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Measurability of real functions defined on the product of metric spaces


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MEASURABILITY OF REAL FUNCTIONS
DEFINED ON THE PRODUCT OF METRIC
SPACES

GRAŻYNA KWIECIŃSKA

Let \((T, d, \mathcal{H}, \lambda)\) be a complete metric space with a metric \(d\), with a \(\sigma\)-finite \(G_b\)-regular complete measure \(\lambda\) defined over a \(\sigma\)-field \(\mathcal{H}\) of subsets of \(T\).

Denote by \(\lambda^*\) the outer measure corresponding to \(\lambda\).

Let \(\mathcal{A}\) be a family of \(\lambda\)-measurable sets with nonempty (1) interiors of a positive and finite measure \(\lambda\), the boundaries of which are of \(\lambda\) measure zero.

**Definition 1.** The sequence \(\{I_k\}_{k=1}^{\infty} \subseteq \mathcal{A}\) is said to converge to the point \(t_0 \in T\) iff \(t_0 \in \text{Int}(I_k)\) for \(k = 1, 2, \ldots\) and the sequence of diameters \(\delta(I_k)\) converge to zero as \(k\) approaches infinity.

This will be denoted by \(I_k \to t_0\).

Let us note that according to the definition due to Bruckner ([1], p. 30) the pair \((\mathcal{A}, \to)\) forms a differentiation basis for the space \((T, d, \mathcal{H}, \lambda)\).

**Definition 2.** Let \(A \subseteq T\) and \(t_0 \in T\). The upper (lower) bound of the set of numbers \(\lim_{k \to \infty} \frac{\lambda^*(A \cap I_k)}{\lambda(I_k)}\) taken from all the sequences \(I_k \to t_0\) (for which this limit exists) is called the upper (lower) external density of \(A\) at \(t_0\) with respect to \(\mathcal{A}\) and is denoted by \(D^*_u(t_0, A)(D^*_l(t_0, A))\).

If \(D^*_u(t_0, A) = D^*_l(t_0, A)\), then their common value is called the external density of \(A\) at \(t_0\) with respect to \(\mathcal{A}\) and is denoted by \(D^*(t_0, A)\).

If \(A \in \mathcal{H}\), then the respective external densities are called densities with respect to \(\mathcal{A}\) and denoted by \(D_u(t_0, A)\), \(D_l(t_0, A)\) and \(D(t_0, A)\) respectively.

A point \(t_0\) is called a density point of the set \(A\) with respect to \(\mathcal{A}\) if there exists a set \(B \in \mathcal{H}\) such that \(B \subseteq A\) and \(D(t_0, B) = 1\).

Assume that (2) the family \(\mathcal{A}\) is countable and for every \(t_0 \in T\) there is a sequence of sets \(\{I_k\}_{k=1}^{\infty}\) from \(\mathcal{A}\) converging to \(t_0\).

Moreover assume that
(3) \( \mathcal{A} \) has the density property, i.e. for every set \( A \subseteq T \) the \( \lambda \) measure of set 
\{ \( t \in A : D^\dagger(t, A) < 1 \) \} is equal to zero.

**Definition 3.** The function \( g : T \to \mathbb{R} \) is called approximately upper (lower) 
semicontinuous at the point \( t_0 \in T \) with respect to \( \mathcal{A} \) iff for every \( a \in \mathbb{R} \) if \( f(t_0) < a \) 
\( (f(t_0) > a) \), then there exists a set \( F \in \mathcal{H} \) such that \( F \subseteq \{ t \in T : f(t) < a \} \) 
\( (F \subseteq \{ t \in T : f(t) > a \}) \) and \( D(t_0, F) = 1 \).

A function that is simultaneously approximately lower and upper semicontinuous 
at \( t_0 \in T \) with respect to \( \mathcal{A} \) is called approximately continuous at \( t_0 \) with respect to \( \mathcal{A} \).

A function that is approximately continuous (approximately lower semicontinuous) (approximately upper semicontinuous)) in any point \( t_0 \in T \) with respect to \( \mathcal{A} \) is called approximately continuous (approximately lower semicontinuous) (approximately upper semicontinuous)) with respect to \( \mathcal{A} \).

**Lemma 1.** If the function \( g : T \to \mathbb{R} \) is \( \lambda \)-measurable, then \( g \) is \( \lambda \)-almost 
everywhere approximately continuous with respect to \( \mathcal{A} \).

**Proof.** Indeed, by Lusin's theorem for every positive \( \varepsilon \) there exists a closed set 
\( F \subseteq T \) such that the function \( g|_F \) is continuous and \( \lambda(T - F) < \varepsilon \). Since \( \mathcal{A} \) has the 
density property almost every point of the set \( F \) is the density point of this set with 
respect to \( \mathcal{A} \). Therefore the function \( g \) is \( \lambda \)-almost everywhere approximately 
continuous with respect to \( \mathcal{A} \).

**Definition 4.** The \( \lambda \)-measurable function \( g : T \to \mathbb{R} \) is said to be degenerate 
(positively degenerate) at the point \( t_0 \in T \) with respect to \( \mathcal{A} \) when there exists 
a open interval \( U \subseteq \mathbb{R} \) such that \( g(t_0) \in U \) and the upper (lower) density of the 
counterimage \( g^{-1}(U) \) at \( t_0 \) with respect to \( \mathcal{A} \) is equal to zero.

**Definition 5.** ([4], definition 4). The function \( g : T \to \mathbb{R} \) has the property \( (G) \) 
with respect to \( \mathcal{A} \) iff for every positive \( \varepsilon \) there exists a set \( I \in \mathcal{A} \) such that 
\( \lambda(A \cap I) > 0 \) and \( \text{osc}_{U} g \leq \varepsilon \), where \( U \) is the set of density points of \( A \cap I \) with respect to \( \mathcal{A} \) belonging to \( A \cap I \).

**Theorem 1.** Let the \( \lambda \)-measurable function \( g : T \to \mathbb{R} \) be positively nondegener-
earate at every point of the closed set \( A \subseteq T \). Then the \( \lambda \)-measurable function

\[
f(x) = \begin{cases} 
g(x) & \text{for } x \in A \\ 0 & \text{for } x \notin A \end{cases}
\]

has the property \( (G) \) with respect to \( \mathcal{A} \).

