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Mathematica Slovaca, Vol. 27 (1977), No. 1, 3--7

Persistent URL: http://dml.cz/dmlcz/129020

MATHEMATICA SLOVACA

VOLUME 27 1977 NUMBER 1

ON DISJOINT COVERING OF GROUPS BY THEIR COSETS

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A system of residue classes of the additive group of integers Z

(1) $$a_i(\text{mod } n_i) \quad i=1, \ldots, k(\geq 2)$$

is said to be disjoint covering if every integer belongs to exactly one of the residue classes (1).

Among other facts the following properties of disjoint covering systems are known (see [1]):

A. $\sum_{i=1}^{k} \frac{1}{n_i} = 1$;

B. $(n_i, n_j) > 1$ for any $i, j = 1, \ldots, k$.

Our article is devoted to the study of the disjoint covering of any group by its cosets (an obvious generalization of the above mentioned problem), and properties analogous to A and B are shown. Further we show that for studying this problem it is sufficient to consider finite groups only.

I

Let G be a group, $G_1, \ldots, G_k$, $k>1$ be its subgroups (not necessary distinct) and $a_1, \ldots, a_k$ some elements of G.
The sequence

(2) $$a_1G_1, \ldots, a_kG_k$$

will be called a disjoint covering system (DCS) of G if every element of G belongs to exactly one coset of (2) (see [3]).

Remark. For any subgroup H and any element x of G the equality $Hx = x(x^{-1}Hx)$ holds and therefore every right coset of G is also a left coset. Hence it is sufficient to consider DCS consisting of left cosets only.

Denote by $[G: H]$ the index of the subgroup H in G.

Lemma 1. *Let G be a group, $G_1, \ldots, G_k$ its subgroups, $H = G_1 \cap \ldots \cap G_k$. Then*

$$[G: H] \leqq [G: G_1] \dots [G: G_k]\,.$$

Proof. Every coset xH can be written as $xG_1 \cap \dots \cap xG_k$. For every $i \in \{1, \dots, k\}$ there are $[G: G_i]$ possibilities for the choice of xG_i.

Theorem 1. *If* (2) *is a DCS of G, then the indices* $[G: G_i]$ $(i = 1, \dots, k)$ *are* finite.

Proof. Let k be the smallest natural for which there exists a DCS (2) of some group G in which $[G: G_r]$ is infinite for some r. Consider two possibilities:

a) there exist some $i, j \in \{1, \dots, k\}$ such that $[G_i: G_i \cap G_j]$ is infinite;
b) all the indices $[G_i: G_i \cap G_j]$ are finite.

a) Without loss of generality we can suppose $a_i = e$. Consider the sequence

(3) $$G_i \cap a_1 G_1, \dots, G_i \cap a_j G_j, \dots, G_i \cap a_k G_k\,.$$

Obviously $G_i \cap a_j G_j = G_i \cap G_j \neq \emptyset$. The index $[G_i: G_i \cap G_j]$ is infinite, therefore the non-empty elements of (3) form a DCS of G_i with less than k members and with at least one infinite index. This is a contradiction.

b) Denote $H = G_1 \cap \dots \cap G_k$. At first we prove that $[G_s: H]$ is finite for all $s = 1, \dots, k$. Obviously $H = (G_1 \cap G_s) \cap \dots \cap (G_k \cap G_s)$ holds. By Lemma 1 we get

$$[G_s: H] \leqq [G_s: G_1 \cap G_s] \dots [G_s: G_k \cap G_s]\,.$$

All factors on the right-hand side are finite and hence $[G_s: H]$ is finite, too.

For every $i \in \{1, \dots, k\}$ the coset $a_i G_i$ consists of $[G_i: H]$ cosets by H and hence the group G consists of $\sum_{i=1}^{k} [G_i: H]$ cosets by H, *i.e.*

(4) $$[G: H] = \sum_{i=1}^{k} [G_i: H]\,.$$

From (4) it follows that $[G: H]$ is finite (all summands on the right are finite). Then from

(5) $$[G: H] = [G: G_r]\,.\,[G_r: H] \quad \text{(see [2])}$$

we get a contradiction.

Theorem 2. *Let* (2) *be a DCS of a group G; denote* $n_i = [G: G_i]$. *Then*

$$\sum_{i=1}^{k} \frac{1}{n_i} = 1\,.$$

Proof. Denote $H = G_1 \cap \dots \cap G_k$. By Theorem 1 and Lemma 1 the index $[G: H]$ is finite. Then the indices $[G_1: H], \dots, [G_k: H]$ are also finite and similarly as in the proof of Theorem 1 we can obtain (4). Thus we get

$$1=\sum_{i=1}^{k}\frac{[G_i:H]}{[G:H]}=\sum_{i=1}^{k}\frac{[G_i:H]}{[G:G_i]\,.\,[G_i:H]}=\sum_{i=1}^{k}\frac{1}{n_i}\,.$$

Remark. Theorem 2 obviously generalizes the property A of the DCS of Z.

