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LEX-IDEALS OF $DR\ell$-MONOIDS AND GMV-ALGEBRAS

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ABSTRACT. The notion of a GMV-algebra is a non-commutative generalization of that of an MV-algebra. Close connections between GMV-algebras and $DR\ell$-monoids are used for studying lexicographic extensions of ideals of GMV-algebras via those of $DR\ell$-monoids.

1. Introduction

MV-algebras have been introduced by C. C. Chang in [2] as an algebraic counterpart of the Łukasiewicz infinite valued propositional logic. G. Georgescu and A. Iorgulescu in [4] and [5], and independently J. Rachůnek, in [11], have introduced non-commutative generalization of MV-algebras (pseudo MV-algebras in [4] and [5] and non-commutative MV-algebras in [11]). We will use for these algebras the name *generalized MV-algebras*, briefly: *GMV-algebras.*

Recall that an intensive development of the theory of MV-algebras was made possible by the fundamental result of D. Mundici in [10] that gave a representability of MV-algebras by means of intervals of unital abelian lattice ordered groups (ℓ-groups). A. Dvurečenskij in [3] has generalized this result also for GMV-algebras, i.e., he has proved that every GMV-algebra is isomorphic to a GMV-algebra introduced by the standard method on the unit interval of a unital (non-abelian, in general) ℓ-group. This representation enable us to use essentially some methods and techniques of widely developed theory of ℓ-groups also for problems in the theory of GMV-algebras.

This approach was applied by D. Hort and J. Rachůnek in [6]. They described the ordered sets of prime and regular ideals of GMV-algebras induced on principal ideals which are generated by additive idempotent elements and studied lexicographic extensions of ideals of GMV-algebras there. However,

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this technique allowed the results excluding the case of proper lex-extensions of ideals of GMV-algebras, which are comparable and different, contrary to [1] for ℓ-groups. It follows from the fact that ideals of a GMV-algebra need not be GMV-algebras, with the exception of principal ideals generated by an additive idempotent element.

However, GMV-algebras are in a one-to-one correspondence with some type of bounded dually residuated lattice ordered monoids ($DR\ell$-monoids). In the paper, lex-extensions and lex-ideals, in a class of $DR\ell$-monoids involving also all such which are induced by GMV-algebras, are studied. By methods of the theory of $DR\ell$-monoids, the results, already corresponding to analogous those for ℓ-groups in [1], are deduced here. Then one can obtain some propositions in [6] as special cases.

2. Definitions and basic properties

DEFINITION. An algebra $\mathcal{M} = (M, +, 0, \vee, \wedge, \rightharpoonup, \leftharpoondown)$ of signature $\langle 2, 0, 2, 2, 2, 2 \rangle$ is called a *dually residuated (non-commutative) lattice ordered monoid* (a *$DR\ell$-monoid*) if and only if

(M1) $(M, +, 0, \vee, \wedge)$ is a lattice ordered monoid (ℓ-monoid), that is, $(M, +, 0)$ is a (non-commutative) monoid, $(M, \vee, \wedge)$ is a lattice, and for any $x, y, u, v \in M$, the following identities are satisfied:

$$u + (x \vee y) + v = (u + x + v) \vee (u + y + v),$$
$$u + (x \wedge y) + v = (u + x + v) \wedge (u + y + v);$$

(M2) if $\le$ denotes the order on M induced by the lattice $(M, \vee, \wedge)$ then, for any $x, y \in M$,

$x \rightharpoonup y$ is the least element $s \in M$ such that $s + y \ge x$,
$x \leftharpoondown y$ is the least element $t \in M$ such that $y + t \ge x$;

(M3) $\mathcal{M}$ fulfils the identities

$$\big((x \rightharpoonup y) \vee 0\big) + y \le x \vee y, \qquad y + \big((x \leftharpoondown y) \vee 0\big) \le x \vee y,$$
$$x \rightharpoonup x \ge 0, \qquad x \leftharpoondown x \ge 0.$$

Commutative $DR\ell$-monoids (called $DR\ell$-semigroups) were introduced by K. L. N. Swamy in [13] as common generalizations of commutative ℓ-groups and Brouwerian algebras. The present definition of a non-commutative extension of $DR\ell$-monoids is due to [7]. Also, for basic properties of non-commutative $DR\ell$-monoids, see [7].