**Proof.** Let \( E \in \mathcal{H} \) be a set of a positive \( \lambda \) measure and let \( \varepsilon > 0 \) be fixed.

Assume that \( \lambda(E - A) > 0 \). Then there is a point \( t_0 \in T \) such that \( t_0 \in E - A \) and
As the set $A$ is closed, it follows from property (2) of the family $\mathcal{A}$ that there exists a set $I \in \mathcal{A}$ such that $t_0 \in \text{Int}(I)$ and $I \cap A = \emptyset$. Therefore for $t \in T$ we have $f(t) = 0$. Hence $\text{osc} f = 0 \leq \epsilon$ and $\lambda(E \cap I) > 0$.

Assume now that $\lambda(E - A) = 0$. Then we notice that all density points of $E$ belong to $A$. In order to show that

A6 1) there exists a set $I \in \mathcal{A}$ such that $\lambda(I \cap E \cap A) > 0$ and $\text{osc} f \leq \epsilon$, where $V$ is the set of density points of $I \cap E \cap A$ with respect to $\mathcal{A}$ belonging to $I \cap E \cap A$, assume that 1) does not hold. Then we have:

2) if for the set $J \in \mathcal{A}$ the inequality $\lambda(J \cap E \cap A) > 0$ holds, then $\text{osc} f \leq \epsilon$, where $W$ is the set of density points of $J \cap E \cap A$ with respect to $\mathcal{A}$ belonging to $J \cap E \cap A$.

We shall construct a sequence of points $\{t_k\}_{k=1}^{\infty} \subset E \cap A$ and a sequence $\{I_k\}_{k=1}^{\infty} \subset \mathcal{A}$ such that the condition 2) leads to a contradiction.

Let $t_1 \in E \cap A$ be a point such that

3) $D(t_1, E \cap A) = 1$

4) the function $f$ is approximately continuous at $t_1$ with respect to $\mathcal{A}$.

The existence of point $t_1$ follows from the density property of $\mathcal{A}$ and from lemma 1.

Let $I_1 \in \mathcal{A}$ be the set such that

5) $t_1 \in \text{Int}(I_1)$ and

6) $\frac{\lambda(I_1 \cap E \cap A)}{\lambda(I_1)} > \frac{1}{2}$

and

$$\lambda \left( I_1 \cap \left\{ t \in E \cap A : |f(t) - f(t_1)| < \frac{\epsilon}{8} \right\} \right) > \frac{1}{2} \lambda(I_1).$$

The existence of the set $I_1$ follows from 3) and 4).

Let $G_1 = \left\{ t \in E \cap A : |f(t) - f(t_1)| \geq \frac{\epsilon}{2} \right\}$. Then

7) $\lambda(G_1) > 0$.

Indeed. Assume that

8) $\lambda(G_1) = 0$.

Then for points $t \in (I_1 \cap E \cap A) - G_1$ the inequality $|f(t) - f(t_1)| < \frac{\epsilon}{2}$ holds and therefore $\text{osc}_{(I_1 \cap E \cap A) - G_1} f < \epsilon$.

If $|f(t) - f(t_1)| \leq \frac{\epsilon}{2}$ for the points $t \in I_1 \cap E \cap A \cap G_1$ such that $D(t, I_1 \cap E \cap A) = 1$, then $\text{osc} f \leq \epsilon$ on the set of the density points of the set $I_1 \cap E \cap A$, which contradicts 2). Therefore there exists a point $s_1 \in I_1 \cap E \cap A \cap G_1$ such that $D(s_1, I_1 \cap E \cap A) = 1$ and $|f(s_1) - f(t_1)| > \frac{\epsilon}{2}$. But the function $f$ is positively non-
degenerate at the point \( s_i(s_i \in A) \) and \( D(s_i, I_i \cap E \cap A) = 1 \), thence
\[ \lambda \left( \left\{ t \in I_i \cap E \cap A: \ |f(t_i) - f(t)| > \frac{\epsilon}{2} \right\} \right) > 0 \], which is contradictory with 8). Therefore 7) holds true.

Let \( t_2 \in G_i \cap \text{Int} (I_i) \) be a point such that
9) \( D(t_2, G_i) = 1 \) and
10) \( f \) is approximately continuous at \( t_2 \) with respect to \( A \).

Again the existence of point \( t_2 \) follows from the density property of \( A \) and from lemma 1.

Let \( I_2 \in A \) be such that
11) \( t_2 \in \text{Int} (I_2) \), \( \text{Cl} (I_2) \subset \text{Int} (I_1) \), \( \delta(I_2) < \frac{1}{2} \) and

\[ \lambda (I_2 \cap E \cap A) > \frac{2}{3} \quad \text{and} \quad \lambda (I_2 \cap \left\{ t \in E \cap A: |f(t_2) - f(t)| < \frac{\epsilon}{8} \right\}) > \frac{2}{3}. \]

The existence of set \( I_2 \) follows from 9) and 10). Similarly as before the set
\[ G_2 = \left\{ t \in I_2 \cap E \cap A: |f(t_2) - f(t)| \geq \frac{\epsilon}{2} \right\} \] is \( \lambda \)-measurable and has a positive measure \( \lambda \).

Proceeding analogously we define the sequence \( \{I_k\}_{k=1}^{\infty} \) of the sets from \( A \) and the sequence \( \{t_k\}_{k=1}^{\infty} \) such that
13) \( t_k \in G_{k-1} \cap \text{Int} (I_{k-1}) \), \( D(t_k, G_{k-1}) = 1 \) and \( f \) is approximately continuous at the point \( t_k \) with respect to \( A \), where

\[ G_{k-1} = \left\{ t \in I_{k-1} \cap E \cap A: |f(t_{k-1}) - f(t)| \geq \frac{\epsilon}{2} \right\}, \]

14) \( t_k \in \text{Int} (I_k) \), \( \text{Cl} (I_k) \subset \text{Int} (I_{k-1}) \), \( \delta(I_k) < \frac{1}{2^{k-1}} \) and

\[ \frac{\lambda (I_k \cap E \cap A)}{\lambda (I_k)} > \frac{k}{k+1} \quad \text{and} \quad \frac{\lambda \left( I_k \cap \left\{ t \in E \cap A: |f(t_k) - f(t)| < \frac{\epsilon}{8} \right\} \right)}{\lambda (I_k)} > \frac{k}{k+1} \]

for \( k = 1, 2, \ldots \).

