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DUAL POINT-PARTITION NUMBER OF COMPLEMENTARY GRAPHS

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ABSTRACT. Dual point-partition number of a graph G with respect to a hereditary property P is the maximum number of disjoint point-induced subgraphs contained in G such that any subgraph does not have the property P. In this article, problems of the Nordhaus-Gaddum type for the dual point-partition number are investigated.

Introduction

In this paper all graphs are finite, undirected, and without loops or multiple lines. The notation and the terminology follow [4]. The point set of a graph G is denoted by $V(G)$, the line set of a graph G is denoted by $E(G)$. The complement of a graph G is denoted by $\bar{G}$. For a subset V of $V(G)$ (E of $E(G)$), the symbol $\langle V\rangle$ ($\langle E\rangle$) denotes the subgraph of the graph G induced by $V(E)$, respectively. The symbol $\{u, v\}$ means the line with endpoints u, v and $N_G(u) = \{w \in V(G): \{u, w\} \in E(G)\}$ for an arbitrary point u in the graph G. The maximum degree $\Delta(G)$ of a graph G is defined as $\max\{\deg_G(v): v \in V(G)\}$. A graph G is bipartite if its set of points $V(G)$ can be partitioned into two sets U, W such that every line in $E(G)$ has one endpoint in U and the other in W. We shall write $G = (U, W)$ accordingly. A subset E of $E(G)$ is said to be independent if two arbitrary lines of E are not adjacent. For any real x we denote the lower and upper integer part of x by $\lfloor x \rfloor$ and $\lceil x \rceil$, respectively. Let $\mathbb{Z}$ be the set of all integers and consider the closed interval with real endpoints a, b. Define $[a, b]$ as $\langle a, b\rangle \cap Z$. The symbol $\mathbb{N}$ means the set of all non-negative integers.

Let $\mathcal{G}$ denote the set of all graphs. Define as in [1] a subset P of $\mathcal{G}$ to be a property if $K_0, K_1 \in P$; P is hereditary if $G \in P$, $H \subset G$ implies $H \in P$ and nontrivial if $P \neq \mathcal{G}$. A graph G has a property P if $G \in P$. The dual point-partition number of a graph G with respect to a special hereditary property P (we shall denote this by $\bar{\chi}_P(G)$) was defined in [2] as the maximum number of disjoint

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point-induced subgraphs contained in G such that any subgraph does not have the property $P(\bar{\chi}_P(G) = 0$ if $G \in P)$. Define a *$\bar{P}$-partition* of $V(G)$ as a partition $V_1, V_2, \ldots, V_r$ of $V(G)$ such that $\langle V_i \rangle \notin P$ for $i \in [1, r]$. Further we denote $\max\{m \in \mathbb{N}: K_{m+1} \in P\}$ by $c(P)$ for any nontrivial hereditary property P.

In this article we observe the following hereditary properties:

$O(k) = \{G$: if H is a connected subgraph of G, then $|V(H)| < k + 2\}$,

$S(k) = \{G: \Delta(G) < k + 1\}$,

$Q(k) = \{G$: the length of any path in the graph G is at most $k\}$.

In 1956 Nordhaus and Gaddum [6] proved the following famous result for chromatic number of a graph G and of its complement $\bar{G}$:

$$2\sqrt{n} \leqq \chi(G) + \chi(\bar{G}) \leqq n + 1$$

$$n \leqq \chi(G) . \chi(\bar{G}) \leqq \lfloor (n+1)^2/4 \rfloor, \text{ where } |V(G)| = n.$$

Since then the relations of some parameters between a graph and its complement are continuosly discussed, they are called Nordhaus-Gaddum problems (see [3, 5]). In this paper, Nordhaus-Gaddum problems are investigated for dual point-partition numbers. The upper and lower bounds for $\bar{\chi}_P(G) + \bar{\chi}_{P'}(\bar{G})$, $\bar{\chi}_P(G) . \bar{\chi}_{P'}(\bar{G})$ are given, where $P, P' \in \{O(k), Q(k), S(k)\}$.