$DR\ell$-monoids are in a close connection with generalized MV-algebras (briefly: GMV-algebras). Recall that GMV-algebras were introduced by J. Rachůnek in [11], and independently by G. Georgescu and A. Iorgulescu in [5], as a non-commutative generalization of MV-algebras (non-commutative MV-algebras in [11] and pseudo MV-algebras in [5]).

DEFINITION. Let $\mathcal{A} = (A, \oplus, \neg, \sim, 0, 1)$ be an algebra of type $\langle 2,1,1,0,0 \rangle$. Set $x \odot y = \sim(\neg x \oplus \neg y)$ for any $x, y \in A$. Then $\mathcal{A}$ is called a *generalized MV-algebra* (briefly: *GMV-algebra*) if for any $x, y, z \in A$ the following conditions are satisfied:

(A1) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$;
(A2) $x \oplus 0 = x = 0 \oplus x$;
(A3) $x \oplus 1 = 1 = 1 \oplus x$;
(A4) $\neg 1 = 0 = \sim 1$;
(A5) $\neg(\sim x \oplus \sim y) = \sim(\neg x \oplus \neg y)$;
(A6) $x \oplus (y \odot \sim x) = y \oplus (x \odot \sim y) = (\neg y \odot x) \oplus y = (\neg x \odot y) \oplus x$;
(A7) $(\neg x \oplus y) \odot x = y \odot (x \oplus \sim y)$;
(A8) $\sim\neg x = x$.

If the operation $\oplus$ is commutative, then the unary operations $\neg$ and $\sim$ coincide and $\mathcal{A}$ is an MV-algebra.

If we put $x \leq y$ if and only if $\neg x \oplus y = 1$, then $\leq$ is an order on A. Moreover, $(A, \leq)$ is a bounded distributive lattice in which $x \vee y = x \oplus (y \odot \sim x)$ and $x \wedge y = x \odot (y \oplus \sim x)$ for each $x, y \in A$, and 0 is the least and 1 is the greatest element in A, respectively. For basic properties of GMV-algebras, see [5].

As shown in [11; Theorem 13], if $(A, \oplus, \neg, \sim, 0, 1,)$ is a GMV-algebra and if we put $x \rightharpoonup y = \neg y \odot x$, and $x \leftharpoondown y = x \odot \sim y$, then $(A, \oplus, 0, \vee, \wedge, \rightharpoonup, \leftharpoondown)$ is a bounded $DR\ell$-monoid with the greatest element 1 (then 0 is the least element) satisfying the conditions

(i) $(\forall x \in A)\big(1 \leftharpoondown (1 \rightharpoonup x) = x = 1 \rightharpoonup (1 \leftharpoondown x)\big)$,
(ii) $(\forall x, y \in A)\big(1 \rightharpoonup ((1 \leftharpoondown x) + (1 \leftharpoondown y)) = 1 \leftharpoondown ((1 \rightharpoonup x) + (1 \rightharpoonup y))\big)$.

Also conversely (see [11; Theorem 12]), if $(M, +, 0, \vee, \wedge, \rightharpoonup, \leftharpoondown)$ is a bounded $DR\ell$-monoid with the greatest element 1 satisfying the previous conditions (i) and (ii) and if we set $\neg x = 1 \rightharpoonup x$, $\sim x = 1 \leftharpoondown x$ for any $x, y \in M$, then $(M, +, \neg, \sim, 0, 1)$ is a GMV-algebra.

3. Ideals

Further, for our purpose, we will consider only bounded $DR\ell$-monoids. In accordance to [8], we define an ideal of such a $DR\ell$-monoid.

DEFINITION. Let $\mathcal{M}$ be a bounded $DR\ell$-monoid and $\emptyset \neq I \subseteq M$. Then I is called an *ideal of* $\mathcal{M}$ if the following conditions are satisfied:

($\mathrm{I1_M}$) if $x, y \in I$, then $x + y \in I$;
($\mathrm{I2_M}$) if $x \in I$, $y \in M$ and $y \leq x$, then $y \in I$.

