Mathematica Slovaca

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Mathematica Slovaca, Vol. 44 (1994), No. 1, 95--98

Persistent URL: http://dml.cz/dmlcz/129016

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A RELATIONSHIP BETWEEN INTERSECTION CONDITIONS AND POROSITY CONDITIONS FOR LOCAL SYSTEMS

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(Communicated by Ladislav Mišík)

ABSTRACT. In the paper, we prove that if a local system S fulfils an intersection condition, then the porosity of the sets from $S(x)$ at x cannot be too large.

First, we recall some definitions from B. Thomson's book [1]. Let A be a subset of the real line $\mathbb{R}$. Then by $|A|^e$ we denote the *exterior Lebesgue measure of* A and we put $x + A = \{x + a;\ a \in A\}$, $-A = \{-a;\ a \in A\}$, $A^+ = A \cap [0,\infty)$ and $A^- = A \cap (-\infty, 0]$.

By the *right-hand porosity of* A *at* x we mean

$$p^+(A,x) = \limsup_{h\to 0^+} \frac{\lambda(A,x,x+h)}{h},$$

where $\lambda(A,x,x+h)$ denotes the length of the largest open subinterval of $(x,x+h)\setminus A$.

Similarly, the *left-hand porosity of* A *at* x is defined as

$$p^-(A,x) = \limsup_{h\to 0^+} \frac{\lambda(A,x-h,x)}{h}.$$

By a *local system*, we mean a family $\mathbb{S} = \{\mathbb{S}(x);\ x \in \mathbb{R}\}$ of nonempty collections of subsets of the real line such that, for any $x \in \mathbb{R}$,

(i) $\{x\} \notin \mathbb{S}(x)$,
(ii) if $S \in \mathbb{S}(x)$, then $x \in S$,
(iii) if $S \in \mathbb{S}(x)$ and $S' \supset S$, then $S' \in \mathbb{S}(x)$,
(iv) if $S \in \mathbb{S}(x)$ and $\delta > 0$, then $S \cap (x-\delta, x+\delta) \in \mathbb{S}(x)$.

AMS Subject Classification (1991): Primary 26A99, 26A21.
Key words: Local system, Intersection condition, Porosity.

We say that a local system $\mathbb{S}$ satisfies the *parametric intersection condition of the form* $S_x \cap S_y \cap [x - \lambda(y-x),\, y + \lambda(y-x)] \neq \emptyset$ if, for each choice of sets $\{S_x;\ x \in \mathbb{R}\}$ with $S_x \in \mathbb{S}(x)$, there is a positive function δ on $\mathbb{R}$ such that

$$S_x \cap S_y \cap [x - \lambda(y-x),\, y + \lambda(y-x)] \neq \emptyset \quad \text{whenever} \quad 0 < y - x < \min\{\delta(x), \delta(y)\}.$$

It is easy to see that if $\mathbb{S}$ satisfies the above-mentioned intersection condition, then there exists a sequence $\{E_n\}$ of subsets of the real line such that

$$\mathbb{R} = \bigcup_{n=1} E_n \quad \text{and} \quad S_x \cap S_y \cap [x - \lambda(y-x),\, y + \lambda(y-x)] \neq \emptyset$$

whenever $x, y \in E_n$.

THEOREM. *Let $\mathbb{S}$ be a local system such that:*

(a) $\mathbb{S}(x) = \{x + S;\ S \in \mathbb{S}(0)\}$ *for every* x,

(b) $S^+ \cup -S^+ \in \mathbb{S}(0)$ *and* $S^- \cup -S^- \in \mathbb{S}(0)$ *whenever* $S \in \mathbb{S}(0)$,

(c) $\mathbb{S}$ *satisfies the parametric intersection condition*

$S_x \cap S_y \cap [x - \lambda(y-x),\, y + \lambda(y-x)] \neq \emptyset$ *for some* $\lambda > -\frac{1}{2}$.

Then, for any $x \in \mathbb{R}$ *and* $S \in \mathbb{S}(x)$,

(d) $p^+(S, x) \leq \dfrac{\frac{1}{2} + \lambda}{1 + \lambda}$ *and* $p^-(S, x) \leq \dfrac{\frac{1}{2} + \lambda}{1 + \lambda}$.

Remark. It is easy to observe that if a system $\mathbb{S}$ is filtering (i.e. $S_1 \cap S_2 \in \mathbb{S}(x)$ for $S_1, S_2 \in \mathbb{S}(x)$), then condition (b) from Theorem is equivalent to:

(b') $-S \in \mathbb{S}(0)$ whenever $S \in \mathbb{S}(0)$.