Theorem 3. *Let* (2) *be a DCS of* G; *let* $[G: G_i]=n_i$. *Then*

$$(n_i, n_j)>1 \quad \text{for all } i, j=1, \ldots, k\,.$$

Proof. Denote $n_{ij}=[G: G_i \cap G_j]$. By Lemma 1

(6) $$n_{ij} \leqq n_i n_j \quad \text{holds.}$$

From $[G: G_i \cap G_j]=[G: G_i]\,.\,[G_i: G_i \cap G_j]$ it follows that $n_i \mid n_{ij}$. Similarly $n_j \mid n_{ij}$. Suppose $(n_i, n_j)=1$; then from the preceeding relations it follows that $n_i n_j \mid n_{ij}$ and hence by (6) we get

$$n_{ij}=n_i n_j\,.$$

Every coset by $G_i \cap G_j$ is an intersection of a coset by G_i and a coset by G_j. There are at most $n_i n_j$ such intersections; however there exists at least one empty intersection ($a_i G_i \cap a_j G_j=\emptyset$) and so $n_{ij}<n_i n_j$, which is a contradiction.

Remark. Theorem 3 generalizes the property B.

II

Definition. *If* (2) *is a DCS of* G *and* $n_i=[G: G_i]$, *then the sequence*

(7) $$n_1, \ldots, n_k$$

will be called the indexing of (2).

In this part we shall show that to study completely the problem of a disjoint covering of groups by their cosets it is sufficient to consider the finite groups only, because if (7) is the indexing of a DCS of some group then there exists also a finite group having a DCS with the indexing (7).

Lemma 2. *Let* G *be a group and* K *its subgroup with* $[G: K]=n<\infty$. *Then there exists such normal subgroup* H *of* G *that* $H \subset K$ *and* $[G: H] \leqq n^n$.

Proof. By [2] there exist exactly $[G: N_G(K)]$ conjugate subgroups to K, where $N_G(K)$ is the normalizer of K in G. Let $K_1, \ldots, K_m$ be the list of all distinct conjugate subgroups to K in G. Obviously $m \leqq n$. For all $i=1, \ldots, m$ $[G: K]=[G: K]$ holds. Now, denote

(8) $$H=K_1 \cap \ldots \cap K_m\,.$$

By Lemma 1 we have $[G: H] \leqq n^m$. It remains to prove that H is a normal subgroup of G. But from (8) for any $x \in G$ we have

$$x^{-1}Hx = (x^{-1}K_a x) \cap \ldots \cap (x^{-1}K_m x) .$$

On the right we have m distinct conjugate subgroups to K and hence all of them. Thus we get $x^{-1} Hx = H$ for every $x \in G$.

Theorem 4. *Let the finite sequence* (7) *be the indexing of a DCS of a group G. Then there exists also a finite group F (of the order at most n^n, where $n = n_1 \ldots n_k$) having a DCS with the indexing* (7).

Proof. Denote $K = G_1 \cap \ldots \cap G_k$; by Lemma 1 $[G: K] \leqq n$. By Lemma 2 there exists such a normal subgroup H of G that $H \subset K$ and $[G: H] \leqq n^n$. Denote $F = G/H$, $F_i = G_i/H$, $b_i = a_i H$. The order of the factor-group F is equal to $[G: H]$, hence it is not greater than n^n. Obviously

$$b_1 F_1, \ldots, b_k F_k$$

is a DCS of the group F with the indexing (7).

Theorem 5. *Let the finite sequence* (7) *be the indexing of a DCS* (2) *of an Abelian group G. Then there exists also a finite Abelian group F (of the order at most $n = n_1 \ldots n_k$) having a DCS with the indexing* (7).

The proof is similar to that of Theorem 4 (put $H = K$).

Remark. Theorem 5 is an analogon of the following known fact. Denote Z_m the additive group of integers modulo m. Now, if a DCS of Z with moduli $n_1, \ldots, n_k$ exists, then there exists a DCS with the same moduli also for Z_m, where $m = [n_1, \ldots, n_k]$. It can be shown that in Theorem 5 the product $n_1 \ldots n_k$ cannot be in general replaced by $[n_1, \ldots, n_k]$.

From Theorems 4 and 5 we get:

Theorem 6. *The following problems are recursively solvable*

a) *whether to a given sequence* (7) *of naturals there exists a group G having a DCS with the indexing* (7);

b) *whether to a given sequence* (7) *of naturals there exists an Abelian group G having a DCS with the indexing* (7).

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Received February 28, 1975

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О ТОЧНО ПОКРЫВАЮЩИХ СИСТЕМАХ ГРУПП СОСТОЯЩИХ ИЗ СМЕЖНЫХ КЛАССОБ

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Резюме

Пусть G группа, $a_1, \ldots, a_p$ $(k \geqq 2)$ элементы G и $G_1, \ldots, G_k$ подгруппы G. Конечная последовательностъ (2) смежых классов группы G называется точно покрывающей системой группы G, если всякий элемент G принадлежит одному и только одному классу из (2). Индексированием системы (2) называется конечная последовательность (7), где $n_i = [G : G_i]$ обозначает индекс G_i в G. Доказывается, что

1) все n_i натуральные числа и $\sum_{i=1}^{k} \frac{1}{n_i} = 1$;
2) среди элементов последовательности (7) не существуют два взаимно простых элемента;
3) проблема, ядляется-ли данная конечная последовательность натуральных чисел индексированием любой точно покрывающей системы, алгоритмически разрешима.