LEMMA 3.1. ([8; Theorem 13]) *Let $\mathcal{M}$ be a bounded $DR\ell$-monoid and $\emptyset \neq I \subseteq M$. Then I is an ideal of $\mathcal{M}$ if and only if I is a convex subalgebra in $\mathcal{M}$.*

DEFINITION. Let $\mathcal{A}$ be a GMV-algebra and $\emptyset \neq H \subseteq A$. Then H is called an *ideal of* $\mathcal{A}$ if the following conditions are satisfied:

($\mathrm{I1_A}$) if $x, y \in H$, then $x \oplus y \in H$;
($\mathrm{I2_A}$) if $x \in H$, $y \in A$ and $y \leq x$, then $y \in H$.

It can be easily seen that the intersection of any family of ideals of a $DR\ell$-monoid $\mathcal{M}$ (a GMV-algebra $\mathcal{A}$, respectively) is still an ideal. For any $K \subseteq M$ ($K \subseteq A$, respectively), the smallest ideal containing K, i.e. the intersection of all ideals I such that $K \subseteq I$, is called the *ideal generated by* K. We will denote it by $I(K)$. In particular, for any element a of a $DR\ell$-monoid $\mathcal{M}$ (a GMV-algebra $\mathcal{A}$, respectively), the ideal $I(\{a\}) =: I(a)$ is said to be the *principal ideal generated by* a.

Denote by $\mathcal{C}(\mathcal{M})$ and $\mathcal{C}(\mathcal{A})$ the set of all ideals in a $DR\ell$-monoid $\mathcal{M}$ and a GMV-algebra $\mathcal{A}$, respectively. Then $(\mathcal{C}(\mathcal{M}), \subseteq)$ and $(\mathcal{C}(\mathcal{A}), \subseteq)$ are complete Brouwerian lattices in which infima coincide with set intersections ([8; Theorem 14] and [5; Proposition 2.11], respectively).

PROPOSITION 3.2. *Let $\mathcal{A}$ be a GMV-algebra and $\emptyset \neq H \subseteq A$. Then H is an ideal in $\mathcal{A}$ if and only if H is a convex subalgebra of the $DR\ell$-monoid induced by $\mathcal{A}$.*

Proof. Let $\mathcal{A} = (A, \oplus, \neg, \sim, 0, 1)$ be a GMV-algebra. Suppose H to be an ideal in $\mathcal{A}$. Then it holds that:

1. $0 \in H$.
2. If $a, b \in H$, then $a \rightharpoonup b = \neg b \odot a \leq a$, hence $a \rightharpoonup b \in H$. Similarly, $a \leftharpoondown b = a \odot {\sim} b \leq a$, therefore $a \leftharpoondown b \in H$.
3. If $a, b \in H$, then $a \wedge b \leq a \vee b \leq a \oplus b \in H$, hence $a \wedge b \in H$ and $a \vee b \in H$.

That means H is a convex subalgebra of the induced $DR\ell$-monoid $(A, \oplus, 0, \vee, \wedge, \rightharpoonup, \leftharpoondown)$.

Conversely, let I be a convex subalgebra of the $DR\ell$-monoid induced by $\mathcal{A}$. Then $0 \in I$ and I is closed under the operation $\oplus$. If $a \in I$, $x \in A$ and $x \leq a$, then $x \in I$ from convexity of I. □

Again in accordance to [8], we define a normal ideal of a bounded $DR\ell$-monoid.

DEFINITION. An ideal I of a bounded $DR\ell$-monoid $\mathcal{M}$ is said to be *normal* if it satisfies the condition:

$$(\forall x, y \in M)(x \rightharpoonup y \in I \iff x \leftharpoondown y \in I).$$

Recall the definition of a normal ideal of a GMV-algebra (see [5]).

DEFINITION. An ideal H of a GMV-algebra $\mathcal{A}$ is called *normal* if it satisfies the condition:

$$(\forall x, y \in A)(\neg x \odot y \in H \iff y \odot {\sim} x \in H).$$

The above definitions and Proposition 3.2 entail the following lemma.

