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Regulators of type \(\alpha\) of lattice ordered groups


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The purpose of the present paper is to investigate the lattice ordered groups \( (\mathbb{L}, \mathbb{G}) \) having a base by using the algebraic and topological methods. (Note that in [9, 10, 12], the \( \mathbb{L} \)-groups having a base are called \( \mathbb{L} \)-groups of kind \( \alpha \); see Definition 1.2 and Lemma 1.4.) The algebraic examination is carried out by means of the so-called regulators, i.e. the indexed systems of prime subgroups having the zero meet and the topological examination by means of the topology induced on a regulator (structure space). For terminology and notations, cf. [13] I and [10]. A short review is also given in sec. 0 of the present paper. Other structure spaces were dealt with by S. J. Bernau [1]. His spaces are defined on the systems of all prime \( z \)-subgroups. Similar considerations will be included in another paper. Prime subgroups need not be \( z \)-subgroups, while minimal prime subgroups do it. The regulators of type \( \alpha \) are formed by minimal prime subgroups and are equipped with a topology inherited from the hull-kernel topology defined in [1].

In the present paper it is proved that there exists (up to equivalence) at most one regulator of type \( \alpha \) of an \( \mathbb{L} \)-group, namely the set of all maximal polars (1.9). The existence of the regulator of type \( \alpha \) characterizes the \( \mathbb{L} \)-groups having a base (1.10). A topological characterization to a regulator of be of type \( \alpha \) is given in 1.13 (the induced space is discrete). A topological characterization of \( \mathbb{L} \)-groups having a base is given in 3.5 (the set of all isolated points is dense in \( (\mathbb{R}, \mathbb{G}) \) provided that the standard regulator \( (\mathbb{R}, \cup) \) is similar to a reduced one) and in 2.4 (the union of all atoms of the lattice \( \mathbb{M}(\mathbb{R}, \mathbb{G}) \) is a dense subset of \( (\mathbb{R}, \mathbb{G}) \) assuming only the standardness of \( (\mathbb{R}, \cup) \)).

The similarity of a standard regulator \( (\mathbb{R}, \cup) \) to a regulator of type \( \alpha \) is described by the relation \( \mathbb{R}(\mathbb{R}, \mathbb{G}) = \mathbb{M}(\mathbb{R}, \mathbb{G}) \) (3.1). The property \( \mathbb{R}(\mathbb{R}, \mathbb{G}) = \mathbb{M}(\mathbb{R}, \mathbb{G}) \) is then characterized by a number of equivalent conditions in 2.10 and 3.6. In Theorem 3.7, where the results of Theorem 3.6 are specified for a completely regular regulator \( (\mathbb{R}, \cup) \), this equality is described by conditions of various kinds.

* The symbol \( \cup \) has the same meaning as the symbol \( \bigcup \) in the preceeding papers [10], [13].
An algebraic condition reads: The set of all minimal prime subgroups \( I \) with \( \mathbb{Z}(I) \neq \emptyset \) is equal to the set of all maximal pohrs of \( G \). A set condition: Every ultraantifilter \( x \) on \( I' \) with \( \mathbb{Z}(\cup x) \neq \emptyset \) is principal. A topological condition: The space \((\mathfrak{M}, G)\) is locally connected. If the regulator \((\mathfrak{M}, \cup)\) is reduced, the above condition reads: The space \((\mathfrak{M}, G)\) is discrete. In sec. 4 conditions are studied under which the regulators \( \mathfrak{M}_n \) and \( \mathfrak{M}_r \) are of type \( \alpha \) or finite and of type \( \alpha \). The results are given in 4.5, 4.7 and 4.8.

0.1 A regulator \((\mathfrak{M}, \cup)\) of an \( l \)-group \( G \) is a set \( \mathfrak{M} \neq \emptyset \) and a mapping \( \cup \). \( \mathfrak{M} \to \mathcal{P}(G) \), the family of all prime subgroups of \( G \) such that \( \cap \{ \cup x: x \in \mathfrak{M} \} = \{0\} \). \((\mathfrak{M}, \cup)\) is called standard if \( \cup x \neq G \) for every \( x \in \mathfrak{M} \). reduced if \( x, y \in \mathfrak{M}, x \neq y \Rightarrow \cup x \cap \cup y \neq \emptyset \) and completely regular if it has the following property: \( x \in \mathfrak{M}, f \in G, f \in \cup x \) implies that there exists \( g \in G \) such that \( f \delta g \) and \( g \in \cup x \) (\( f \delta g \) means \( |f| \wedge |g| = 0 \)). \((\mathfrak{M}, \cup)\) is said to be finite if the set \( \mathfrak{M} \) is finite. Two special types of regulators (the \( II' \)-regulator and \( \Gamma \)-regulator) are defined in 0.5.