Assume P is a nontrivial hereditary property. The following assertions are obtained directly from preceding definitions:

— $\bar{\chi}_P(K_n) = \lfloor n/(c(P) + 2) \rfloor$,

— if H is a subgraph of a graph G, then $\bar{\chi}_P(H) \leqq \bar{\chi}_P(G)$,

— if G is a graph with n points, then $\bar{\chi}_P(G) \leqq \lfloor n/(c(P) + 2) \rfloor$,

— if $k \in \mathbb{N}$, $P' \in \{O(k), Q(k), S(k)\}$, then P' is a nontrivial hereditary property and $c(P') = k$. Let k be a non-negative integer. It is easy to see that if G is a graph, $P \in \{O(k), Q(k), S(k)\}$, $V_1, V_2, \ldots, V_r$ is a $\bar{P}$-partition of $V(G)$, then there exists a $\bar{P}$-partition $W_1, W_2, \ldots, W_r$ of $V(G)$ such that $|W_i| = k + 2$ for $i \in$ $\in [1, r - 1]$.

Preparatory Results

Lemma 1. *Let $G = (U, W)$ be a bipartite graph with $2n$ points, $n \geqq 3$, such that $|U| = |W|$, $\deg_G(u) \geqq \lceil n/2 \rceil$ for each point u belonging to U and $G \neq 2K_{q,q}$ for any $q \in \mathbb{N}$. Then a path P of length n in G exists.*

Proof. Let E be an independent set of lines in G with maximal number of elements. Suppose that $U_1 \subset U$, $W_1 \subset W$ are the sets of points of G such that $U_1 \cup W_1 = V(\langle E \rangle)$. If the set $U - U_1$ is empty, then we easily form the desired path. So we suppose that $U - U_1 \neq \emptyset$. Consider a path P' in G with maximal length, say s, such that the initial point of P' belongs to $U - U_1$ such that the lines of P' are alternately not in and in E. Assume $s < n$. Let V be the endpoint of P'. Distinguish the possibilities:

1. The number s is even. It is easy to see that a point $w \in W$ adjacent to v satisfying $\{w, v\} \notin P'$ exists. Then the path P' may be extended, a contradiction.

2. The number s is odd. Then the point v belongs to $W - W_1$. Hence define the set E' as $E - E(P') \cup E(P') - E$. Evidently, E' is the independence set of lines in G and $|E'| = |E| + 1$, which contradicts with maximality of E. The proof is complete.

Lemma 2. *Let $P, P' \in \{Q(k), O(k), S(k)\}$. Then the following statements hold:*

(1) *if G is a graph with $2k + 2$ points, $G \in P$, then $\bar{\chi}_{P'}(\bar{G}) = 1$.*

(2) *if G is a graph with $k + 2$ points, $G \in O(k)$, then $\bar{\chi}_{O(k)}(\bar{G}) = 1$.*

Proof. Evidently, (2) holds. It is a routine matter to verify (1) for $P, P' \in \{Q(k), O(k), S(k)\}$ satisfying $P \neq Q(k)$ or $P' \neq Q(k)$. Now we prove that $G \in Q(k)$ implies $\bar{\chi}_{Q(k)}(\bar{G}) = 1$ for a graph G with $2k + 2$ points. Use the induction on the number k. Evidently, Lemma 2 holds for $k = 0$ 1. Assume $\bar{\chi}_{Q(l)}(\bar{G}) = 1$ for arbitrary graph G, $G \in Q(l)$, having $2l + 2$ points, $l < k$. Consider the graph G with $2k + 2$ points, $G \in Q(k)$. If a path of length k in G exists, then the graph $\bar{G}$ contains a path of length $k + 1$ by Lemma 1 (the graph $\bar{G}$ contains a subgraph fulfilling the assumptions of Lemma 1). In the other case, remove two arbitrary different points from G resulting in a graph G'. The path of length at least k in $\bar{G}'$ exists by the induction hypothesis. Suppose the length of each path in $\bar{G}$ is less than $k + 1$. Then the graph G contains a path with length $k + 1$ by Lemma 1, which contradicts to $G \in Q(k)$. The proof is complete.

