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A note on the intersection multiplicity


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A NOTE ON THE INTERSECTION MULTIPLICITY

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Let $V$ and $W$ be irreducible algebraic projective varieties of the projective space $\mathbb{P}^n_k$ over any (algebraically closed) field $k$ with associated prime ideals $p_V$, $p_W \subset k[X_0, \ldots, X_n]$. Let $C$ be an irreducible component of the intersection $V \cap W$ with the property $\dim(C) = \dim(V) + \dim(W) - n$ (that is to say, $V$ and $W$ are cutting proper in $C$). Let $i(V, W; C)$ denotes a multiplicity of the component $C$ in $V \cap W$ (see [7]). Without loss of generality we can assume, that $W$ is a complete intersection (see [3], [5]). That means algebraically

$$p_W = (F_1, \ldots, F_d)$$

and $\dim(W) = n - d$.

This implies that

$$R = : (k[X_0, \ldots, X_n]/p_V)_{p_C \cdot (k[X_0, \ldots, X_n]/p_V)}$$

is a Noetherian local ring of dimension $d$ with a maximal ideal $p_C \cdot R$ and $p_C \cdot R$-primary ideal $p_W \cdot R$ generated by a system of parameters $f_1, \ldots, f_d$ ($f_i = : F_i \cdot R$ for any $1 \leq i \leq d$). The Samuel’s Theorem of reduction ([5], Chap. II, §7, b) says, that

$$i(V, W; C) = e_0(n, R)$$

$(n = : (f_1, \ldots, f_d)$ and $e_0(n, R)$ denotes the leading coefficient of the Hilbert—Samuel polynomial $l(R/n^n)$ $n \gg 0$). The last equation shows that the intersection multiplicity can be counted by the multiplicity in local algebra.

There exists a various methods to count this multiplicity (see [1], [3], [6], [8]). In this note we give further method of the calculation of $i(V, W; C)$. I would like to thank Prof. W. Vogel (Halle) for helpful discussions.

Let $(A, m)$ be a commutative Noetherian local ring of dimension $d$. For any ideal $a$ of $A$, $\dim(a)$ means the dimension of the ring $A/a$. Let $\text{Ass}(a)$ denotes the set of all the prime ideals which belong to any irredundant primary decomposition of an ideal $a$. $\text{Assi}(a)$ indicate the set of all the isolated prime ideals of $\text{Ass}(a)$. Let $\text{Assh}(a)$ be defined by

$$\text{Assh}(a) = : \{p \in \text{Ass}(a) ; \dim(p) = \dim(a)\}$$

and $U(a)$ indicate the intersection of all the primary ideals whose associated prime
ideals belong to Assh (a). At last Ui (a) denotes the intersection of all the primary ideals, whose associated prime ideals belong to Assi (a).

Let $q$ be an ideal generated by a system of parameters $a_1, ..., a_d$ in $A$. Let us construct the following ideals $q_i$ for all $k = 0, ..., d$ by

$$
q_0 = (0)
$$

$$
q_i = (a_k) + U_i(q_{i-1})
$$

**Lemma 1.** For all $k = 0, ..., d$ it holds

(i) $(a_1, ..., a_k) \subseteq q_k$

(ii) Assi $((a_1, ..., a_k)) = $ Assi $(q_k)$.

**Proof.** Part (i) is clear. (ii) will follow by induction on $k$. If $k = 0$ it is obvious. Assume that

$\text{Assi} ((a_1, ..., a_k)) = $ Assi $(q_k)$, $0 < k \leq d - 1$

and $p \in \text{Assi} ((a_1, ..., a_{k+1}))$. Then we have

$p \supseteq p' \in \text{Assi} ((a_1, ..., a_k))$,

so by the induction hypothesis

$p \supseteq (a_{k+1}, U_i(q_i)) = q_{k+1}$.

If $p \in \text{Assi} (q_{k+1})$, then

$p' \supseteq p_0 \supseteq q_{k+1}$, so

$p \supseteq p_0 \supseteq (a_1, ..., a_{k+1})$.

This is a contradiction, since $p \in \text{Assi} ((a_1, ..., a_{k+1}))$, so it holds $p \in \text{Assi} (q_{k+1})$. Conversely let $p \in \text{Assi} (q_{k+1})$. Then $p \supseteq (a_1, ..., a_{k+1})$. If $p \notin \text{Assi} ((a_1, ..., a_{k+1}))$, then

$p \supseteq p' \supseteq U_i(q_i)$

(by the induction hypothesis). This implies

$p \supseteq p' \supseteq (a_{k+1}, U_i(q_i)) = q_{k+1}$,

which is a contradiction. So we have $p \in \text{Assi} ((a_1, ..., a_{k+1}))$. This completes the proof.

**Corollary 1.** Assh $((a_1, ..., a_k)) = $ Assh $(q_k)$ for all $k = 0, ..., d$.

Let us return to a ring $R$ and an ideal $n = (f_1, ..., f_d)$ of $R$. Construct ideals $n_k$ for all $k = 0, ..., d$ by process (1). We are going to show, that

$i(V, W; C) = l(R/n_k)$.

The following Proposition is implied by [1] and [2].
**Proposition 1.** Let $q_k$ be ideals which are constructed from any ideal $q$ generated by a system of parameters $a_1, \ldots, a_d$ (of any local Noetherian ring $A$) by

$$q_0 = (0)$$

$$q_k = (a_k) + U(q)_{i-1}$$

for all $k = 0, \ldots, d$. Then

(i) $(a_1, \ldots, a_k) \subseteq q_k$

(ii) $\text{Assh}(q_k) = \{ p \in \text{Assh}((a_1, \ldots, a_k)) ; h(p) = k \}$

(iii) $e_0(q, A) = l(A/q_d)$

for all $k = 0, \ldots, d$.

