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ON EXTENSION OF BAIRE VECTOR MEASURES

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It is a well-known fact that every Baire positive measure can be extended uniquely to a regular Borel positive measure [1, Theorem 65.1; 6, Theorem 54.D]. Similar propositions are stated for set functions on relatively compact Baire and Borel sets with values in Banach spaces [3, p. 354, vector measures with finite variation] and more generally for set functions with values in complete locally convex spaces [4]. It has been asked by some persons if it is possible to reduce the assumption concerning completeness of the range space of vector-valued measure. We answer this question in the positive: Every Baire vector-valued measure with values in a metrisable and somewhat more general locally convex space X — not necessarily complete — can be extended uniquely to a regular Borel vector-valued measure with values in the same space X — more precisely in the closed convex cover of the values of the given Baire vector-valued measure. Some other results concerning extension and regularity of vector-valued measures are also added.

1. Extension of vector measures

Let T be a set, $\boldsymbol{D}$ a ring of subsets of T. Let X be a Hausdorff locally convex space with the topology defined by the system of continuous seminorms, $P=(p)$. Denote by $\check{X}$ and $\tilde{X}$ the quasi-completion and the completion of X [10], $\check{p}$ or $\tilde{p}$ being the extension of p to $\check{X}$ and $\tilde{X}$, respectively.

We shall make use of the following

Lemma 1. *If $\boldsymbol{m}$: $\boldsymbol{D}\to X$ is an additive set function, and if for every p in P there exists a positive finite measure v_p on $\boldsymbol{D}$ such that*

$$\lim_{v_p(A)\to 0} p(\boldsymbol{m}(A))=0, \quad A\in \boldsymbol{D},$$

then $\boldsymbol{m}$ is sigma additive [4, p. 506].

Let $\boldsymbol{N}$ be a set of positive finite subadditive and increasing set functions v defined on $\boldsymbol{D}$ with $v(\emptyset)=0$. Consider on $\boldsymbol{D}$ the uniform structure $t(\boldsymbol{N})$ defined by the family $(\mathrm{d}_v)_{v\in \boldsymbol{N}}$ of semi-distances defined by

$$\mathrm{d}_v(A, B)=v(A-B)+v(B-A), \quad A, B\in \boldsymbol{D}.$$

Now we can state the result that is easy to prove [4, p. 506].

Lemma 2. *Let* $\boldsymbol{D}_0 \subseteq \boldsymbol{D}$ *be a ring and* $\boldsymbol{m}$: $\boldsymbol{D}_0 \to X$ *a set function. If for every* p *in* P *there exists* v_p *in* $\boldsymbol{N}$ *such that*

$$\lim_{v_p(A)\to 0} p(\boldsymbol{m}(A)) = 0, \quad A \in \boldsymbol{D}_0,$$

and if either $\boldsymbol{m}$ *is additive or* $\boldsymbol{m}$ *is positive, subadditive and increasing, then* $\boldsymbol{m}$ *is uniformly continuous on* $\boldsymbol{D}_0$.

It follows, in particular, that every set function v in $\boldsymbol{N}$ is uniformly continuous on $\boldsymbol{D}_0$.

We shall need the "bounded" analogue of [4, p. 506, Theorem 2], interesting in itself.

Theorem 1. *Let* $\boldsymbol{D}_0 \subseteq \boldsymbol{D}$ *be a ring dense in* $\boldsymbol{D}$ *for the topology induced by* $t(\boldsymbol{N})$ *and* $\boldsymbol{m}$: $\boldsymbol{D}_0 \to X$ a bounded additive set function such that for every p *in* P *there exists* v_p *in* $\boldsymbol{N}$ *such that*

$$\lim_{v_p(A)\to 0} p(\boldsymbol{m}(A)) = 0, \quad A \in \boldsymbol{D}_0.$$

Then $\boldsymbol{m}$ *can be extended to a bounded additive set function* $\boldsymbol{m}_1$: $\boldsymbol{D} \to \check{X}$ *such that for every* p *in* P *we have*

$$\lim_{v_p(A)\to 0} \check{p}(\boldsymbol{m}_1(A)) = 0, \quad A \in \boldsymbol{D}.$$