Lemma 3. *Let $P, P' \in \{O(k), Q(k), S(k)\}$, and let G be a graph with n points. Then the following statements hold:*

(1) *if $G \in P$, then $\bar{\chi}_{P'}(\bar{G}) \geqslant \lfloor n/(k + 2) \rfloor - 1$,*

(2) *if $G \in O(k)$, then $\bar{\chi}_{O(k)}(\bar{G}) = \lfloor n/(k + 2) \rfloor$,*

(3) *if $G \in S(k)$, $k \in \{0, 1\}$, then $\bar{\chi}_{S(k)}(\bar{G}) = \lfloor n/(k + 2) \rfloor$,*

(4) *if $G \in S(2)$, $n \neq 4, 5$, then $\bar{\chi}_{S(2)}(\bar{G}) = \lfloor n/4 \rfloor$.*

Proof. We prove only the case (1). Analogously we can proceed the other cases. Use the induction on the number n. It is easy to see that (1) holds for $n \leqslant 2k + 3$. Now suppose that (1) holds for every graph H with m points, $m < n$, belonging to P. Consider a graph G with n points such that $G \in P$. Since $n \geqslant 2k + 4$, we can take a subset W of $V(G)$ with $2k + 2$ points. By Lemma 2, we have a subset U of W with $k + 2$ points such that $\langle \bar{U} \rangle \notin P'$. Further consider the graph $G' = G - U$. By induction hypothesis, we have $\bar{\chi}_{P'}(\bar{G}') \geqslant \lfloor n/(k + 2) \rfloor - 2$. The fact $\bar{\chi}_{P'}(\bar{G}) \geqslant \bar{\chi}_{P'}(\bar{G}') + 1$ concludes the proof. In the case (4), the induction starts from $n = 9$. Considering all possibilities we can prove (4) for $n \in \{1, 2, 3, 6, 7, 8\}$.

Corollary 1. *If G is a graph with n points, $P, P' \in \{S(k), Q(k), O(k)\}$, $\bar{\chi}_P(G) = 1$, then*

(1) $\bar{\chi}_{P'}(\bar{G}) \geqslant \lfloor n/(k+2) \rfloor - 2$,
(2) *if* $P = P' = O(k)$, *then* $\bar{\chi}_{P'}(\bar{G}) \geqslant \lfloor n/(k+2) \rfloor - 1$,
(3) *if* $P = P' = S(k)$, $k \in \{0, 1, 2\}$, *then* $\bar{\chi}_{P'}(\bar{G}) \geqslant \lfloor n/(k+2) \rfloor - 1$.

Proof. The assumption $\bar{\chi}_P(G) = 1$ implies the existence of W, $W \subset V(G)$, such that $|W| = k + 2$ and $\langle W \rangle \notin P$. Denote $G' = G - W$. Then $G' \in P$ and $|V(G')| = n - k - 2$. Now we employ Lemma 3 to obtain the desired results. The proof is complete.

Lemma 4. *If* $m \in \mathbb{N}$, $m \geqslant 2$, *G is a graph with* $m.(k+2)$ *points*, $G \in S(k)$ *and there exists* $U \subset V(G)$ *such that* $|U| = m$ *and* $\bigcap_{u \in U} N_G(u) = \emptyset$, *then* $\bar{\chi}_{S(k)}(\bar{G}) = m$.

Proof. We use the induction on the number m. It is easy to verify Lemma 4 for $m = 2$. Let m be at least 3. As the induction hypothesis assume $\bar{\chi}_{S(k)}(\bar{G}) = l$ for each graph G with $l.(k+2)$ points, $G \in S(k)$, for which there exists $W \subset V(G)$ with l points, $l < m$, satisfying $\bigcap_{w \in W} N_G(w) = \emptyset$. Consider a graph G with $m.(k+2)$ points such that $G \in S(k)$ and consider $U \subset V(G)$ with m points satisfying $\bigcap_{u \in U} N_G(u) = \emptyset$. The assumption $G \in S(k)$ implies $w \in U$ with property $|N_{\bar{G}}(w)| > (m-1).(k+2)$ exists. Denote the set $U - w$ by U' and denote $\bigcap_{u \in U'} N_G(u)$ by M. Assume $|M| = s$. Notice that $0 \leqslant s \leqslant k$ and then $|N_{\bar{G}}(w) - (U \cup M)| > m(k+2) - 2k > k + 1$. It follows from $\bigcap_{u \in U} N_G(u) = \emptyset$ that the fact $v \in M$ implies $\{w, v\} \in E(\bar{G})$. Consider a subset V of $N_{\bar{G}}(w) - (U \cup M)$ such that $|V| = k + 1 - s$. Define the set V_m as $V \cup \{w\} \cup M$. It is simple that $\langle V_m \rangle \notin S(k)$. Further denote $G - V_m$ by G'. By the induction hypothesis it is $\bar{\chi}_{S(k)}(\bar{G}') = m - 1$. Since $\bar{\chi}_{S(k)}(\bar{G}) \geqslant \bar{\chi}_{S(k)}(\bar{G}') + 1$, the proof is concluded.