Since \( t_k \in G_{k-1} \), we have
15) \( |f(t_{k-1}) - f(t_k)| \geq \frac{\epsilon}{2} \) for \( k = 1, 2, \ldots \).
The set \( \bigcap_{k=1}^{\infty} I_k \) consists of one point \( t_0 \). As the function \( f \) is positively nondegenerate at \( t_0 \) with respect to \( \mathcal{A} \left( t_0 \in \bigcap_{k=1}^{\infty} I_k \cap A \right) \) we have shown that

\[
D_1 \left( t_0, \left\{ t : |f(t_0) - f(t)| < \frac{\varepsilon}{8} \right\} \right) > 0.
\]

Denote by \( \alpha \) this density. Moreover the sequence of sets \( \{ I_k \}_{k=1}^{\infty} \) is a convergence to \( t_0 \), hence there exists a natural number \( n \) such that for \( k > n \)

\[
\frac{\lambda \left( I_k \cap \left\{ t : |f(t_0) - f(t)| < \frac{\varepsilon}{8} \right\} \right)}{\lambda(I_k)} > \frac{\alpha}{2} \quad \text{and}
\]

\[
\frac{\lambda \left( I_k \cap \left\{ t \in E \cap A : |f(t_0) - f(t)| < \frac{\varepsilon}{8} \right\} \right)}{\lambda(I_k)} > 1 - \frac{\alpha}{2} .
\]

Therefore for every \( k > n \)

\[
\left\{ t : |f(t_0) - f(t)| < \frac{\varepsilon}{8} \right\} \cap \left\{ t \in E \cap A : |f(t_0) - f(t_0)| < \frac{\varepsilon}{8} \right\} \cap I_k \neq \emptyset.
\]

Hence for \( k > n \) the following inequality holds \( |f(t_0) - f(t_0)| < \frac{\varepsilon}{8} \), which contradicts 16. Thus the negation of 1) leads to a contradiction. Therefore 1) holds true. The proof of the theorem is completed.

**Lemma 2** ([2], lemma 2). Let \((X, \mathcal{M}, \mu)\) be a measurable space with the \(\sigma\)-finite measure \(\mu\). Let \(g: X \rightarrow \mathbb{R}\) be such that for any \(\varepsilon > 0\) for a class of sets \(\mathcal{D}_\varepsilon = \{ D \in \mathcal{M} : \text{osc } g \leq \varepsilon \} \) satisfies the following condition:

(d) for any set \(B \in \mathcal{M}\) with a positive measure there exists a set \(D \in \mathcal{D}_\varepsilon\) such that \(D \subset B\) and \(\mu(D) > 0\).

Then the function \(g\) is \(\bar{\mu}\)-measurable, where \(\bar{\mu}\) stands for the completion of \(\mu\).

(Davies has proved the lemma under the assumption that is finite, whereas \(\sigma\)-finiteness is sufficient).

Let for every \(i = 1, \ldots, n\), \((X_i, \varrho_i, \mathcal{M}_i, \mu_i)\) be a space as \((T, d, \mathcal{K}, \lambda)\) was, i.e. let every \((X_i, \varrho_i, \mathcal{M}_i, \mu_i)\) be a complete space with a \(\sigma\)-finite \(G_\delta\)-regular complete measure \(\mu_i\) defined over the \(\sigma\)-field \(\mathcal{M}_i\) of subsets of \(X_i\).

Moreover let for every \(i = 1, \ldots, n\) \(\mathcal{F}_i \subset \mathcal{M}_i\) be a family which satisfies the conditions (1), (2) and (3) of family \(\mathcal{A}\).

Let \((X, \varrho, \mathcal{M}, \mu) = (X_1 \times \ldots \times X_n, \varrho_1 \times \ldots \times \varrho_n, \mathcal{M}_1 \times \ldots \times \mathcal{M}_n, \mu_1 \times \ldots \times \mu_n)\) where
\(\mu_1 \times \ldots \times \mu_n\) denotes the completion of the measure \(\mu_1 \times \ldots \times \mu_n\). Moreover let 
\[\mathcal{F} = \mathcal{F}_1 \times \ldots \times \mathcal{F}_n = \{F: F = F_1 \times \ldots \times F_n, F_i \in \mathcal{F}_i \text{ for } i = 1, \ldots, n\}.

We note that \(\mathcal{F}\) has the density property because every family \(\mathcal{F}_i\) has the density property (see [1], p. 2 and 34).

Let \(A \subseteq \mathcal{X} = X_{i-1} \times X_i \times X_{i+1}\), where \(X_{i-1} = X_1 \times \ldots \times X_{i-1}\) and \(X_{i+1} = X_{i+1} \times \ldots \times X_n\). Then the sets \(A_{x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n} = \{x_i \in X_i: (x_1, \ldots, x_n) \in A\}\) and \(A_{x_1, \ldots, x_i, \ldots, x_n} = \{(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \in X_{i-1} \times X_{i+1}: (x_1, \ldots, x_n) \in A\}\) are called a section of the set \(A\) with respect to \((x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)\) and a section of the set \(A\) with respect to \(x_i\) respectively.

**Lemma 3.** Let \(A \in \mathcal{M}\). For every fixed \(i = 1, \ldots, n\) there exists a set \(B \subseteq A\) and \(B \in \mathcal{M}\) such that \(\mu(A - B) = 0\), every point \((x_1, \ldots, x_n) \in B\) is the density point of \(B\) with respect to \(\mathcal{F}\) and for every point \((x_1, \ldots, x_n) \in B\)
(i) \(D(x_i, B_{x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n}) = 1\) and
(ii) \(D((x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n), B_{x_1, x_i, \ldots, x_n}) = 1\).

The proof of this lemma is analogous to the proof of lemma 2 of [7].

**Lemma 4.** Let \(A \in \mathcal{M}\). There exists a set \(B \subseteq A\) and \(B \in \mathcal{M}\) such that \(\mu(A - B) = 0\) and for every point \((x_1, \ldots, x_n) \in B\)
(i) \(D((x_1, \ldots, x_n), B) = 1\),
(ii) for every \(i = 1, \ldots, n\) \(D(x_i, B_{x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n}) = 1\),
(iii) \(D((x_2, \ldots, x_n), B_{x_1, x_2, x_n}) = 1\).