Proof. Since $\boldsymbol{m}$ is uniformly continuous on $\boldsymbol{D}_0$, it can be uniquely extended to $\boldsymbol{m}_1$ on $\boldsymbol{D}$ with values in $\tilde{X}$. This extension is additive on $\boldsymbol{D}$ as can be easily shown. However, by assumption $\boldsymbol{m}$ is bounded on $\boldsymbol{D}_0$ and for each A in $\boldsymbol{D}$ we have $\lim \boldsymbol{m}(B) = \boldsymbol{m}_1(A) \in \tilde{X}$ when $\lim B = A$, $B \in \boldsymbol{D}_0$, in the uniform structure $t(\boldsymbol{N})$. Since $\boldsymbol{m}(\boldsymbol{D}_0) = \{\boldsymbol{m}(B)\colon B \text{ in } \boldsymbol{D}_0\}$ is a bounded subset of X, $\boldsymbol{m}_1(A)$ is a strict closure point of $\boldsymbol{m}(\boldsymbol{D}_0)$ in $\tilde{X}$ and hence $\boldsymbol{m}_1(A)$ is in the quasi-completion $\check{X}$ of X [10, §23].

Remark 1. Since every non-empty closed, convex subset of a locally convex space is the intersection of all closed semi-spaces containing it [11, II.9.2] we can see that $\boldsymbol{m}_1(\boldsymbol{D})$ is contained in the $\check{X}$-closed convex cover of $\boldsymbol{m}(\boldsymbol{D}_0)$. For every closed semi-space in $\check{X}$ containing $\boldsymbol{m}(\boldsymbol{D}_0)$ contains also $\boldsymbol{m}_1(\boldsymbol{D})$.

Corollary 1. *Let* $\boldsymbol{R}$ *be a ring and* $\boldsymbol{S}(\boldsymbol{R})$ *the sigma ring generated by* $\boldsymbol{R}$. *A vector measure* $\boldsymbol{m}$: $\boldsymbol{R} \to X$ *can be extended to a measure* $\boldsymbol{m}_1$: $\boldsymbol{S}(\boldsymbol{R}) \to \check{X}$ *if and only if for every* p *in* P *there exists a positive bounded measure* v_p *on* $\boldsymbol{R}$ *such that*

$$\lim_{v_p(A)\to 0} p(\boldsymbol{m}(A)) = 0, \quad A \in \boldsymbol{R}.$$

Since v_p is bounded on $\boldsymbol{R}$ it can be extended to a positive bounded measure μ_p on $\boldsymbol{S}(\boldsymbol{R})$ and $\boldsymbol{m}$: $\boldsymbol{R} \to X$ is also bounded on $\boldsymbol{R}$. In this case $\boldsymbol{D}_0 = \boldsymbol{R}$ and $\boldsymbol{D} = \boldsymbol{S}(\boldsymbol{R})$. In

[4, p. 507] it is proved that $\boldsymbol{m}_1$: $\boldsymbol{S}(\boldsymbol{R}) \to \tilde{X}$. The "only if" part follows from [4, Theorem 1].

Recall that many important locally convex spaces are quasi-complete, however not complete.

Corollary 2. *If X is sequentially complete, then the extension $\boldsymbol{m}_1$ takes its values in X* [9, Theorem 4.2].

For the set of those A in $\boldsymbol{S}(\boldsymbol{R})$ for which $\boldsymbol{m}_1(A)$ is in X forms a monotone system containig $\boldsymbol{R}$ [6, p. 27].