Lemma 5. *If G is a graph with* $m.(k+2)$ *points*, $m \geqslant 2$, *then the following conditions are equivalent*:

(1) $\bar{\chi}_{S(k)}(G) = 0 = \bar{\chi}_{S(k)}(\bar{G}) - m + 1$,
(2) $\Delta(G) \leqslant k$ *and if U is a subset of* $V(G)$ *such that* $|U| = m$, then $\bigcap_{u \in U} N_G(u) \neq \emptyset$.

Proof. Using Lemma 4 it is easy to prove that (2) follows from (1). Conversely, suppose that (2) holds. The equality $\bar{\chi}_{S(k)}(G) = 0$ follows immediately from $\Delta(G) \leqslant k$. We have $\bar{\chi}_{S(k)}(\bar{G}) \geqslant m - 1$ by Lemma 3. To get the contradiction suppose $\bar{\chi}_{S(k)}(\bar{G}) = m$. Let $V_1, V_2, \ldots, V_m$ be a $\overline{S(k)}$-partition of $V(G)$. Hence $v_i \in V_i$ such that $\deg_{\overline{\langle V_i \rangle}}(v_i) \geqslant k + 1$ exists for $i \in [1, m]$. Consider the set of points $U = \{v_1, v_2, \ldots, v_m\}$. Then $\bigcap_{i=1}^{m} N_G(v_i) \neq \emptyset$ by (2). Take any x from $\bigcap_{i=1}^{m} N_G(v_i)$. The line $\{v_i, x\}$ does not belong to $E(\bar{G})$ for $i \in [1, m]$. An index $j \in [1, m]$ such that $x \in V_j$ exists, too. Since $\langle V_j \rangle \notin S(k)$, $|V_j| = k + 2$, it is clear that the line $\{v_j, x\}$ belongs to $E(\bar{G})$, a contradiction. The proof is complete.

Lemma 6. *If $k \geqslant 3$, $m \geqslant k$, G is a graph with $m \,.\, (k + 2)$ points, $G \in S(k)$, then $\bar{\chi}_{S(k)}(\bar{G}) = m$.*

Proof. Again we know that $\bar{\chi}_{S(k)}(\bar{G}) \geqslant m - 1$ by Lemma 3. Assume $\bar{\chi}_{S(k)}(\bar{G}) = m - 1$. Consider two different points x, y of $V(G)$. Let $|N_G(x) \cap N_G(y)|$ be equal j. Lemma 5 implies:

(1) every point of $(V(G)$ must be adjacent to some point of $N_G(x) \cap N_G(y)$,

(2) $j \geqslant m - 1$.

We can obtain the inequality $j \geqslant 2$ by $m \geqslant k \geqslant 3$ and by (2). Let s denote the number of lines joining a point of $V(G)$ and a point of $N_G(x) \cap N_G(y)$. By (1), the inequality $s \geqslant m(k + 2) - j/2$ holds. On the other hand the maximum number of points of G which may be adjacent to points of $N_G(x) \cap N_G(y)$ is $j(k - 2) + 2$. Hence $j(k - 2) \geqslant m(k + 2) - 2 - j/2$. The fact $G \in S(k)$ implies $j \leqslant k$. Then $k(k - 2) \geqslant m(k + 2) - 2 - k/2$ which is impossible. So $\bar{\chi}_{S(k)}(\bar{G}) = m$ and the proof is complete.