Proof. In accordance with lemma 3 \((i = 1)\) for the set \(A\) there exists a \(\mu\)-measurable subset \(A_1\) of \(A\) such that \(\mu(A - A_1) = 0\), for every point \((x_1, \ldots, x_n) \in A_1\) \(D((x_1, \ldots, x_n), A_1) = 1\), \(D(x_1, (A_1)_1, x_2, \ldots, x_n) = 1\) and \(D((x_2, \ldots, x_n), (A_1)_1, x_3, \ldots, x_n) = 1\). Again in accordance with lemma 3 \((i = 2)\) for the set \(A_1\) there exists a \(\mu\)-measurable subset \(A_2\) of \(A_1\) such that \(\mu(A_1 - A_2) = 0\) and for every point \((x_1, \ldots, x_n) \in A_2\) \(D((x_1, \ldots, x_n), A_2) = 1\), \(D(x_2, (A_2)_1, x_3, \ldots, x_n) = 1\) and
\[D((x_1, x_3, \ldots, x_n), (A_2)_1, x_2, \ldots, x_n) = 1.\]
Let \(C_1 = A_1 - A_2\). It is clear that
\[\mu_2 \times \ldots \times \mu_n(((x_2, \ldots, x_n): \mu_1(C_1)_{x_2, x_3, \ldots, x_n} > 0)) = 0.\]

Let \(D_1 = \{(x_2, \ldots, x_n): \mu_1(D_1)_{x_2, x_3, \ldots, x_n} > 0\}\) and let
\[F_1 = \{(x_3, \ldots, x_n): \mu_2(D_1)_{x_3, x_4, \ldots, x_n} > 0\}(\mu_3 \times \ldots \times \mu_n(F_1) = 0)\]
and \(H_1 = \{x_1: \mu_2 \times \ldots \times \mu_n((C_1)_{x_1, x_2, \ldots, x_n}) > 0\}(\mu_1(H_1) = 0)\). For \(B_1\) take \(B_1 = A_2 - [(X_1 \times D_1) \cup (X_1 \times X_2 \times F_1) \cup (H_1 \times X_2 \times \ldots \times X_n)]\). Evidently for every point
(x_1, \ldots, x_n) \in B_1$ we have \( D((x_1, \ldots, x_n), B_1) = 1 \), \( D(x_1, (B_1)_{x_2, \ldots, x_n}) = 1 \), \( D(x_2, (B_1)_{x_1, x_3, \ldots, x_n}) = 1 \) and \( D((x_2, \ldots, x_n), (B_1)_{x_1, \ldots}) = 1 \).

As a sequel to the set \( B_1 \) in accordance with lemma 3 \((i = 3)\) there exists a \( \mu \)-measurable subset \( A_3 \) of \( B_1 \) such that \( \mu(B_1 - A_3) = 0 \) and for every point \((x_1, \ldots, x_n) \in A_3 \) \( D((x_1, \ldots, x_n), A_3) = 1 \), \( D(x_3, (A_3)_{x_1, x_2, \ldots, x_n}) = 1 \) and \( D((x_3, x_2, x_1, \ldots, x_n), (A_3)_{x_1, x_2, \ldots}) = 1 \). Let \( C_2 = B_1 - A_3 \) and let \( D_2, 1 = \{(x_2, \ldots, x_n): \mu^*(\{(x_2)_{x_3, \ldots, x_n}\}) > 0\} \) and \( D_2, 2 = \{(x_1, x_3, \ldots, x_n): \mu^*(\{(x_1)_{x_2, \ldots, x_n}\}) > 0\} \) because \( \mu(C_2) = 0 \).

Let \( F_2, 1, 1 = \{(x_3, \ldots, x_n): \mu^*(\{(D_2, 1)_{x_3, \ldots, x_n}\}) > 0\} \),
\[ F_2, 1, 2 = \{(x_2, x_4, \ldots, x_n): \mu^*(\{(D_2, 1)_{x_2, x_4, \ldots, x_n}\}) > 0\}, \]
\[ F_2, 1, 3 = \{(x_3, \ldots, x_n): \mu^*(\{(D_2, 2)_{x_3, \ldots, x_n}\}) > 0\}, \]
\[ F_2, 2, 2 = \{(x_1, x_4, \ldots, x_n): \mu^*(\{(D_2, 2)_{x_1, x_4, \ldots, x_n}\}) > 0\} \]
and
\[ H_2 = \{x_1: \mu^* \times \mu^*_n((A_1 - A_3)_{x_1, \ldots}) > 0\}. \]

Evidently all these sets are of respective measure zero.

Let \( B_2 = A_3 - [(X_1 \times D_2, 1) \cup \{(x_1, \ldots, x_n): (x_1, x_3, \ldots, x_n) \in D_2, 2 \text{ and } x_2 \in X_2 \}\cup (X_1 \times X_2 \times F_2, 1, 1) \cup \{(x_1, \ldots, x_n): x_1 \in X_1 \text{ and } x_3 \in X_3 \text{ and } (x_2, x_4, \ldots, x_n) \in F_2, 1, 2\}\cup (X_1 \times X_2 \times F_2, 1, 2) \cup \{(x_1, \ldots, x_n): (x_1, x_4, \ldots, x_n) \in F_2, 2, 2 \text{ and } (x_2, x_3) \in X_2 X_3 \}\cup (H_2 \times X_2 \times X_3 \times X_n)] \).