2. Regular vector-valued measures

Let S be a Hausdorff locally compact space. Recall that the class of relatively compact Baire sets in S is the delta ring generated by the compact sets which are G_δ, and is denoted $\boldsymbol{B}_a^r(S)$. The class of relatively compact Borel sets in S is the delta ring generated by the compact sets in S, and is denoted $\boldsymbol{B}^r(S)$. Clearly S is in $\boldsymbol{B}^r(S)$ if and only if S is compact. In this case $\boldsymbol{B}^r(S)$ is a sigma algebra. The class of Baire sets in S is the sigma ring generated by the compact G_δ sets, and is denoted $\boldsymbol{B}_a(S)$. The class of Borel sets in S is the sigma ring generated by the compact sets, and is denoted $\boldsymbol{B}(S)$. The class of weakly Borel sets in S is the sigma ring generated by the closed or equivalently open sets in S; it is a sigma algebra, and is denoted $\boldsymbol{B}_w(S)$. The Borel sets are precisely the sigma bounded weakly Borel sets [1, p. 181]. When S is metrisable, $\boldsymbol{B}_a(S) = \boldsymbol{B}(S)$, but there exist non-metrisable compact spaces S for which the equality holds [8]. Clearly $\boldsymbol{B}(S) = \boldsymbol{B}_w(S)$ if and only if S is sigma compact. Our terminology is drawn from [1], [3], [6].

Let $\boldsymbol{R}(S)$ be a ring of subsets of S and $\boldsymbol{m}$: $\boldsymbol{R}(S) \to X$ an additive set function. We say that $\boldsymbol{m}$ is regular if for each E in $\boldsymbol{R}(S)$ and every $d > 0$, for all p in P there exist a comact set C in $\boldsymbol{R}(S)$ and an open set O in $\boldsymbol{R}(S)$, $C \subset E \subset O$, such that we have $p(\boldsymbol{m}(H)) < d$ for every H in $\boldsymbol{R}(S)$ with $H \subset O - C$. Recall that if $\boldsymbol{m}$: $\boldsymbol{R}(S) \to X$ is additive and regular, then $\boldsymbol{m}$ is countably additive [4, p. 510, Theorem 3].

By a Baire vector measure on S we mean a vector measure $\boldsymbol{m}_a$: $\boldsymbol{B}_a(S) \to X$. By a Borel vector measure, a weakly Borel vector measure we mean a vector measure $\boldsymbol{m}$: $\boldsymbol{B}(S) \to X$, $\boldsymbol{m}_w$: $\boldsymbol{B}_w(S) \to X$, respectively.

In [4, p. 511] it is proved that every vector measure $\boldsymbol{m}_a^r$: $\boldsymbol{B}_a^r(S) \to X$ is regular. However, a slightly more general result is true.

Theorem 2. *Every Baire vector measure $\boldsymbol{m}_a$: $\boldsymbol{B}_a(S) \to X$ is regular.*

Proof. From [4, Theorem 1] we deduce that for every p in P there is a non-negative finite measure v_p^a on $\boldsymbol{B}_a(S)$ such that $v_p^a(B) \to 0$ implies $p(\boldsymbol{m}_a(B)) \to 0$ B in $\boldsymbol{B}_a(S)$. Since every v_p^a is a Baire measure on $\boldsymbol{B}_a(S)$, v_p^a is regular [1], therefore [4, Lemma 3] $\boldsymbol{m}_a$ is regular.

Theorem 3. *Let X be a normed space. Every Baire vector measure $\boldsymbol{m}_a$: $\boldsymbol{B}_a(S)\to X$ can be extended uniquely to the regular Borel vector measure $\boldsymbol{m}$: $\boldsymbol{B}(S)\to X$.*

The proof is based on the following.

Lemma 3. *If μ: $\boldsymbol{B}(S)\to R_+$ is a finite regular Borel measure and A is any set in $\boldsymbol{B}(S)$, then there exists a set B in $\boldsymbol{B}_a(S)$ such that*

$$\mathrm{d}_\mu(A, B)=\mu(A-B)+\mu(B-A)=0$$

and $\mu(A)=\mu(B)$ [1, p. 221].