Corollary 2. *If G is a graph with n points, $n \geqslant k \,.\, (k + 2)$, $k \geqslant 3$, $G \in S(k)$, then $\bar{\chi}_{S(k)}(\bar{G}) = \lfloor n/(k + 2) \rfloor$.*

Corollary 3. *If G is a graph with n points, $n \geqslant (k + 1) \,.\, (k + 2)$, $k \geqslant 3$, $\bar{\chi}_{S(k)}(G) = 1$, then $\bar{\chi}_{S(k)}(\bar{G}) \geqslant \lfloor n/(k + 2) \rfloor - 1$.*

Bounds

Theorem 1. *If G is a graph with n points, P, $P' \in \{O(k), Q(k), S(k)\}$, then*

(1) $\lfloor n/(k + 2) \rfloor \leqslant \bar{\chi}_{O(k)}(G) + \bar{\chi}_{O(k)}(\bar{G})$,

(2) $\lfloor n/(k + 2) \rfloor - 1 \leqslant \bar{\chi}_P(G) + \bar{\chi}_{P'}(\bar{G})$,

(3) $\bar{\chi}_P(G) + \bar{\chi}_{P'}(\bar{G}) \leqslant 2 \,.\, \lfloor n/(k + 2) \rfloor$,

(4) if $k \geqslant 3$, $0 \notin \{\bar{\chi}_{S(k)}(G)\} \cup \{\bar{\chi}_{S(k)}(\bar{G})\}$, $[1, \lfloor n/(k + 2) \rfloor - k] \cap \{\bar{\chi}_{S(k)}(G), \bar{\chi}_{S(k)}(\bar{G})\} \neq \emptyset$ *or* $n \geqslant 2k \,.\, (k + 2)$, *then*

$$\lfloor n/(k + 2) \rfloor \leqslant \bar{\chi}_{S(k)}(G) + \bar{\chi}_{S(k)}(\bar{G}),$$

(5) *if* $k \in \{0, 1, 2\}$ *and* $k \neq 2$ *or* $n \notin \{4, 5\}$, *then*

$$\lfloor n/(k + 2) \rfloor \leqslant \bar{\chi}_{S(k)}(G) + \bar{\chi}_{S(k)}(\bar{G}).$$

Proof. The case (3) is evident. Further we prove the case (1). The proof of the cases (2), (5) is similar. Suppose $\bar{\chi}_{O(k)}(G) = 0$. Lemma 3 gives

$\bar{\chi}_{O(k)}(\bar{G}) = \lfloor n/(k+2) \rfloor$ implying the desired result. Suppose $\bar{\chi}_{O(k)}(G) = p > 0$. Consider an $O(k)$-partition $V_1, V_2, \ldots, V_p$ of the point set of the graph G such that $|V_i| = k+2$ for $i \in [1, p-1]$. Since $\bar{\chi}_{O(k)}(\langle V_p \rangle) = 1$, we have $\bar{\chi}_{O(k)}(\langle \bar{V}_p \rangle) \geqslant \geqslant \lfloor n/(k+2) \rfloor - p$ by Corollary 1. Then $\bar{\chi}_{O(k)}(\bar{G}) \geqslant \lfloor n/(k+2) \rfloor - p$, which completes the proof of (1). Now we prove (4). Assume $\bar{\chi}_{S(k)}(G) = p$, $p \in [1, \lfloor n/(k+2) \rfloor - k]$. Consider a $S(k)$-partition $V_1, V_2, \ldots, V_p$ of $V(G)$ satisfying $|V_i| = k+2$ for $i \in [1, p-1]$. Then $|V_p| \geqslant (k+1).(k+2)$. As $\bar{\chi}_{S(k)}(\langle V_p \rangle) = 1$, we obtain $\bar{\chi}_{S(k)}(\langle \bar{V}_p \rangle) \geqslant \lfloor n/(k+2) \rfloor - p$ by Corollary 3. The inequality $\bar{\chi}_{S(k)}(\bar{G}) \geqslant \geqslant \bar{\chi}_{S(k)}(\langle \bar{V}_p \rangle)$ implies the desired result. Hence assume $\bar{\chi}_{S(k)}(G) = p$, $\bar{\chi}_{S(k)}(\bar{G}) = q$, $p \geqslant \lfloor n/(k+2) \rfloor - k$, $q \geqslant \lfloor n/(k+2) \rfloor - k$. If follows from $n \geqslant 2k.(k+2)$ that $p + q \geqslant \lfloor n/(k+2) \rfloor$. The proof of Theorem 1 is complete.