For every point \((x_1, \ldots, x_n) \in B_2 \) \( D((x_1, \ldots, x_n), B_2) = 1 \), \( D(x_1, (B_2)_{x_2, \ldots, x_n}) = 1 \), \( D(x_2, (B_2)_{x_1, x_3, \ldots, x_n}) = 1 \), \( D(x_3, (B_3)_{x_2, x_4, \ldots, x_n}) = 1 \), \( D((x_3, \ldots, x_n), (B_2)_{x_2, \ldots}) = 1 \). Proceeding analogously in accordance with lemma 3 \((i = n)\) we define the set \( B_{n-2} \) a \( \mu \)-measurable set \( A_n \subset B_{n-2} \) such that \( \mu(B_{n-2} - A_n) = 0 \) and for every point \((x_1, \ldots, x_n) \in A_n \) \( D((x_1, \ldots, x_n), A_n) = 1 \), \( D(x_1, (A_n)_{x_3, \ldots, x_n, \ldots, \ldots}) = 1 \) and \( D((x_1, \ldots, x_{n-1}), (A_n)_{x_n, \ldots, \ldots}) = 1 \). Let \( C_{n-1} = B_{n-2} - A_n \). Evidently \( \mu(C_{n-1}) = 0 \). Let
\[ D_{n-1, 1} = \{(x_2, \ldots, x_n): \mu^*_1((C_{n-1})_{x_2, \ldots, x_n}) > 0\}, \]
\[ D_{n-1, 2} = \{(x_1, x_3, \ldots, x_n): \mu^*_2((C_{n-1})_{x_1, x_3, \ldots, x_n}) > 0\}, \]
\[ \vdots \]
\[ D_{n-1, n-1} = \{(x_1, \ldots, x_{n-2}, x_n): \mu^*_n((C_{n-1})_{x_1, \ldots, x_{n-2}, x_n}) > 0\}. \]

Evidently all these sets are of respective measure zero.

Moreover the sets
\[ F_{n-1, 1, 1} = \{(x_3, \ldots, x_n): \mu^*_2((D_{n-1, 1})_{x_3, \ldots, x_n}) > 0\}, \]
\[ F_{n-1, 1, 2} = \{(x_2, x_4, \ldots, x_n): \mu^*_2((D_{n-1, 1})_{x_2, x_4, \ldots, x_n}) > 0\}, \]
\[ \vdots \]
\[ F_{n-1, 1, n-1} = \{(x_2, \ldots, x_{n-1}): \mu^*_n((D_{n-1, 1})_{x_2, \ldots, x_{n-1}, \ldots}) > 0\} \] and
Let $f: X \to R$ be a function. Then the function $f_{x_1, \ldots, x_{n-1}, \bullet, x_{n+1}, \ldots, x_n}(x_i) = f(x_1, \ldots, x_n)$ is called as usually a section of $f$ with respect to $(x_1, \ldots, x_{n-1}, \bullet, x_{n+1}, \ldots, x_n)$.

Let $\Phi(f) = \{(x_1, \ldots, x_n): \exists i, f_{x_1, \ldots, x_{i-1}, \bullet, x_{i+1}, \ldots, x_n} \text{ is not approximately continuous at } x_i \in X_i \text{ with respect to } \mathcal{F}_i\}$.

**Lemma 5** ([7], lemma 5). Let $f: X \to R$ be a $\mu$-measurable function. Then $\mu(\Phi(f)) = 0$.

For the function $f: X \to R$ we denote by $A(f) = \{(x_1, \ldots, x_n): \exists i, f_{x_1, \ldots, x_{i-1}, \bullet, x_{i+1}, \ldots, x_n} \text{ is positively degenerate at the point } x_i\}$ with respect to $\mathcal{F}_i$ or the section $f_{x_1, \ldots, x_{n-1}, \bullet}$ is degenerate at the point $x_n\text{ with respect to } \mathcal{F}_n$.

**Theorem 2.** Let $f: X \to R$ be a function such that all its sections $f_{x_1, \ldots, x_{i-1}, \bullet, x_{i+1}, \ldots, x_n}$ are $\mu$-measurable $(i = 2, \ldots, n)$ and all its sections $f_{\bullet, x_2, \ldots, x_n}$ have...
the property (G) with respect to $\mathcal{F}_1$. Then the function $f$ is $\mu$-measurable iff $\mu(A(f)) = 0$.

Proof. This theorem holds true for $n = 2$ (see [4], theorem 4).

Assume that

(1) if for the function $f: X_2 \times \ldots \times X_n \to \mathbb{R}$ all its sections $f_{x_2, \ldots, x_{i-1}, \bullet, x_{i+1}, \ldots, x_n}$ are $\mu_i$-measurable for every $i = 3, \ldots, n$ and all its sections $f_{\bullet, x_3, \ldots, x_n}$ have the property (G) with respect to $\mathcal{F}_2$, then $f$ is $\mu_2 \times \ldots \times \mu_n$-measurable iff $\mu_2 \times \ldots \times \mu_n (A_i(f)) = 0$.

\[ A_i(f) = \{(x_2, \ldots, x_n): \exists f_{x_2, \ldots, x_{i-1}, \bullet, x_{i+1}, \ldots, x_n} \]

is positively degenerate at the point $x_i \in X_i$ with respect to $\mathcal{F}_i$ or $f_{x_2, \ldots, x_{n-1}, \bullet}$ is degenerate at $x_n$ with respect to $\mathcal{F}_n$.

Let $f$ be such as in this theorem. If $f$ is the $\mu$-measurable function, then $\mu(A(f)) = 0$ because $A(f) \subset \Phi(f)$ and in accordance with lemma 5 $\mu(\Phi(f)) = 0$.

Assume that $\mu(A(f)) = 0$. It is sufficient to show that the function $f$ satisfies the assumptions concerning the function $g$ of lemma 2.

Let $E \in \mathcal{M}$, $\mu(E) > 0$, $\varepsilon > 0$ and let $\{I_k\}_{k=1}^\infty$ be the sequence of all sets belonging to $\mathcal{F}_1$ and let $\{K_k\}_{k=1}^\infty$ be the sequence of all closed intervals with rational ends and lengths smaller then $\varepsilon$.

Let $Q = \{(x_2, \ldots, x_n): (x_2, \ldots, x_n) \in X_2 \times \ldots \times X_n \in \mathcal{M}_1$ and $\mu_1(E_{x_3, \ldots, x_n}) > 0\}$. The set $Q$ is $\mu_2 \times \ldots \times \mu_n$-measurable and $\mu_2 \times \ldots \times \mu_n (Q) > 0$. Let $Q_{r,s}$ be a set of points $(x_2, \ldots, x_n) \in Q$ such that

(i) $\mu_1(I \cap E_{x_2, \ldots, x_n}) > 0$

(ii) if $D(x_1, I \cap E_{x_2, \ldots, x_n}) = 1$ and $x_1 \in I \cap E_{x_3, \ldots, x_n}$, then $f(x_1, \ldots, x_n) \in K_r$.