**Remark 1.** Part (iii) shows one of the methods to count multiplicity.

**Proposition 2.** For all $k = 0, \ldots, d$ there holds

$$q^i_k \subseteq q_k.$$  

**Proof.** We use the induction on $k$. The case $k = 0$ is easy. Let now $q^i_k \subseteq q_k,$ $0 < k \leq d - 1$. Since $\text{Assh}(q_k) \subseteq \text{Assh}(q^i_k)$, the induction hypothesis implies

$$Ui(q^i_k) \subseteq U(q^i_k) \subseteq U(q_k).$$

Then we have $(a_{k+1}) + Ui(q^i_k) \subseteq (a_{k+1}) + U(q_k)$, so

$$q^i_{k+1} \subseteq q_{k+1}.$$  

**Corollary 2.** $q \subseteq q'_d \subseteq q_d$.  

**Corollary 3.** $e_0(q, A) = l(A/q'_d)$ if and only if $q_d = q'_d$.

**Proposition 3.** There is a local Noetherian ring $(A, m)$ and an ideal $q$ generated by a system of parameters of $A$, for which

$q_d \neq q'_d$.  

**Proof.** Let $k$ be any (algebraically closed) field. Let us observe the ideal $a = (X_2, X_1X_3, X_1X_4)$ of the polynomial ring $Q = k[X_1, X_2, X_3, X_4]$. Let

$$A = Q(X_1, x_3, x_4)/a \cdot Q(X_1, x_3, x_4).$$

Since the ideal $(X_3, X_1 + X_4, a) = (X_2, X_3, X_1X_4, X_1 + X_4)$ is $(X_1, X_2, X_3, X_4)$-primary and $\dim(A) = 2$, the ideal $(X_3, X_1 + X_4) \cdot A$ is generated by a system of parameters. We count immediately

$$(0) = (X_1 + X_2) \cap (X_2, X_3, X_4) \cdot A$$

$q_1 = (X_1, X_2, X_3) \cdot A$  

$q'_1 = ((X_1, X_2, X_3) \cap (X_2, X_3, X_4)) \cdot A$  

$q_2 = (X_1, X_2, X_3, X_4) \cdot A$  

$q'_2 = (X_2, X_3, X_4, X_1 + X_4) \cdot A$  

We are going to give a sufficient condition for equality of ideals $q_d$ and $q'_d$ now.
**Theorem 1.** Let \((A, m)\) be a commutative Noetherian local ring of dimension \(d\) and \(q\) an ideal of \(A\) generated by a system of parameters \(a_1, \ldots, a_d\). Denote \(q_i(q')\) the ideals, which are constructed from \(q\) by processes (2) ((1)). If \(h(p) = k\) for all \(p \in \text{Assh} ((a_1, \ldots, a_k))\) and \(k = 0, \ldots, d\), then

(i) \(\text{Assh} (q_k) = \text{Assh} (q'_k) = \text{Assh} ((a_1, \ldots, a_k))\)

(ii) \(U(q_k) = U(q'_k)\)

for all \(k = 0, \ldots, d\).

**Proof.** (i) follows from Proposition 1 and Corollary 1. (ii) follows by induction on \(k\). If \(k = 0\), it is obvious. Let now \(U(q_k) = U(q'_k) 0 < k \leq d - 1\).

Proposition 2 implies \(U(q'_{k+1}) \subseteq U(q_{k+1})\). Take an element \(x \in U(q_k)\). Then \(x \in U(q'_k)\) by the induction hypothesis, so

\[
q'_k: x \notin p \quad \text{for all} \quad p \in \text{Assh} (q'_k).
\]

Hence by the assumption of the theorem we have

\[
U(i(q'_k): x \notin p \quad \text{for all} \quad p \in \text{Assh} (q'_{k+1}),
\]

so \(x \in U(q'_{k+1})\).

We have proved that \(U(q_k) \subseteq U(q'_{k+1})\). Hence we get

\[
q_{k+1} = (a_{k+1}) + U(q_k) \subseteq U(q'_{k+1})
\]

and, using part (i),

\[
U(q_{k+1}) \subseteq U(q'_{k+1})\]

**Corollary 4.** On the assumptions in Theorem 1 it holds

\[
q_d = q'_d, \quad \text{so} \quad l(A/q_d) = l(A/q'_d).
\]

Let us return to an ideal \(n = (f_1, \ldots, f_d)\) of a ring

\[
R = (k[X_0, \ldots, X_n]/p^\infty)_{pc} \cdot (k[X_0, \ldots, X_n]/p^\infty)
\]
again. Since the ring \(R\) is equidimensional ([4], Chap. II, §3), it obviously satisfies the conditions of Theorem 1, which yields

**Theorem 2.** \(i(V, W; C) = l(R/n'_d)\).

Proposition 1 and Theorem 2 show, that both processes (1) and (2) are applicable for calculating of the intersection multiplicity.
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ЗАМЕТКА О КРАТНОСТИ ПЕРЕСЕЧЕНИЯ

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Резюме

Пусть i(V, W; C) обозначает кратность компоненты C в пересечении неприводимых алгебраических проективных многообразий V и W. В силу теоремы Самюэля об редукции i(V, W; C) = e_d(n, R) для определенного параметрического идеала n в определенном локальном нетеровом кольце R(e_d(n, R) — старший член многочлена Гильберт—Самюэля I(R/n), т.н. кратность идеала n в кольце R). В работе вводится один практический метод для вычисления e_d(n, R).