Let \((\mathfrak{M}_1, \cup_1)\) be a regulator of an \( l \)-group \( G \), \( (i = 1, 2) \). The regulator \((\mathfrak{M}_2 \cup_1)\) is said to be similar (equivalent) to the regulator \((\mathfrak{M}_1, \cup_1)\) if there exists an \( l \)-isomorphism \( \alpha \) of \( G_1 \) onto \( G_2 \) and a surjection \( (a \ bijection) \beta: \mathfrak{M}_1 \) onto \( \mathfrak{M}_2 \) such that \( f \in \cup_1 \beta x = af \in \cup_2 x \) for every \( f \in G_1 \) and every \( x \in \mathfrak{M}_2 \) or equivalently \( \alpha \cup_1 \beta x = \cup_2 x \) for every \( x \in \mathfrak{M}_1 \). (The mapping \( \beta \) is continuous, open and closed (a homeomorphism) with respect to the induced topology defined in sec. 0.2 below, [13] II 4.2.)

An equivalence of \((\mathfrak{M}_1, \cup_1)\) and \((\mathfrak{M}_2, \cup_2)\) with \( G_1 = G, (-G) \) and \( \alpha = \text{id}_G \) is called an equality.

Let \((\mathfrak{M}, \cup)\) be a regulator of \( G \). Take \( x \in \mathfrak{M} \) and define \( \hat{x} = \{ y \in \mathfrak{M}: \cup x = \cup y \} \), \( \mathfrak{M} = \{ \hat{x}: x \in \mathfrak{M} \} \) and \( \cup x = \cup \hat{x} \). Then \( \cup \) is a mapping of \( \mathfrak{M} \) into \( \mathcal{P}(G) \) and \((\hat{\mathfrak{M}}, \cup)\) is a regulator similar to \((\mathfrak{M}, \cup)\), the so-called simplification of \((\mathfrak{M}, \cup)\).

0.2 For \( f \in G \) define \( \mathbb{Z}(\mathcal{f}) = \{ x \in \mathfrak{M}: f \in \cup x \} \). If \((\mathfrak{M}, \cup)\) is a standard regulator of \( G \), then \( (G \neq \{0\}) \) and the set \( \mathfrak{N} = \{ \mathbb{Z}(\mathcal{f}): f \in G \} \) is a base of closed sets for a topology on the set \( \mathfrak{M} \) ([13] I 1.2). This topology (in the sense of Bourbaki) is called the topology induced on \( \mathfrak{M} \) by the \( l \)-group \( G \). The corresponding topological space is denoted by \((\mathfrak{M}, G)\).

0.3 Let \((\mathfrak{M}, \cup)\) be a regulator of an \( l \)-group \( G \). We define

\[
\Psi(A) = \{ f \in G: f \in \cup x \text{ for every } x \in A \} \quad (\emptyset \subseteq A \subseteq \mathfrak{M}),
\]

\[
\mathcal{Z}(P) = \{ x \in \mathfrak{M}: f \in \cup x \text{ for every } f \in P \} \quad (\emptyset \subseteq P \subseteq G).
\]

If \( A = \{ x \} \) or \( P = \{ f \} \) is a singleton, we write \( \Psi(x) \) or \( \mathcal{Z}(f) \) instead of \( \Psi(\{ x \}) \) or \( \mathcal{Z}(\{ f \}) \), respectively. \( \Psi \) and \( \mathcal{Z} \) are evidently dual isotone mappings between the sets \( \exp \mathfrak{M} \) and \( \exp G \) ordered by inclusion. \( \Psi(x) - \cup x \) and \( \mathcal{Z}(f) \) coincides with the notation in 0.2. We denote by \( \mathfrak{M}(\mathfrak{M}, G) \) or \( \mathfrak{M}(\mathfrak{M}, G) \) or \( \mathfrak{C}(\mathfrak{M}, G) \) the system of all closed or regular closed or clopen sets of \((\mathfrak{M}, G)\), respectively.

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0.4 The Boolean algebra of all polars of $G$ is denoted by $\Gamma(G)$. Being $\emptyset \neq A \subseteq G$, we define $A' = \{g \in G: g \in f \text{ for every } f \in A\}$. Then the complement of a polar $K$ in $\Gamma(G)$ is $K'$. $\Pi'(G) := \{f': f \in G\}$ or $\Pi(G) := \{f': f \in G\}$ is the system of all dual principal or principal polars of $G$, respectively. $\Pi'(G)$ and $\Pi(G)$ are sublattices of the lattice $\Gamma(G)$.