LEMMA 3.3. *Let $\mathcal{A}$ be a GMV-algebra. A subset $H \subseteq A$ is a normal ideal of $\mathcal{A}$ if and only if H is a normal ideal of the induced $DR\ell$-monoid.*

4. Lex-extensions of $DR\ell$-monoids

An ideal H of a GMV-algebra $\mathcal{A}$ is called *prime* (see [5]) if H is a finitely meet-irreducible element in the lattice $\mathcal{C}(\mathcal{A})$. The same property of an element of the lattice $\mathcal{C}(\mathcal{M})$ is used for the definition of a *prime ideal* of an $DR\ell$-monoid $\mathcal{M}$ (see [9]).

Let $0 \neq a \in A$ and $H \in \mathcal{C}(\mathcal{A})$. Then H is called a *value of a* if it is maximal with respect to the property "not containing a". Denote by $\mathrm{val}_{\mathcal{A}}(a)$ the set of values of a. Further, $H \in \mathcal{C}(\mathcal{A})$ is called a *regular ideal* of $\mathcal{A}$ if H is meet-irreducible in $\mathcal{C}(\mathcal{A})$. By [5], $H \in \mathcal{C}(\mathcal{A})$ is regular if and only if $H \in \mathrm{val}_{\mathcal{A}}(a)$ for some $0 \neq a \in A$. Denote by $\mathrm{V}(\mathcal{A})$ the set of regular ideals of $\mathcal{A}$. Then $\mathrm{V}(\mathcal{A})$ is a root system and, moreover, $\bigcap \mathrm{V}(\mathcal{A}) = \{0\}$.

If $\mathcal{M}$ is a $DR\ell$-monoid, then a *regular ideal* and *values* of $0 \neq a \in M$ will be defined in a similar way as in GMV-algebras.

An ideal H of $\mathcal{A}$ is said to be *special* if H is the unique value of some $0 \neq a \in A$. Such an element which has only one value is called a *special element.* We define a *special* ideal and a *special element* of a $DR\ell$-monoid $\mathcal{M}$ analogously.

Let $\mathcal{A}$ be a GMV-algebra and $X \subseteq A$. The set

$$X^{\perp} = \{a \in A: \ a \wedge x = 0 \text{ for each } x \in X\}$$

is called the *polar of* X *in* $\mathcal{A}$. For any $a \in A$, we write $a^{\perp}$ instead of $\{a\}^{\perp}$. A subset Y of A is a *polar in* $\mathcal{A}$ if $Y = X^{\perp}$ for some $X \subseteq A$.

If $\mathcal{M}$ is a $DR\ell$-monoid, then a *polar in* $\mathcal{M}$ is defined in the same way.

Further, let us consider $DR\ell$-monoids satisfying the inequalities

$$\begin{aligned} (x \rightharpoonup y) \wedge (y \rightharpoonup x) \le 0\,, \\ (x \leftharpoondown y) \wedge (y \leftharpoondown x) \le 0\,. \end{aligned} \tag{$*$}$$

Obviously, for a bounded $DR\ell$-monoid, the inequalities $(*)$ can be written in the following way:

$$\begin{aligned} (x \rightharpoonup y) \wedge (y \rightharpoonup x) = 0\,, \\ (x \leftharpoondown y) \wedge (y \leftharpoondown x) = 0\,. \end{aligned}$$

Any bounded $DR\ell$-monoid induced by a GMV-algebra satisfies $(*)$.

THEOREM 4.1. *Let* $\mathcal{M}$ *be a bounded* $DR\ell$*-monoid satisfying* $(*)$ *and* $I \in \mathcal{C}(\mathcal{M})$. *Then the following conditions are equivalent:*

(1) I *is a prime ideal and it holds that* $x \ge y$ *for each* $x \in M \setminus I$ *and* $y \in I$.
(2) I *is a prime ideal and* I *is comparable with every* $J \in \mathcal{C}(\mathcal{M})$.
(3) I *contains all proper polars in* $\mathcal{M}$.
(4) I *contains all minimal prime ideals.*
(5) $x^{\perp} = \{0\}$ *for any* $x \in M \setminus I$.
(6) *Every element in* $M \setminus I$ *is special.*

Proof.

(1) $\Longrightarrow$ (2): Let $K \in \mathcal{C}(\mathcal{M})$, $K \nsubseteq I$ and $x \in K \setminus I$. Then $I \subseteq I(x) \subseteq K$.

