r\}$.

Then $F^-(V)$ is not a neighbourhood of 0 in $A$ and that is a contradiction.
The following simple example shows that for \( F: X \to Y \) the lower almost continuity does not imply the existence of an almost-neighbourhood \( A \) of \( x \) such that \( F/A \) is lower continuous at \( x \).

**Example 4.** Let \( X = Y = \mathbb{R} \), where \( \mathbb{R} \) is the set of real numbers with the usual topology. Let \( F \) be defined as \( F(0) = \{1, 2\} \), \( F(x) = \{1\} \) for \( x \) rational and \( F(x) = \{2\} \) for \( x \) irrational. Then \( F \) is lower almost continuous at 0 and there is no almost open set \( A \) containing 0 for which \( F/A \) is lower semicontinuous at 0.

**Theorem 2.** Let \( X \) be a topological space with a \( \sigma \)-discrete base. Let \( F: X \to Y \) be upper almost continuous. Let there exist for any \( x \in X \) a countable base of neighbourhoods of \( F(x) \). Then there exists a dense set \( D \) in \( X \) such that \( F/D \) is upper semicontinuous.

**Proof.** Let \( \{ \mathcal{V}_n : n \in \mathbb{N} \} \) be discrete systems of nonempty open sets such that \( \mathcal{V} = \bigcup \{ \mathcal{V}_n : n \in \mathbb{N} \} \) is a base for \( X \). For any \( V \in \mathcal{V} \) choose \( x_V \in V \) and put \( D_1 = \{ x_V : V \in \mathcal{V}_1 \} \). For any \( x_V \in D_1 \) denote \( A_{x_V} \) an almost-neighbourhood of \( x_V \) such that \( A_{x_V} \subset V \) and \( F/A_{x_V} \) is upper almost continuous and upper semicontinuous at \( x_V \). (See Theorem 1) Put \( A_1 = \bigcup \{ A_{x_V} : x_V \in D_1 \} \) and \( X_1 = (X \setminus \overline{A_1}) \cup A_1 \). Then \( X_1 \) is dense in \( X \). Since \( \mathcal{V} \) is a discrete family, \( F/X_1 \) is upper almost continuous and upper semicontinuous at every \( x \in D_1 \).

By induction we will construct sequences \( \{ D_n \} \), \( \{ X_n \} \) with the following properties: (a) \( X_n \) is a dense subset of \( X_{n-1} \), (b) \( D_n \subset X_n \), (c) \( D_{n-1} \subset D_n \), (d) for any \( V \in \mathcal{V} \) \( V \cap D_n \neq \emptyset \), (e) there exists a pairwise disjoint locally finite family of open neighbourhoods of points of \( D_n \), (f) \( F/X_n \) is upper almost continuous and upper semicontinuous at every \( x \in D_n \).

Suppose \( D_1 \), \( D_2 \), ..., \( D_n-1 \), \( X_1 \), \( X_2 \), ..., \( X_{n-1} \) were constructed. Put \( \mathcal{B}_n = \mathcal{V} \setminus \{ V \in \mathcal{V} : V \cap D_{n-1} \neq \emptyset \} \). For any \( V \in \mathcal{B}_n \) choose \( x_V \in V \cap X_{n-1} \) and put \( C_n = \{ x_V : V \in \mathcal{B}_n \} \). For any \( x \in D_{n-1} \) there exists an open neighbourhood \( U_x \) such that \( U_x \cap C_n = \emptyset \) and that the family \( \{ U_x : x \in D_{n-1} \} \) is pairwise disjoint. By assumption there exists a pairwise disjoint locally finite family \( \{ V_x : x \in D_{n-1} \} \) of open neighbourhoods of points of \( D_{n-1} \). Let \( x \in D_{n-1} \). Since \( \mathcal{B}_n \) is a discrete family in \( X \) there exists an open neighbourhood \( O \) of \( x \) such that \( O \cap V \neq \emptyset \) for at most one member \( V \) from \( \mathcal{B}_n \). Since \( X \) is Hausdorff, there exists an open set \( O_1 \) such that \( x \in O \subset O_1 \) and \( x \notin \overline{O_1} \). Put \( U_x = V \setminus O_1 \). For any \( x \in C_n \) put \( U_{x_V} = V \cap (X \setminus \{ U_x : x \in D_{n-1} \}) \). Since \( \{ U_x \} \) is a locally finite family, we have \( \bigcup \{ U_{x_V} : x \in D_{n-1} \} = \bigcup \{ U_x : x \in D_{n-1} \} \) and thus \( U_{x_V} \) is an open neighbourhood of every \( x_V \) from \( C_n \). Put \( D_n = D_{n-1} \cup C_n \). The family \( U_x : x \in D_n \) is pairwise disjoint and locally finite. \( D_n \subset X_{n-1} \) and \( F/X_{n-1} \) is upper almost continuous. For any \( x \in D_n \) denote \( A_x \) an almost open set in \( X_{n-1} \) such that \( x \in A_x, A_x \subset U_x \) and \( F/A_x \) is upper almost continuous and upper semicontinuous at \( x \).