Theorem 2. *If G is a graph with n points, P, $P' \in \{O(k), Q(k), S(k)\}$ and $0 \notin \{\bar{\chi}_P(G)\} \cup \{\bar{\chi}_{P'}(\bar{G})\}$, then*

(1) $\lfloor n/(k+2) \rfloor - 1 \leqq \bar{\chi}_{O(k)}(G) . \bar{\chi}_{O(k)}(\bar{G})$,

(2) $\lfloor n/(k+2) \rfloor - 2 \leqq \bar{\chi}_P(G) . \bar{\chi}_{P'}(\bar{G})$,

(3) $\bar{\chi}_P(G) . \bar{\chi}_{P'}(\bar{G}) \leqq (\lfloor n/(k+2) \rfloor)^2$,

(4) *if* $k \geqq 3$, $[1, \lfloor n/(k+2) \rfloor - k] \cap \{\bar{\chi}_{S(k)}(G), \bar{\chi}_{S(k)}(\bar{G})\} \neq \emptyset$ *or* $n \geqq 2k.(k+2)$, *then* $\lfloor n/(k+2) \rfloor - 1 \leqq \bar{\chi}_{S(k)}(G) . \bar{\chi}_{S(k)}(\bar{G})$,

(5) *if* $k \in \{0, 1, 2\}$, *then* $\lfloor n/(k+2) \rfloor - 1 \leqq \bar{\chi}_{S(k)}(G) . \bar{\chi}_{S(k)}(\bar{G})$.

Proof. The case (3) is evident. We now verify the case (1) only. The proof of the other cases is similar. Assume $\bar{\chi}_{O(k)}(G) = r$, $\lfloor n/(k+2) \rfloor = a$. We have found $\bar{\chi}_{O(k)}(\bar{G}) \geqq a - r$ according to Theorem 1 and $\bar{\chi}_{O(k)}(\bar{G}) \geqq 1$ on the other hand. So $\bar{\chi}_{O(k)}(\bar{G}) \geqq \max\{a - r, 1\}$ for $r \in [1, a]$. Then $\bar{\chi}_{O(k)}(G) . \bar{\chi}_{O(k)}(\bar{G}) \geqq \geqq \max\{a.r - r^2, r\}$ where $r \in [1, a]$. Hence $\bar{\chi}_{O(k)}(G) . \bar{\chi}_{O(k)}(\bar{G}) \geqq \min \max_{r \in [1, a]} \{a.r - r^2, r\} = a - 1$. The proof is complete.

Theorem 3. *If $k \in \mathbb{N}$; P, $P' \in \{O(k), Q(k), S(k)\}$, then there exist $n \in \mathbb{N}$ and a graph G with n points such that the sum $\bar{\chi}_P(G) + \bar{\chi}_{P'}(\bar{G})$ attains the corresponding bounds of Theorem 1.*

Proof. By Theorem 1, introduce the best lower and upper bounds of $\bar{\chi}_P(G) + \bar{\chi}_{P'}(\bar{G})$ for P, $P' \in \{O(k), Q(k), S(k)\}$, $k \in \mathbb{N}$. Distinguish the following possibilities:

1. Suppose $k \in \mathbb{N}$; P, $P' \in \{O(k), S(k), Q(k)\}$. The corresponding upper bound is given in Theorem 1.3. Define $n = 2l(k+2)$ for an arbitrary $l \in \mathbb{N}$, $G = 2lK_{k+2}$.
2. Let $k \in \mathbb{N}$, $P = P' = O(k)$. The lower bound is introduced in Theorem 1.1. K_n is the desired graph for $n \in \mathbb{N}$.

3. Assume $k \in \mathbb{N}$, $k \geqq 2$, $P \in \{O(k), Q(k), S(k)\}$, $P' = Q(k)$. The lower bound is determined by Theorem 1.2. The number n is defined as $l(k+2)$ for $l \in \mathbb{N}$, $l \geqq 2$ and $G = \bar{H}_{(l-1)(k+2)+k+1}$.