Evidently $Q = \bigcup_{r,s} Q_{r,s}$. Moreover $Q \subset \bigcup_{r,s} Q_{r,s}$ because all sections $f_{x_2, \ldots, x_n}$ have the property (G) with respect to $\mathcal{F}_1$. Therefore $Q = \bigcup_{r,s} Q_{r,s}$. Thus there exists a couple of positive integers $(r_0, s_0)$ such that $\mu_2 \times \ldots \times \mu_n^* (Q_{r_0, s_0}) > 0$ because $\mu_2 \times \ldots \times \mu_n (Q) > 0$. Let

\[ P = \{(x_2, \ldots, x_n): D^*((x_2, \ldots, x_n), Q_{r_0, s_0}) = 1\}.

The measure $\mu_2 \times \ldots \times \mu_n$ is $G_3$ regular and $\mathcal{F}_2 \times \ldots \times \mathcal{F}_n$ has the density property, hence $P \in \mathcal{M}_2 \times \ldots \times \mathcal{M}_n$ and $\mu_2 \times \ldots \times \mu_n (P) = \mu_2 \times \ldots \times \mu_n^* (Q_{r_0, s_0}) > 0$.

Let $F = E \cap (I_k \times P)$. Evidently $F \in \mathcal{M}$ and $\mu(F) > 0$ because for all points $(x_2, \ldots, x_n) \in Q_{r_0, s_0}$ $\mu_2 \times \ldots \times \mu_n (F_{x_2, \ldots, x_n}) > 0$. Let $M = F - A(f)$. For the set $M$, in
accordance with lemma 4, there exists a set $H \subset M$ such that $\mu(M - H) = 0$, for every point $(x_1, \ldots, x_n) \in H$

$$D((x_1, \ldots, x_n), H) = 1 \text{ and for every } i = 1, \ldots, n$$
$$D(x_i, H_{x_1, \ldots, x_{i-1}, \bullet, x_{i+1}, \ldots, x_n}) = 1 \text{ and}$$
$$D((x_2, \ldots, x_n), H_{x_1, \bullet, \ldots, \bullet}) = 1.$$  

Evidently $H \subset E$ and $\mu(H) > 0$. To prove the theorem, in accordance with lemma 2 it is sufficient to show that $f(x_1, \ldots, x_n) \in K_{x_0}$ for every point $(x_1, \ldots, x_n) \in H$.

Let $(x_1^0, \ldots, x_n^0)$ be a point of the set $H$ such that $f(x_1^0, \ldots, x_n^0) \in K_{x_0}$. Every point of $H_{x_1, \bullet, \ldots, \bullet}$ is the density point of $H_{x_1, \bullet, \ldots, \bullet}$, therefore

$$H_{x_1, \bullet, \ldots, \bullet} \in M_2 \times \ldots \times M_n \quad \text{and} \quad \mu_2 \times \ldots \times \mu_n (H_{x_1, \bullet, \ldots, \bullet}) > 0.$$  

Moreover every subset of $H_{x_1, \bullet, \ldots, \bullet}$ of positive measure and the set $Q_{x_0, x_0}$ have common points. Let $f_{x_1^0, \bullet, \ldots, \bullet} : X_{x_1^0, \bullet, \ldots, \bullet} \to R$. For every $i = 2, \ldots, n$ $f_{x_1^0, \ldots, x_{i-1}^0, \bullet, x_{i+1}^0, \ldots, x_n^0}$ is the $\mu$-measurable function. Moreover by theorem 1 all sections $f_{x_1^0, \bullet, \ldots, x_n}^0$ have the property (G) with respect to $\mathcal{F}_2$ because the functions $f_{x_1^0, \bullet, x_3, \ldots, x_n}$ are positively nondegenerate at every point $x_2$ with respect to $\mathcal{F}_2$ ($(x_0^0, x_2, \ldots, x_n) \notin A(f)$). Notice that $\mu_2 \times \ldots \times \mu_n (H_{x_1^0, \bullet, \ldots, \bullet} \cap A(f_{x_1^0, \bullet, \ldots, \bullet})) = 0$. Then if we assume that

$$f(x_2, \ldots, x_n) = \begin{cases} f_{x_1^0, \bullet, \ldots, \bullet}^0 (x_2, \ldots, x_n) & \text{for } (x_2, \ldots, x_n) \in H_{x_1^0, \bullet, \ldots, \bullet}, \\ 0 & \text{for } (x_2, \ldots, x_n) \notin H_{x_1^0, \bullet, \ldots, \bullet}, \end{cases}$$

then, according to ($*$), the function $f_{x_1^0, \bullet, \ldots, \bullet}^0$ is $\mu_2 \times \ldots \times \mu_n$-measurable. In result the set

$$(f_{x_1^0, \bullet, \ldots, \bullet})^{-1} (K_{x_0}) \in M_2 \times \ldots \times M_n \quad \text{and as} \quad f_{x_1^0, \bullet, \ldots, \bullet} (Q_{x_0, x_0}) \subset K_{x_0} \quad \text{then}$$

$$(**)
\overline{\mu_2 \times \ldots \times \mu_n (H_{x_1^0, \bullet, \ldots, \bullet} - (f_{x_1^0, \bullet, \ldots, \bullet})^{-1} (K_{x_0}))} = 0.$$  

On the other hand $f(x_1^0, \ldots, x_n^0) \in K_{x_0}$ and the function $f_{x_1^0, \bullet, x_3^0, \ldots, x_n^0}$ is positively nondegenerate at the point $x_2^0$ with respect to $\mathcal{F}_2$, thence we infer that

$$\mu_2^0 (H_{x_1^0, \bullet, x_3^0, \ldots, x_n^0} \cap (f_{x_1^0, \bullet, x_3^0, \ldots, x_n^0})^{-1} (R - K_{x_0})) > 0.$$  

For every point

$$x_2 \in H_{x_1^0, \bullet, x_3^0, \ldots, x_n^0} \cap (f_{x_1^0, \bullet, x_3^0, \ldots, x_n^0})^{-1} (R - K_{x_0})$$

the sections $f_{x_1^0, x_2^0, x_3^0, \ldots, x_n^0}$ are nondegenerate at $x_2^0$ with respect to $\mathcal{F}_3$, thence
\[
\mu_2 \times \mu_3 \left( H_{x_0} \cap (f_{x_0} \cap \cdots) \cap (R - K_0) \right) > 0.
\]

Proceeding analogously we infer that for every point \((x_2, \ldots, x_{n-1}) \in H_{x_1} \cap \cdots \cap (f_{x_1} \cap \cdots) \cap (R - K_0)\)
the sections \(f_{x_1, x_2, \ldots, x_{n-1}}\) are nondegenerate at the point \(x_1^0\) with respect to \(\mathcal{F}_n\), therefore
\[
\mu_2 \times \cdots \times \mu_{n-1} \left( H_{x_0} \cap (f_{x_0} \cap \cdots) \cap (R - K_0) \right) > 0,
\]
which contradicts \((**)\). The function \(f\) of \(n\) variables is \(\mu\)-measurable. Thence by the mathematical induction theorem 2 holds true.