0.5 By an ultraintifilter on a $\vee$-semilattice $\Lambda$ there is meant a maximal antifilter on $\Lambda$ and an antifilter is a dual notion to that of a filter. The family of all ultraintifilters on $\Lambda$ is denoted by $\mathcal{U}(\Lambda)$. If $\Lambda$ is a $\vee$-semilattice of subsets of $G$ (e.g. $\Lambda = \Gamma(G)$ or $= \Pi'(G)$ or $= \Pi(G)$) and $x \in \mathcal{U}(\Lambda)$, we define $\cup x = \cup\{K \in \Lambda: K \in x\}$. An ultraintifilter $x$ is called standard if $\cup x \neq G$. If $G \neq \{0\}$, every $x \in \mathcal{U}(\Pi'(G))$ is standard and every $x \in \mathcal{U}(\Lambda)$, where $\Lambda = \Gamma(G)$ or $\Pi(G)$, is standard iff $G$ has a weak unit. The set of all standard ultraintifilters on $\Gamma(G)$ is denoted by $\mathcal{U}_S(\Gamma(G))$. Assuming $\Lambda = \Gamma(G)$ or $\Pi'(G)$ or $\Pi(G)$ and $x \in \mathcal{U}(\Lambda)$, then $\cup x$ is a prime subgroup of $G$. $(\mathcal{U}(\Gamma(G)), \cup)$ and $(\mathcal{U}(\Pi'(G)), \cup)$ — briefly denoted by $\mathfrak{M}_\Gamma$ and $\mathfrak{M}_{\Pi'}$, respectively, are standard regulators of $G$, the latter is reduced and completely regular. $\mathfrak{M}_\Gamma$ or $\mathfrak{M}_{\Pi'}$ is called the $\Gamma$-regulator or the $\Pi'$-regulator of $G$, respectively.

Put $\Lambda = \mathcal{U}_S(\Gamma)$ or $= \mathfrak{M}_\Gamma$, respectively. Then the set

$$
\Sigma' = \{\cup f': f \in G\} \quad \text{or} \quad \Sigma = \{\cup K: K \in \Lambda\},
$$

where $\cup K = \{x \in \mathcal{U}(\Lambda): K \in x\}$ ($K \in \Lambda$), is a base of open sets for a topology on $\mathcal{U}_S(\Gamma(G))$ or $\mathcal{U}(\Lambda)$, respectively.

$$
(\mathcal{U}_S(\Gamma(G)), \Sigma') \quad \text{or} \quad (\mathcal{U}(\Lambda), \Sigma),
$$

respectively, is the notation of the corresponding space.

1.

1.1 Definition. A regulator $(\mathfrak{M}, \cup)$ of an $\ell$-group $G$ is called a regulator of type $\alpha$ (of type $\beta$) if $\cap \{\cup y: y \in \mathfrak{M}, y \neq x\} \neq \{0\}$ ($= \{0\}$) for every $x \in \mathfrak{M}$. If $(\mathfrak{M}, \cup)$ is a regulator of type $\alpha$ of $G$, then $G \neq \{0\}$ and $(\mathfrak{M}, \cup)$ is clearly reduced (and hence standard).

1.2 Definition. An $\ell$-group $G$ is said to be an $\ell$-group of kind $\alpha$ (of kind $\beta$) if an arbitrary polar of $G$ different from $G$ is contained in a maximal polar of $G$ (if in $G$ no maximal polar exists). A representable $\ell$-group is of kind $\alpha$ iff $G$ has an irreducible representation (see the following Proposition 1.3 and [5] 3.11). In [9] p. 407, I called the corresponding realization a realization of type $\alpha$.

By a maximal polar of $G$ there is meant a dual atom of the lattice $\Gamma(G)$ of polars of $G$. Dually, a minimal polar of $G$ is defined.
1.3 Lemma. The set of all dual atoms of $\Gamma(G)$ is equal to the set of all dual atoms of $\Pi'(G)$.
Proof. $\subseteq$: A dual atom of $\Gamma(G)$ (a maximal polar of $G$) is a dual principal polar because its disjoint complement, a minimal polar of $G$, is a principal polar.
$\supseteq$: If $K$ is a dual atom of $\Pi'(G)$, then $K'$ is a minimal polar of $G$. If not, there exists $a \in G$ such that $a^n K'$, $\{0\} \neq a \neq K'$, hence $a K, G \neq a' \neq K$, a contradiction. Consequently, $K$ is a maximal polar of $G$ (a dual atom of $\Gamma(G)$).