4. If $k \in \{0, 1\}$; P, P' are as in 3., then n and G are defined as in 2, because $O(k) = Q(k) = S(k)$.

5. Suppose $k \in \mathbb{N}$, $k \geqq 3$, $P = P' = S(k)$. The lower bound is given in Theorem 1.2. Show that $\mathscr{S}(k) = \{G\colon \bar{\chi}_{S(k)}(G) + \bar{\chi}_{s(k)}(\bar{G}) = \lfloor n/(k+2) \rfloor - 1\}$ is finite. Assume to get a contradiction that $\mathscr{S}(k)$ is infinite. By Theorem 1.4 the following condition holds:
$0 \in \{\bar{\chi}_{S(k)}(G)\} \cup \{\bar{\chi}_{S(k)}(\bar{G})\}$ or ($n < 2k(k+2)$ and $\bar{\chi}_{S(k)}(H) > \lfloor n/(k+2) \rfloor - k$) for $H \in \{G, \bar{G}\}$). Hence $\mathscr{S}(k) = \mathscr{S}_1(k) \cup \mathscr{S}_2(k)$ where $\mathscr{S}_1(k) = \{G\colon G \in \mathscr{S}(k), \bar{\chi}_{S(k)}(G) = 0\}$, $\mathscr{S}_2(k) = \mathscr{S}(k) - \mathscr{S}_1(k)$. If $G \in \mathscr{S}_2(k)$, then $|V(G)| < 2k(k+2)$. Then $\mathscr{S}_1(k)$ is infinite. Then a graph G with at least $(k+1)(k+2)$ points belonging to $\mathscr{S}_1(k)$ exists. Consider $G' \subset G$ with $m(k+2)$ points for $m \in \mathbb{N}$, $m > k$. It is clear that $G' \in S(k)$. Hence by Lemma 4 we have:

$$(\forall U \subset V(G'))(|U| = m \rightarrow \bigcap_{u \in U} N_{G'}(u) \neq \emptyset.$$

Consider U a subset of $V(G')$ with m elements. Then there is $w \in N_{G'}(u)$ which implies $\deg_G(w) > k$. It is a contradiction and $\mathscr{S}(k)$ is finite. The graph $2K_k$ belongs to $\mathscr{S}(k)$. It is the open problem to charakterize the set $\mathscr{S}(k)$.

6. Let $k \in \{0, 1, 2\}$, $P = P' = S(k)$. The lower bound is given in Theorem 1.5. The number n and the graph G are defined as in 2 (in the case $k = 2$, $n \in \in \{4, 5\}$ $\bar{\chi}_{S(k)}(C_n) + \bar{\chi}_{S(k)}(\bar{C}_n) = 0 = \lfloor n/(k+2) \rfloor - 1$).

7. Suppose $k \in \mathbb{N}$, $P = O(k)$, $P' = S(k)$. The lower bound is given in Theorem 1.2. The number n is defined as $2k$ and $G = 2k_k$. Denote $\mathscr{S}'(k) = \{G\colon \bar{\chi}_{O(k)}(G) + \bar{\chi}_{S(k)}(\bar{G}) = \lfloor n/(k+2) \rfloor - 1\}$. The characterization of $\mathscr{S}'(k)$ is an open problem.

REFERENCES

[1] BROWN, J. I.—CORNEIL, D. G.: On Generalized Graphs Colorings. Journal of Graph Theory 11, 1987, 87—99.

[2] CHARTRAND, G.—GELLER, D.—HEDETNIEMI, S.: Graphs with Forbidden Subgraphs. Journal of Combinatorial Theory 10, 1971, 12—41.

[3] CHARTRAND, G.—MITCHEM, J.: Graphical Theorems of Nordhaus and Gaddum. Recent Trends in Graph Theory, Lectures Notes in Mathematics 186, 1971, 55—61.

[4] HARARY, F.: Graph Theory. Addison-Wesley. Reading. 1969.
[5] Xu, Shao-ji: Some Parameters of Graph and its Complement. Discrete Mathematics 65, 1987, 197—207.
[6] NORDHAUS, E. A.—GADDUM, J. W.: On Complementary Graphs. Amer. Math. Monthly 63, 1956, 175—177.

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