Proof of Theorem 3. It is well-known [cf. e.g. 4] that there exists a non-negative finite Baire measure μ_a: $\boldsymbol{B}_a(S)\to R_+$ such that

$$\lim_{\mu_a(B)\to 0} \|\boldsymbol{m}_a(B)\|=0, \quad B \text{ in } \boldsymbol{B}_a(S).$$

The Baire measure μ_a can be extended uniquely to the non-negative finite regular Borel measure μ: $\boldsymbol{B}(S)\to R_+$ [1]. According to Lemma 3 for every A in $\boldsymbol{B}(S)$ there exists a set B in $\boldsymbol{B}_a(S)$ such that $\mathrm{d}_\mu(A, B)=\mu(A-B)+\mu(B-A)=0$, hence $\boldsymbol{B}(S)$ is dense in $\boldsymbol{B}_a(S)$ for the topology induced by $\mathrm{d}_\mu(A, B)$. From Theorem 1 we deduce that there exists a unique extension of $\boldsymbol{m}_a$ to a Borel vector measure $\boldsymbol{m}$: $\boldsymbol{B}(S)\to\check{X}$ such that

$$\lim_{\mu(A)\to 0} \|\boldsymbol{m}(A)\|=0, \qquad A\in\boldsymbol{B}(S),$$

and $\boldsymbol{m}$ is regular because μ is regular [4, Lemma 3]. Further, according to Lemma 3, if A is in $\boldsymbol{B}(S)$, there is a set B in $\boldsymbol{B}_a(S)$ such that $\mathrm{d}_\mu(A, B)=0$, hence $\boldsymbol{m}(A-B)=\boldsymbol{m}(B-A)=0$ and so $\boldsymbol{m}(A)=\boldsymbol{m}(B)=\boldsymbol{m}_a(B)$ and thus the element $\boldsymbol{m}(A)$ belogs to X, that is $\boldsymbol{m}$: $\boldsymbol{B}(S)\to X$.

Proposition 1. *Let X be a normed space. Every (restricted) Baire vector measure $\boldsymbol{m}_a^r$: $\boldsymbol{B}_a^r(S)\to X$ can be extended uniquely to the regular (restricted) Borel vector measure $\boldsymbol{m}^r$: $\boldsymbol{B}^r(S)\to X$.*

Proof. If A is in $\boldsymbol{B}^r(S)$, there is a compact set K such that $A\subset K$. Then A belongs to $K\cap\boldsymbol{B}^r(S)=\boldsymbol{B}^r(K\cap S)$. $\boldsymbol{B}^r(K\cap S)$ is a sigma ring of subsets of K and we may go on as in proving Theorem 3 and obtain the unique regular Borel extension $\boldsymbol{m}_K^r$: $\boldsymbol{B}(K\cap S)\to X$. Then we put

$$\boldsymbol{m}^r(A)=\boldsymbol{m}_K^r(A)$$

Then $\boldsymbol{m}^r$ is unambiguously defined, $\boldsymbol{m}^r(A)$ belongs to X and $\boldsymbol{m}^r$ extends $\boldsymbol{m}_a^r$. In [4, Theorem 5] it is proved that $\boldsymbol{m}^r$ takes its values in $\check{X}=\tilde{X}$.

The preceding theorem remains to be true if X is metrisable, $P=(p_k)$ being a countable family of continuous seminorms definig the topology in X.

Theorem 4. *Let X be a metrisable locally convex space, $P=(p_k)$. Every Baire vector measure $\boldsymbol{m}_a$: $\boldsymbol{B}_a(S)\to X$ can be extended in a unique way to a regular Borel vector measure $\boldsymbol{m}$: $\boldsymbol{B}(S)\to X$.*

Proof. For every p_k there is a finite non-negative Baire vector measure μ_a^k: $\boldsymbol{B}_a(S)\to R_+$ such that

$$\lim_{\mu_a^k(B)\to 0} p_k(\boldsymbol{m}_a(B))=0, \qquad B\in \boldsymbol{B}_a(S).$$

Denote by μ_k the unique regular Borel extension of μ_a^k, by $\boldsymbol{m}$ the unique regular Borel extension of $\boldsymbol{m}_a$, $\boldsymbol{m}$: $\boldsymbol{B}(S)\to\check{X}$ and note that

$$\lim_{\mu_k(A)\to 0} \check{p}_k(\boldsymbol{m}(A))=0, \qquad A\in \boldsymbol{B}(S).$$

This follows from Theorem 1, Lemma 1 and Lemma 2.