(2) $\Longrightarrow$ (3): Let B be a polar such that $B \nsubseteq I$. Then $I \subset B$ (because $B \in \mathcal{C}(\mathcal{M})$). Let us consider $y \in B \setminus I$. If $z \in y^{\perp}$, then $z \wedge y = 0$, and hence $z \in I$. That means $y^{\perp} \subseteq I$ and therefore also $B^{\perp} \subseteq I$. From this we get $B^{\perp} \subset B$. It means $B = M$.

(3) $\Longrightarrow$ (4): By [9; Proposition 26], every minimal prime ideal of a $DR\ell$-monoid is a join of polars.

(4) $\Longrightarrow$ (5): Let $x \notin I$. If P is a minimal prime ideal in $\mathcal{M}$, then $P \subseteq I$, hence $x \notin P$, and so $x^{\perp} \subseteq P$. Since the intersection of all regular ideals in $\mathcal{M}$ is $\{0\}$ and every regular ideal is a prime ideal, it holds also that the intersection of all minimal prime ideals is $\{0\}$. Therefore $x^{\perp} = \{0\}$, too.

(5) $\Longrightarrow$ (6): Assume $x \in M \setminus I$ and $P \in \operatorname{val}(x)$ to be such a value for which $I \subseteq P$. Let $N \in \operatorname{val}(x)$, $N \neq P$. Consider $x \in P \setminus N$, $y \in N \setminus P$. It holds that

$$x = \big(x \rightharpoonup (x \wedge y)\big) + (x \wedge y)\,, \qquad y = \big(y \rightharpoonup (x \wedge y)\big) + (x \wedge y)\,,$$

and at the same time,

$$\begin{aligned}&\big(x \rightharpoonup (x \wedge y)\big) \wedge \big(y \rightharpoonup (x \wedge y)\big)\\ &= \big((x \rightharpoonup x) \vee (x \rightharpoonup y)\big) \wedge \big((y \rightharpoonup x) \vee (y \rightharpoonup y)\big)\\ &= \big(0 \wedge (y \rightharpoonup x)\big) \vee (0 \wedge 0) \vee \big((x \rightharpoonup y) \wedge (y \rightharpoonup x)\big) \vee \big((x \rightharpoonup y) \wedge 0\big).\end{aligned}$$

Since $\mathcal{M}$ fulfills the conditions $(*)$, we have

$$\big(x \rightharpoonup (x \wedge y)\big) \wedge \big(y \rightharpoonup (x \wedge y)\big) = 0 .$$

Moreover, $x \rightharpoonup (x \wedge y) \notin N$, $y \rightharpoonup (x \wedge y) \notin P$. Thus $y \rightharpoonup (x \wedge y) \notin I$, but $\big(y \rightharpoonup (x \wedge y)\big)^{\perp} \neq \{0\}$, a contradiction. Therefore, each element from $M \setminus I$ is special.

(6) $\implies$ (5): Let $x \in M \setminus I$ and P be the unique value of x. Then $I \subseteq P$. Consider $y \in x^{\perp}$. If $x \vee y \in P$, then $0 \le x \le x \vee y$ entails $x \in P$, which is a contradiction. Hence $x \vee y \in M \setminus P$ and therefore $P \subseteq N$ where N is the unique value of the element $x \vee y$. At the same time, from $x \wedge y = 0$ and $x \notin P$ we have $y \in P$.

If it held $x \vee y \notin I(x)$, then it would be $I(x) \subseteq N$ and therefore $x \vee y \in I(x) \vee P \subseteq N$, a contradiction. Hence $x \vee y \in I(x)$, and so also $y \in I(x)$. But then $I(x)^{\perp} = x^{\perp} \subseteq I(x)$ and from this it follows that $x^{\perp} = \{0\}$.

(5) $\implies$ (1): Let $x \in M \setminus I$, $a \in I$. It holds that $x = \big(x \rightharpoonup (x \wedge a)\big) + (x \wedge a)$, $a = \big(a \rightharpoonup (x \wedge a)\big) + (x \wedge a)$, and since $x \wedge a \in I$, it holds that $x \rightharpoonup (x \wedge a) \notin I$, thus $\big(x \rightharpoonup (x \wedge a)\big)^{\perp} = \{0\}$. Moreover, from the assumption of validity of the conditions $(*)$ we obtain $\big(x \rightharpoonup (x \wedge a)\big) \wedge \big(a \rightharpoonup (x \wedge a)\big) = 0$, and so $a \rightharpoonup (x \wedge a) = 0$. Therefore $a = x \wedge a < x$. □

DEFINITION. Let $\mathcal{M}$ be a bounded $DR\ell$-monoid with the properties $(*)$ and let $I \in \mathcal{C}(\mathcal{M})$ satisfy any of the conditions from Theorem 4.1. Then $\mathcal{M}$ is said to be a *lex-extension* of the ideal I.