Remark 1. The following theorem is not true:

Theorem ([5], theorem 1). Let the function \(f: \mathbb{R}^n \to \mathbb{R}\) be such that all its sections \(f_{x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n}\) \((i = 1, \ldots, n)\) are measurable in the sense of Lebesgue and all its sections \(f_{x, x_2, \ldots, x_n}\) have the property \((G)\). Then the function \(f\) is measurable in the sense of Lebesgue iff
\[
m_n(\mathbb{R}^n - D(f)) = 0
\]
where \(m_n\) denoted the Lebesgue measure in \(\mathbb{R}^n\) and
\[
D(f) = \{(x_1, \ldots, x_n): \text{for } i = 1, \ldots, n \text{ } f_{x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n}
\text{ is nondegenerate at the point } x_i\}
\]

This is stated in the example given in the paper [6] by Z. Grande. Indeed, the theorem:

**Theorem 3** ([6] theorem 1). Assume that the continuum hypothesis holds. Then there exists a function \(F: \mathbb{R} \times \mathbb{R} \to \mathbb{R}\) of Lebesgue nonmeasurable such that all its sections \(F_{x, x_2, \ldots, x_n}\) and \(F_{x, x_2, \ldots, x_n}\) are of Lebesgue measurable and nondegenerate at any point \(t \in \mathbb{R}\).

It is sufficient to take the function \(f: \mathbb{R}^3 \to \mathbb{R}\) such that
\[
f(x_1, x_2, x_3) = F(x_2, x_3).
\]

Let \(f: X \to \mathbb{R}\) be a function such that all its sections \(f_{x_1, x_2, \ldots, x_n}\) are \(\mu_1\)-measurable. Denote by \(B(f) = \{(x_1, \ldots, x_n) \in X: f_{x_1, x_2, \ldots, x_n} \text{ is not approximately continuous with respect to } \mathcal{F}_1 \text{ at } x_1 \in X_1\}\) and \(C(f) = \{(x_1, \ldots, x_n) \in X: f_{x_1, x_2, \ldots, x_n} \text{ is positively degenerate at } x_1 \in X_1 \text{ with respect to } \mathcal{F}_1\}\).

**Theorem 4.** Let \(f: X \to \mathbb{R}\) be a function such that for \(i = 1, \ldots, n\) all its sections \(f_{x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n}\) are \(\mu_i\)-measurable.

Then the conditions:

(i) the function \(f\) is \(\mu\)-measurable,
(ii) \( \mu(A(f) \cup B(f)) = 0 \) and
(iii) \( \mu(A(f) \cup C(f)) = 0 \)
are equivalent.

Proof. If the function \( f \) is \( \mu \)-measurable, then \( \mu(A(f) \cup B(f)) = 0 \) because \( A(f) \cup B(f) \subseteq \Phi(f) \) and by lemma 5 (i) implies (ii). Also (ii) implies (iii) because \( A(f) \cup C(f) \subseteq A(f) \cup B(f) \). It is sufficient to show that (iii) implies (i).

Let \( \mu(A(f) \cup C(f)) = 0 \) and let \( A = X - [A(f) \cup C(f)] \). The measure \( \mu \) is \( G_6 \) regular and \( \mathcal{F} \) has the density property, thence there exists a sequence \( \{A_k\}_{k=1}^{\infty} \) of closed sets of positive and finite measure such that \( A_k \subseteq A_{k+1} \) and \( \mu \left( A - \bigcup_{k=1}^{\infty} A_k \right) = 0 \).

Let
\[
f_k(x_1, \ldots, x_n) = \begin{cases} f(x_1, \ldots, x_n) & \text{for } (x_1, \ldots, x_n) \in A_k \\ 0 & \text{for } (x_1, \ldots, x_n) \notin A_k \end{cases}
\]

As almost everywhere \( \lim_{k \to \infty} f_k(x_1, \ldots, x_n) = f(x_1, \ldots, x_n) \) with respect to the measure \( \mu \), it is sufficient to show that the functions \( f_k \) satisfy the assumptions of theorem 2. According to the assumption all sections \( (f_k)_0, x_2, \ldots, x_n \) are \( \mu_1 \)-measurable and at almost every point of the closed set \( (A_k)_0, x_2, \ldots, x_n \) are positively nondegenerate with respect to \( \mathcal{F} \) because \( (x_1, \ldots, x_n) \notin \Phi(f) \). Here we infer from theorem 1 that the function \( f \) has the property (G) with respect to \( \mathcal{F}_1 \). Moreover \( \mu(A(f) \cup C(f)) = 0 \), therefore \( \mu(A(f_k)) = 0 \). Hence by theorem 2 the functions \( f_k \) are \( \mu \)-measurable. The proof of the theorem 4 is completed.

Returning to our space \( (T, d, \mathcal{K}, \lambda) \) let \( \mathcal{K} \) be a \( \sigma \)-field enclosing Borel sets of \( T \).

**Definition 6.** The function \( g: T \to R \) has the property (H) with respect to \( \mathcal{A} \) iff for every point \( t \in T \) there exist two open and nonempty sets \( U(t) \) and \( V(t) \) such that \( D_u(t, U(t)) > 0, D_u(t, V(t)) > 0, f|_{U(t) \cup V(t)} \) is upper semicontinuous and \( f|_{V(t) \cup U(t)} \) is lower semicontinuous at \( t \).

**Theorem 5.** The function \( g: T \to R \) which has property (H) with respect to \( \mathcal{A} \) is \( \lambda \)-almost everywhere continuous.