Define the measure μ on $\boldsymbol{B}(S)$ by the relation

$$\mu(A)=\sum_{k=1}^{\infty} 2^{-k}\frac{\mu_k(A)}{1+M_k(S)}, \qquad M_k(S)=\sup_{A\in\boldsymbol{B}(S)}\mu_k(A).$$

This is a finite non-negative regular Borel measure on $\boldsymbol{B}(S)$. For every Borel set A there is a Baire set B such that $d_\mu(A, B)=\mu(A-B)+\mu(B-A)=0$. Hence $\mu_k(A-B)=\mu_k(B-A)=0$ and so $\check{p}_k(\boldsymbol{m}(A-B))=\check{p}_k(\boldsymbol{m}(B-A))=0$, $k=1, 2, \dots$ and thus $\boldsymbol{m}(A)=\boldsymbol{m}(B)=\boldsymbol{m}_a(B)$. Hence $\boldsymbol{m}(A)$ belongs to X for every Borel set A in $\boldsymbol{B}(S)$.

Analogously we have the following.

Proposition 2. *Let X be a metrisable locally convex space. Every restricted Baire vector measure $\boldsymbol{m}_a^r$: $\boldsymbol{B}_a^r(S)\to X$ can be extended uniquely to a regular restricted Borel vector measure $\boldsymbol{m}^r$: $\boldsymbol{B}^r(S)\to X$.*

In [4, p. 511] it is stated that $\boldsymbol{m}^r$ has its values in the completion $\tilde{X}=\check{X}$ of X.

Let now X be a Hausdorff locally convex space with a system $P=(p)$ of continuous seminorms on X corresponding to a base of absolutely convex neighbourhoods of zero in X. We recall the following, see e.g. [7; 10]. The seminorms p in P form a directed set when we define $p\leq q$ for p, q in P if $p(x)\leq q(x)$ for all x in X. If $N_p=p^{-1}(0)$, then we denote by X_p the normed space which we obtain if in X/N_p we put, for the coset $\hat{x}_p$ (of x from X) in X_p,

$$\|\hat{x}_p\|_p=p(x) \quad \text{for } x \text{ in } \hat{x}_p \text{ in } X/N_p.$$

Then by setting

$$\hat{x}_p=f_{pq}(\hat{x}_q), \qquad p\leq q,$$

a continuous linear mapping f_{pq} from the normed space X_q onto the normed space X_p is defined since $\|f_{pq}(\hat{x}_q)\|_p=\|\hat{x}_p\|_p\leq\|\hat{x}_q\|_q$. Moreover for $p\leq q\leq r$ we have

$f_{pr}=f_{pq}\circ f_{qr}$. Hence a system (X_p, f_{pq}), p, $q\in P$ forms a projective system and we can form its projective limit

$$\hat{X}=\text{lim proj }(X_p, f_{pq})$$

as a subspace of the topological product $\prod_{p\in P} X_p$ consisting of all $\hat{x}=(x_p)$, $x_p\in X_p$, for which $f_{pq}(x_q)=x_p$ for all $p\leq q$. Assigning

$$\bar{x}\rightarrow\check{x}=(\hat{x}_p)$$

an isomorphism j of the space X onto the subspace $\bar{X}$ of $\hat{X}$ is defined. This is well defined since for $p\leq q$ we have $f_{pq}(\hat{x}_q)=\hat{x}_p$. Moreover to every $\bar{x}\in\bar{X}$ there corresponds some x in X for which $\bar{x}=(\hat{x}_p)=j(x)$. An isomorphism $x\in X\rightarrow(\hat{x}_p)\in\bar{X}$ is topological as follows from the fact that the topology of the space $\hat{X}$ is determined by the system of seminorms

$$\hat{P}=\{\|\ \|_p\circ f_p,\ p \text{ in } P\},$$

where f_p is a restriction to $\hat{X}$ of the projection of $\prod_{p\in P} X_p$ into X_p. So if $\hat{p}=\|\ \|_p\circ f_p$ and $\hat{x}\in\hat{X}$, then

$$\hat{p}(\hat{x})=(\|\ \|_p\circ f_p)(\hat{x})=\|f_p(\hat{x})\|_p=\|\hat{x}_p\|_p=p(x)$$

for all x in X.