Proof of Theorem. Suppose that conditions (a) – (c) hold, but condition (d) is not true. We can assume that there is a set $S \in \mathbb{S}(0)$ such that $p^+(S, 0) > \dfrac{\alpha + \lambda}{1 + \lambda}$ for some α with $\dfrac{1}{2} < \alpha < \min\{1, 1 + \lambda\}$ (in the case $p^-(S, 0) > \dfrac{\alpha + \lambda}{1 + \lambda}$ the proof is analogous). Hence, there are sequences $\{a_n\}$, $\{b_n\}$ of positive numbers, converging to zero, such that $b_{n+1} < a_n < b_n$, $S \cap \bigcup_{n=1}^{\infty} (a_n, b_n) = \emptyset$ and $\dfrac{a_n}{b_n} = 1 - \dfrac{\alpha + \lambda}{1 + \lambda} = \dfrac{1 - \alpha}{1 + \lambda}$. Put $S_0 = S^+ \cup -S^+$ and $S_x = x + S_0$ for $x \in \mathbb{R}$. Conditions (a) and (b) guarantee that $S_x \in \mathbb{S}(x)$, and

$$S_x \cap \left[\bigcup_{n=1}^{\infty} (x + a_n,\, x + b_n) \cup \bigcup_{n=1}^{\infty} (x - b_n,\, x - a_n)\right] = \emptyset. \tag{1}$$

From (c) it follows that we can find a sequence $\{E_n\}$ of subsets of the real line such that $\mathbb{R} = \bigcup_{n=1}^{\infty} E_n$ and, for every positive integer n,

$$S_x \cap S_y \cap \big[x - \lambda(y-x),\, y + \lambda(y-x)\big] \neq \emptyset \qquad \text{whenever} \quad x, y \in E_n\,. \tag{2}$$

Let n_0 be a positive integer for which $|E_{n_0}|^e > 0$, and let $x_0 \in E_{n_0}$ be an exterior density point of E_{n_0}. Then there is a positive number δ such that

$$\frac{|E_{n_0} \cap (x_0, t)|^e}{t - x_0} > 2 - 2\alpha \qquad \text{for} \quad t \in (x_0, x_0 + \delta)\,. \tag{3}$$

Let k_0 be a positive integer with $b_{k_0} < (1+\lambda)\delta$. Then (3) implies that the set $E_{n_0} \cap \left(x_0 + \frac{2-2\alpha}{1+\lambda} b_{k_0},\, x_0 + \frac{1}{1+\lambda} b_{k_0}\right)$ is nonempty. Choose any point y_0 from this set. Then $y_0 \in E_{n_0}$ and $\frac{2-2\alpha}{1+\lambda} b_{k_0} < y_0 - x_0 < \frac{1}{1+\lambda} b_{k_0}$.

Hence,

$$y_0 - x_0 + \lambda(y_0 - x_0) < b_{k_0}\,, \tag{4}$$

and

$$\frac{1}{2}(y_0 - x_0) > \frac{1-\alpha}{1+\lambda} b_{k_0} = a_{k_0} > (1-\alpha)(y_0 - x_0) > -\lambda(y_0 - x_0)\,. \tag{5}$$

Inequalities (4) and (5) imply that

$$y_0 - b_{k_0} < x_0 - \lambda(y_0 - x_0) < x_0 + a_{k_0} < y_0 - a_{k_0} < y_0 + \lambda(y_0 - x_0) < x_0 + b_{k_0}\,,$$

and, consequently,

$$\big[x_0 - \lambda(y_0 - x_0),\, y_0 + \lambda(y_0 - x_0)\big] \subset (y_0 - b_{k_0},\, y_0 - a_{k_0}) \cup (x_0 + a_{k_0},\, x_0 + b_{k_0})\,.$$

This inclusion and (1) show that

$$S_{x_0} \cap S_{y_0} \cap \big[x_0 - \lambda(y_0 - x_0),\, y_0 + \lambda(y_0 - x_0)\big] = \emptyset\,.$$

However, the last condition contradicts (2) because $x_0, y_0 \in E_{n_0}$. This ends the proof.

COROLLARY 1. *If a local system $\mathbb{S}$ fulfils conditions* (a) – (c) *of Theorem, then any set S from $\mathbb{S}(x)$ is neither right-hand strongly porous at x nor left-hand strongly porous at x (i.e. $p^+(S,x) < 1$ and $p^-(S,x) < 1$).*

COROLLARY 2. *If a local system* $\mathbb{S}$ *fulfils conditions* (a) – (b) *of Theorem and an intersection condition* $S_x \cap S_y \cap [x,y] \neq \emptyset$, *then* $p^+(S,x) \leq \frac{1}{2}$ *and* $p^-(S,x) \leq \frac{1}{2}$ *for any* $S \in \mathbb{S}(x)$.

COROLLARY 3. *If a local system* $\mathbb{S}$ *fulfils conditions* (a) – (b) *of Theorem and*

(c') $\mathbb{S}$ *satisfies the parametric intersection condition*

$$S_x \cap S_y \cap \big[x - \lambda(y-x),\, y + \lambda(y-x)\big] \neq \emptyset \text{ for each } \lambda > -\frac{1}{2},$$

then any set S *from* $\mathbb{S}(x)$ *is neither right-hand porous at* x *nor left-hand porous at* x (*i.e.* $p^+(S,x) = p^-(S,x) = 0$).

REFERENCES

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Received July 17, 1992

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