Proof. Denote by \( D_g \) the set of points of discontinuity of the function \( g \). Assume that \( \lambda(D_g) > 0 \). We can assume that \( g \) is bounded. Let \( A = \{ t \in D_g: D(t, D_g) = 1 \} \) and let \( B \subseteq A \) be a closed set such that:
(a) for every \( I \in \mathcal{A} \); \( \text{Int}(I) \cap B \neq \emptyset \Rightarrow \lambda(I \cap B) > 0 \). Denote by \( m \) the essential infimum of \( g \) on the set \( B \). Let \( t \in B \) be a point such that \( D(t, B) = 1 \) and \( g(t) < m + \frac{1}{4} \). The function \( g \) has the property (H) with respect to \( \mathcal{A} \), therefore for the point \( t \) there exists an open nonempty set \( U(t) \) such that \( D_u(t, U(t)) > 0 \) and \( g|_{U(t) \cup U(t)} \) is upper semicontinuous at \( s \). Therefore \( g(t) - g(t) < \frac{1}{4} \) for \( t \in U(t) \). As
Let \( t \in I_1 \).

As \( D(s_1, B \cap \text{Int} (I_1)) = 1 \) and \( D(s_1, V(s_1)) > 0 \), there exists a set \( J_1 \in \mathcal{A} \) such that \( \text{Cl} (J_1) \subseteq V(s_1) \), \( B \cap \text{Int} (J_1) \neq \emptyset \) and \( \delta(J_1) < 1 \). Evidently \( \text{osc} g < 1 \), because \( g(t) < m + \frac{1}{2} \) and \( g(t) > g(s_i) \frac{1}{4} \). Therefore we have a set \( J_1 \in \mathcal{A} \) such that \( B \cap \text{Int} (J_1) = 0 \), \( \delta(J_1) < 1 \) and \( \text{osc} g < 1 \) on the set \( J_1 \).

Proceeding analogously we define the sequence \( \{J_k\}_{k=1}^{\infty} \) of the sets from \( \mathcal{A} \) such that

(i) \( \text{Cl} (J_k) \subseteq \text{Int} (J_{k-1}) \)

(ii) \( B \cap \text{Int} (J_k) \neq \emptyset \)

(iii) \( \delta(J_k) < \frac{1}{k} \) and \( \text{osc} g < \frac{1}{k} \) on the set \( J_k \).

The set \( B \cap \bigcap_{k=1}^{\infty} \text{Cl} (J_k) \neq \emptyset \). Let \( t_0 \in \bigcap_{k=1}^{m} B \cap \text{Cl} (J_k) \). As for \( k = 1, 2, \ldots \), \( t_0 \in \text{Int} (J_k) \), the oscillation of the function \( g \) at the point \( t_0 \) is equal to zero i.e. \( t_0 \notin D_g \). On the other hand \( t_0 \in B \), therefore \( t_0 \notin D_g \), which is contradictory with \( t_0 \in D_g \). The proof of the theorem is completed. Theorem 5 is a generalization of theorem 1 of [3].

Remark 2. Let \( S \subset T \) be a countable dense set. If the function \( g: T \to R \) has the property (H) with respect to \( \mathcal{A} \), then: (R) \( \lim_{t \to s} \inf_{t \in S} g(t) \leq g(s) \leq \lim_{t \to s} \sup_{t \in S} g(t) \) for every \( s \in S \).

**Theorem 6.** Let \( f: X \to R \) be a function such that all its sections \( f_{s_1, s_2, \ldots, s_k} \) are \( \mu_1 \)-measurable and all its sections \( f_{s_1, s_2, \ldots, s_k-1, s_{k+1}, \ldots, s_n} \) have the property (H) with respect to \( \mathcal{F}_i \) for every \( i = 2, \ldots, n \).

Then \( f \) is a \( \mu \)-measurable function.

**Proof.** This theorem for \( n = 2 \) holds by the theorem given in the paper [8] by E. Marczewski and Cz. Ryll-Nardzewski.

**Theorem 7** ([8], theorem 2). Let \( f: Y \times T \to R \), where \( Y \) is a space with
a measure \( \mu \), be a function such that all its sections \( f_{\bullet,} \) are \( \mu \)-measurable and all its sections \( f_y, \bullet \) are \( \lambda \) — almost everywhere continuous and satisfy the condition (R).

Then the function \( f \) is \( \bar{\mu} \) — measurable, where \( \bar{\mu} = x \times \lambda \).

Assume that if \( g: X_1 \times \ldots \times X_{n-1} \rightarrow R \) is a function such that all its sections \( g_{\bullet, x_2, \ldots, x_{n-1}} \) are \( \mu_i \) — measurable and all its sections \( g_{x_1, \ldots, x_{i-1}, \bullet, x_{i+1}, \ldots, x_{n-1}} \) have the property (H) with respect to \( \mathcal{F} \) for \( i = 2, \ldots, n - 1 \), then \( g \) is \( \mu_1 \times \ldots \times \mu_{n-1} \) — measurable. Let \( f: X_1 \times \ldots \times X_n \rightarrow R \) satisfy the condition of theorem 6. Then the function

\[
f_{\ldots, \bullet, x_n}(x_1, \ldots, x_{n-1}) = g(x_1, \ldots, x_{n-1})
\]

is \( \mu_1 \times \ldots \times \mu_{n-1} \) measurable. Therefore \( f: X_{n-1} \times X_n \rightarrow R \) as the function of two variables is \( \mu \) — measurable. The proof of the theorem is completed.

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ИЗМЕРИМОСТЬ ДЕЙСТВИТЕЛЬНЫХ ФУНКЦИЙ,
ЗАДАННЫХ НА ДЕКАРТОВОМ ПРОИЗВЕДЕНИИ
МЕТРИЧЕСКИХ ПРОСТРАНСТВ

Grażyna Kwiecińska

Резюме

Эта работа состоит из двух частей. В первой части находятся необходимое и достаточное
условия измеримости действительных функций, заданных на декартовом произведении $n \ (n > 2)$
метрических пространств с мерами, которые удовлетворяют некоторым дополнительным усло-
виям. Вторая часть содержит теорему, которая связана с теоремой Лебега о измеримости
функции двух переменных.