PROPOSITION 4.2. *Let $\mathcal{M}$ be a $DR\ell$-monoid, $I \in \mathcal{C}(\mathcal{M})$ and $0 \neq a \in I$. Then a is special in $\mathcal{M}$ if and only if it is special in I.*

Proof. It follows from the fact that the correspondence $\varphi\colon N \mapsto N \cap I$ ($N \in \mathrm{val}_{\mathcal{M}}(a)$) is a bijection of $\mathrm{val}_{\mathcal{M}}(a)$ onto $\mathrm{val}_{I}(a)$. □

For GMV-algebras, an ideal H is a GMV-algebra with the operation $\oplus$, which is the restriction of the operation $\oplus$ from $\mathcal{A}$, if and only if $H = X_e$, where $e \in B(\mathcal{A})$ (i.e. the set of all additively idempotent elements in $\mathcal{A}$) and $X_e = \big([0,e], \oplus, \neg_e, \sim_e, 0, e\big)$, $\neg_e x = \neg x \wedge e$, $\sim_e x = \sim x \wedge e$ (see [12; Lemmas 6, 7]). For this reason, an analogy of [1; Proposition 7.1.3] could not be expressed for arbitrary $C, D \in \mathcal{C}(\mathcal{A})$, $C \subset D$ ([6]).

For $DR\ell$-monoids, an ideal is a subalgebra and the following proposition holds.

PROPOSITION 4.3. *Let $\mathcal{M}$ be a bounded $DR\ell$-monoid with $(*)$, $I, J \in \mathcal{C}(\mathcal{M})$ and $J \subset I$. Then $\mathcal{M}$ is a lex-extension of J if and only if $\mathcal{M}$ is a lex-extension of I and I is a lex-extension of J.*

P r o o f . It follows from Theorem 4.1 (by using the condition (6)) and from Proposition 4.2. □

DEFINITION. The join of all proper polars in the lattice $\mathcal{C}(\mathcal{M})$ is called the *lex-kernel* of $DR\ell$-monoid $\mathcal{M}$ and it will be denoted by $\operatorname{lex} M$.

Remark 4.4. By Theorem 4.1, it holds that:

a) $\operatorname{lex} M$ is the supremum of all minimal prime ideals in $\mathcal{C}(\mathcal{M})$;
b) if $I \in \mathcal{C}(\mathcal{M})$, then $\mathcal{M}$ is a lex-extension of I if and only if $\operatorname{lex} M \subseteq I$.

DEFINITION. A $DR\ell$-monoid $\mathcal{M}$ is said to be *lex-simple* if $\operatorname{lex} M = M$.

PROPOSITION 4.5. *In any bounded $DR\ell$-monoid $\mathcal{M}$ with the property $(*)$, $\operatorname{lex} M$ is the greatest ideal in $\mathcal{M}$ which is lex-simple.*

P r o o f . If $\operatorname{lex} M$ is a lex-extension of $I \in \mathcal{C}(\mathcal{M})$, then, by Proposition 4.3, $\mathcal{M}$ is also a lex-extension of I. Hence $\operatorname{lex} M \subseteq I$, and therefore $\operatorname{lex} M$ is lex-simple.

Let $J \in \mathcal{C}(\mathcal{M})$ be lex-simple and assume $\operatorname{lex} M \subset J$. Then J is a lex-extension of $\operatorname{lex} M$, thus $\operatorname{lex} J \subset J$, which is a contradiction. But $\operatorname{lex} M$ is comparable with every ideal of $\mathcal{M}$ by Theorem 4.1. For this reason, $J \subseteq \operatorname{lex} M$. □

DEFINITION. An ideal $I \in \mathcal{C}(\mathcal{M})$ is called a *lex-ideal* of $\mathcal{M}$ if $\operatorname{lex} I \neq I$.