Put \( A_n = \bigcup \{ A_x : x \in D_n \} \) and \( X_n = (X_{n-1} \setminus \overline{A_n}) \cup A_n \). It is evident that \( X_n \) is
dense in $X_{n-1}$ and $F/X_n$ is upper almost continuous and upper semicontinuous at every $x \in D_n$. It follows from the construction that $D = \bigcup_{i=1}^{x} D_i$ is dense in $X$ and $F/D$ is upper semicontinuous.

Remark 4. Notice that the set $D$ constructed in the proof of Theorem 2 is an $F_\sigma$-set and in spaces without isolated points, $D$ is a set of the first category. The following example shows that this result is the best possible.

Example 5. Let $X$ be the set of real numbers with the usual topology and $Y$ be the set of real numbers with the discrete topology. For any irrational number $p$ put $C_p = p + Q$, where $Q$ is the set of rational numbers. Choose $c_p$ from $C_p$ for any irrational $p$. ($c_p = c_q$ for any $q \in p + Q$)

Define $f: X \to Y$ as $f(x) = 0$ for $x \in Q$ and $f(x) = c_p$ for $x \in C_p$. It is easy to see that $f$ is almost continuous. If $D$ is a set in $X$ such that $f/D$ is continuous, then $D$ is countable. Suppose that $D$ is uncountable. Then there exists $x \in D$ such that for every neighbourhood $V$ of $x$, $V \cap D$ is an uncountable set. It is clear that $f/D$ is not continuous at $x$.

Theorem 3. Let $X$ be a space with a $\sigma$-discrete base and $Y$ be a second countable space with infinitely many points. The following statements are equivalent.

1. $X$ is a Baire space,
2. for every compact-valued multifunction $F: X \to Y$ there is a dense set $D$ in $X$ such that $F/D$ is upper semicontinuous.

Proof. Suppose that $X$ is a Baire space. Then the assertion is clear from Remark 3, Proposition 3 and Theorem 2.

Now assume that $X$ is not Baire and choose a nonempty open set $U$ which is of the first category. Let $C_1, C_2, \ldots$ be a sequence of mutually disjoint nowhere dense sets with $\cup \{C_n : n \in N\} = U$. Let $L$ be an infinite discrete subset of $Y$ and let $(c_n : n \geq 0)$ be an enumeration of $L$. Define $f: X \to Y$ by $f(C_n) = c_n$, $n \geq 1$ and $f(X \setminus U) = c_0$. There is no set $D$ dense in $X$ for which the restriction $f/D$ is continuous. Suppose that there is a dense set $D$ in $X$ such that the restriction $f/D$ is continuous. Choose $x \in D \cap U$. There is $n \geq 1$ such that $f(x) = c_n$. Since $L$ is a discrete set and $f/D$ is continuous at $x$ there is an open neighbourhood $V$ of $x$ in $X$ such that $f(V \cap U \cap D) = c_n$. Thus $C_n = f^{-1}(c_n) \supset V \cap U \cap D$, i.e. $C_n \supset V \cap U$ and that is a contradiction since $C_n$ is nowhere dense.

Remark 4. The question is, whether the assumption on $X$ in Theorem 2 is essential?

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ЗАМЕЧАНИЕ К ПОЧТИ НЕПРЕРЫВНЫМ ОТНОШЕНИЯМ

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Резюме

В этой статье изучается почти непрерывность отношений, дана характеристика сверху почти непрерывных отношений.