We can see that a Hausdorff locally convex space X is topologically isomorphic to the dense subspace $\bar{X}$ of the projective limit $\hat{X}$ of the normed spaces X_p, $p\in P$.

Let $\boldsymbol{R}$ be a ring of subsets of a set S and $\boldsymbol{l}$: $\boldsymbol{R}\rightarrow X$ an additive set function. By setting

$$\boldsymbol{l}_p(A)=\widehat{\boldsymbol{l}(A)}_p$$

an additive set function $\boldsymbol{l}_p$: $\boldsymbol{R}\rightarrow X_p$ is defined, for all p in P. Thus for each A in $\boldsymbol{R}$ the element $\boldsymbol{l}(A)$ of X may be identified with an element $(\boldsymbol{l}_p(A))_{p\in P}$ in $\bar{X}$ with $f_{pq}(\boldsymbol{l}_q(A))=\boldsymbol{l}_p(A)$, $p\leq q$, and we may write $\bar{l}(A)=(\boldsymbol{l}_p(A))_{p\in P}\equiv\boldsymbol{l}(A)$. Moreover, it is clear that if $\boldsymbol{l}$ is countably additive the $\boldsymbol{l}_p$ is countably additive for all p in P. So every additive (countably additive) set function $\boldsymbol{l}$: $\boldsymbol{R}\rightarrow X$ gives a family $(\boldsymbol{l}_p)_{p\in P}$ of the additive (countably additive) set functions every $\boldsymbol{l}_p$ taking its values in the normed space X_p. For all p in P we have

$$p(\boldsymbol{l}(A))=\|\boldsymbol{l}_p(A)\|_p=\hat{p}(\bar{l}(A)).$$

We can now state the following.

Theorem 5. *Let X be a Hausdorff locally convex space. Every Baire vector measure $\boldsymbol{m}_0$: $\boldsymbol{B}_a(S)\rightarrow X$ can be extended uniquely to a regular Borel vector measure $\hat{\boldsymbol{m}}$: $\boldsymbol{B}(S)\rightarrow\hat{X}$.*

Proof. For A in $\boldsymbol{B}_a(S)$ we have $\boldsymbol{m}_a(A) \equiv (\boldsymbol{m}_{ap}(A))_{p \in P}$ Since X_p are the normed spaces, according to Theorem 3 every $\boldsymbol{m}_{ap}$: $\boldsymbol{B}_a(S) \to X_p$ can be extended uniquely to a regular Borel vector measure $\boldsymbol{m}_p$: $\boldsymbol{B}(S) \to X_p$. Define $\hat{\boldsymbol{m}}(A) = (\boldsymbol{m}_p(A))_{p \in P}$, A in $\boldsymbol{B}(S)$. We must show that $\hat{\boldsymbol{m}}(A)$ belongs to $\hat{X}$. We have to prove that $f_{pq}(\boldsymbol{m}_q(A)) = \boldsymbol{m}_p(A)$ for $p \leq q$ and A in $\boldsymbol{B}(S)$. Now $\boldsymbol{m}_p$ and $\boldsymbol{m}_q$ are both regular Borel vector measures and so are $f_{pq}(\boldsymbol{m}_q)$ because f_{pq} are continuous as mappings from X_q onto X_p. Since $f_{pq}(\boldsymbol{m}_q(B)) = \boldsymbol{m}_p(B)$ for all B in $\boldsymbol{B}_a(S)$, the uniqueness of the extension of a Baire vector measure to a regular Borel vector measure implies that $f_{pq}(\boldsymbol{m}_q(A)) = \boldsymbol{m}_p(A)$ for all A in $\boldsymbol{B}(S)$. So indeed, the mapping $A \to \hat{\boldsymbol{m}}(A)$ takes its values in $\hat{X}$. Since

$$\hat{p}(\hat{\boldsymbol{m}}(A)) = \|\boldsymbol{m}_p(A)\|_p = \|f_p(\hat{\boldsymbol{m}}(A))\|_p$$

it follows that $\hat{\boldsymbol{m}}$: $\boldsymbol{B}(S) \to \hat{X}$ so defined is a regular Borel vector measure extending uniquely the Baire measure $\boldsymbol{m}_a$.