PROPOSITION 4.6. *An element $a \in \mathcal{M}$ is special if and only if $I(a)$ is a lex-ideal of $\mathcal{M}$.*

P r o o f . Let a be a special element in $\mathcal{M}$ and N be its only value. Then $N \cap I(a)$ is the only value of a in $I(a)$ and consequently, $N \cap I(a)$ is the greatest proper ideal in $I(a)$. Hence $I(a)$ is a lex-extension of $N \cap I(a)$, i.e. $\operatorname{lex} I(a) \neq I(a)$, by Theorem 4.1 (the condition (2)).

Conversely, suppose $\operatorname{lex} I(a) \neq I(a)$, that is $a \notin \operatorname{lex} I(a)$. By Theorem 4.1 (the condition (6)), we get a to be special in $I(a)$, therefore a is also special in $\mathcal{M}$. □

THEOREM 4.7. *Any two lex-ideals in $\mathcal{M}$ are either comparable or orthogonal or their intersection is a principal ideal generated by an idempotent element.*

P r o o f . Let I and J be lex-ideals in $\mathcal{M}$. If $I \not\subseteq J$, then there exists $0 \neq a \in I$ such that $a \notin J \cup \operatorname{lex} I$. Analogously, if $J \not\subseteq I$, then there exists $0 \neq b \in J$ such that $b \notin I \cup \operatorname{lex} J$. Obviously, $I \cap J \in \mathcal{C}(\mathcal{M})$, therefore $I \cap J$ is comparable with $\operatorname{lex} I$. If $I \cap J \subseteq \operatorname{lex} I$, then $I \cap J < a$. In case that $\operatorname{lex} I \subseteq I \cap J$, then I is a lex-extension of $I \cap J$ and therefore also $I \cap J < a$. We would prove that $I \cap J < b$ analogously.

However, $a \wedge b \in I \cap J$ and $a \wedge b$ is greater or equal to every element from $I \cap J$, therefore $a \wedge b$ is the greatest element in $I \cap J$. Hence $I \cap J = (a \wedge b]$, and so $I \cap J$ is a principal ideal generated by idempotent element $a \wedge b$. (If $a \wedge b = 0$, then I and J are orthogonal.) □

Remark 4.8. Only the first two possibilities from Theorem 4.7 can arise in the case of ℓ-groups, because there does not exist any idempotent element $a \neq 0$ there.

THEOREM 4.9. *Let $I, J \in \mathcal{C}(\mathcal{M})$ and $I \subset J$. Then J is a lex-extension of I if and only if $b^{\perp} = J^{\perp}$ for any $b \in J \setminus I$.*

Proof. Suppose J to be a lex-extension of I and $b \in J \setminus I$. It holds that $J^{\perp} \subseteq b^{\perp}$. Let $z \in b^{\perp}$. Then $b \wedge z \wedge y = 0$ for all $y \in J$. Therefore using Theorem 4.1(5) we obtain $z \wedge y = 0$ for any $y \in J$, that means $z \in J^{\perp}$ and therefore $b^{\perp} \subseteq J^{\perp}$.

Conversely, assume $b^{\perp} = J^{\perp}$ for every $b \in J \setminus I$. Let $b \in J \setminus I$, $c \in J$ and $b \wedge c = 0$. Then $c \in J \cap b^{\perp} = J \cap J^{\perp} = \{0\}$, whence $c = 0$. Therefore $b^{\perp} = \{0\}$, which yields, by Theorem 4.1(5), J is a lex-extension of I. □

THEOREM 4.10. *If $\{0\} \neq I \in \mathcal{C}(\mathcal{M})$ and $J \in \mathcal{C}(\mathcal{M})$ is a lex-extension of I, then $I^{\perp} = J^{\perp}$.*

Proof. Let $0 \neq a \in I$. Consider $b \in I^{\perp}$ and $x \in J \setminus I$. If $b \wedge x \notin I$, then $b \wedge x \geq a$ and hence $a = b \wedge a = 0$, which is a contradiction. Therefore $b \wedge x \in I \cap I^{\perp} = \{0\}$, thus $b \wedge x = 0$. That means $I^{\perp} \subseteq b^{\perp} = J^{\perp}$. □

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