Analogously we can obtain the following.

Proposition 3. *Let X be a Hausdorff locally convex space. Every restricted Baire vector measure $\boldsymbol{m}_a^r$: $\boldsymbol{B}_a^r \to X$ can be extended uniquely to a regular restricted Borel vector measure $\hat{\boldsymbol{m}}^r$: $\boldsymbol{B}^r(S) \to \hat{X}$.*

Remark 2. In [4, p. 511] it is stated that $\boldsymbol{m}^r$: $\boldsymbol{B}^r(S) \to \check{X}$, $\check{X}$ being the completion of X.

Remark 3. According to Remark 1 $\hat{\boldsymbol{m}}(\boldsymbol{B}(S))$ is contained in the closed convex cover of $\bar{\boldsymbol{m}}_a(\boldsymbol{B}_a(S))$ in $\hat{X}$ not only in $\check{X}$.

For the weakly Borel sets we have the following.

Theorem 6. *Let X be a Hausdorff locally convex space. Every regular Borel measure $\boldsymbol{m}$: $\boldsymbol{B}(S) \to X$ can be extended uniquely to a regular weakly Borel measure $\hat{\boldsymbol{m}}_w$: $\boldsymbol{B}_w(S) \to \hat{X}$.*

The proof is based on the

Lemma 4. *If μ_w is a regular weakly Borel measure on S and A is any weakly Borel set, then there exists a Borel set B (even Baire sigma compact set) such that*

$$\mu_w(A - B) + \mu_w(B - A) = 0.$$

This can be proved in the same way as for a regular Borel measure [1, p. 221].

Futher every positive regular Borel measure can be extended uniquely to a regular weakly Borel measure [2].

Now the proof of our theorem proceeds as that of Theorem 5.

Corollary. *Every Baire vector measure $\boldsymbol{m}_a$: $\boldsymbol{B}_a(S) \to X$ can be extended uniquely to a regular weakly Borel vector measure $\hat{\boldsymbol{m}}_w$: $\boldsymbol{B}_w(S) \to \hat{X}$.*

Remark 4. The fact that a regular Borel vector measure extending the Baire vector measure $\boldsymbol{m}_a$ has its values in the same space as $\boldsymbol{m}_a$ is useful, for example, in connection with tensor products of regular Borel vector measures [5] and regular weakly Borel vector measures. Recall that, in general, the tensor product of locally convex spaces fails to be complete even if the factors are complete.

As for Theorem 1 it is useful when the space is not complete bur only quasi-complete, for example, the space of operators on a Banach space with the strong operator topology is quasi-complete.

Remark 5. Modifying the proof of Theorem 5 we could prove that if X is the locally convex projective limit of metrisable locally convex spaces X_q, $q \in Q$ in the sense of [10], then every Baire vector measure $\boldsymbol{m}_a$: $\boldsymbol{B}_a(S) \to X$ can be extended uniquely to a regular Borel vector measure $\boldsymbol{m}$: $\boldsymbol{B}(S) \to X$. It is clear that the space X needs not be, in general, metrisable. The case of an arbitrary locally convex space X has remained open if we do not assume that X is quasicomplete.

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О ПРОДОЛЖЕНИИ ВЕКТОРНЫХ МЕР БЭРА

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Резюме

В работе доказаны некоторые утверждения о продолжении векторных аддитивных функций множества на кольцо из кольца плотного в последнем в некоторой равномерной структуре. При помощи этих результатов доказано следующее утверждение. Каждая векторная мера Бэра со значениями в метрическом даже общем отделимом ликально выпуклом пространстве (никакая полнота не предполагается) может быть продолжена однозначно в регулярную векторную меру Бореля со значениями в том же пространстве, а именно в замкнутой выпуклой оболочке значений данной векторной меры Бэра.