1.4 Proposition. An l-group $G \neq \{0\}$ is of kind $\alpha$ iff $G$ has a base.
Proof follows from [5] Theorem 3.4

1.5 Lemma. Let $(\mathcal{M}, \cup)$ be a regulator of an l-group $G$. If there exists $x \in \mathcal{M}$ such that $M - \cap \{y: y \in \mathcal{M}, y \neq x\}$, then $\cup x - M$ is a maximal polar of $G$.
Proof. Denote $J = \cup x$. Suppose $K \in \Gamma(G)$, $K \neq G$ and $K \neq M'$. There holds $J \cap M - \{0\}$, hence $M' - J$. Thus we have $K \neq J$. If $K \neq J$, then $K' \neq J$, whence $K = G$, a contradiction. Consequently, $K \cup J = M'$ is a maximal polar of $G$ since clearly $M' \neq G$.

1.6 Corollary. Every regulator of an l-group of kind $\beta$ is of type $\beta$.

1.7 Theorem. A standard regulator $(\mathcal{M}, \cup)$ of an l-group $G$ is of type $\alpha$ iff the mapping $\cup$ is injective and $\cup x$ a (maximal) polar of $G$ for every $x \in \mathcal{M}$.
Proof. Every regulator $(\mathcal{M}, \cup)$ of type $\alpha$ is reduced, thus the mapping $\cup$ is injective. By 1.5, $\cup x$ is a maximal polar of $G$ for every $x \in \mathcal{M}$.
Conversely, let the condition of Theorem be fulfilled, $x \in \mathcal{M}$ and $M = \cap \{y: y \in \mathcal{M}, y \neq x\}$. By the definition of a prime subgroup $(\cup x) M$ holds. Since $\cup x \cap M = \{0\}$, we have $(\cup x)' - M$, hence $(\cup x)' = M$. Consequently $M \neq \{0\}$ and $(\mathcal{M}, \cup)$ is of type $\alpha$.

1.8 Note. An analogous assertion as in 1.6 for l-groups of kind $\beta$ is not true, in general, for l-groups having a base, namely there does not hold the following statement:
(•) Every reduced regulator of an l-group having a base is of type $\alpha$.
Indeed, the set of all minimal prime subgroups of an arbitrary l-group $G \neq \{0\}$ is a reduced regulator ([13] II 1 5(1)). If $G$ has a base and if there exists a minimal prime subgroup of $G$, which is not a maximal polar, then by 1.7 this regulator is not of type $\alpha$. A characterization of l-groups whose every minimal prime subgroup is a (maximal) polar is given in 4.6.

1.9 Corollary. (1) Let an l-group $G \neq \{0\}$ have a base. Then the set of all maximal polars of $G$ together with the identical mapping is a regulator of type $\alpha$ of $G$.
(2) If $(\mathcal{M}, \cup)$ is a regulator of type $\alpha$ of an l-group $G$, then $\{\cup x: x \in \mathcal{M}\}$ is the set of all maximal polars of $G$.212
Proof. (1) By 3.4 [5] the intersection of the set of all prime subgroups that are polars is zero. Each of these polars is maximal or equal to $G$, [12] III 7.15. Hence the set of all maximal polars of $G$ together with the identical mapping is a standard regulator of $G$. This regulator is of type $\alpha$ by 1.7.

(2) By 1.7 $\cup x$ is a maximal polar of $G$ for every $x \in \mathfrak{M}$. $G$ has a base. In fact, for $G \neq L \in \Gamma(G)$, $L = L \vee \cap\{\cup x: x \in \mathfrak{M}\} = \cap\{L \vee \cup x: x \in \mathfrak{M}\}$. From the maximality of the polar $\cup x$, $L \vee \cup x = G$ or $L \subseteq \cup x$. The set of $x \in \mathfrak{M}$ with the property $L \subseteq \cup x$ is clearly nonempty, hence $G$ is of kind $\alpha$ and by 1.4 $G$ has a base. Now if $\{\cup x: x \in \mathfrak{M}\}$ is not the set of all maximal polars of $G$, then by (1), $\cap\{\cup x: x \in \mathfrak{M}\} \neq \{0\}$, a contradiction.

1.10 Theorem. Let $G$ be an $l$-group $\neq \{0\}$. Then the following conditions are equivalent.

(1) $G$ has a base.

(2) Every polar is an intersection of maximal polars of $G$.

(3) There exists a regulator of type $\alpha$ of $G$.

Proof. 1 $\Rightarrow$ 3. By 1.9(1).

3 $\Rightarrow$ 2. If $(\mathfrak{M}, \cup)$ is a regulator of type $\alpha$, then $\cup x$ ($x \in \mathfrak{M}$) is a maximal polar of $G$ by 1.7. Since $\cap\{\cup x: x \in \mathfrak{M}\} = \{0\}$ for an arbitrary $L \in \Gamma(G)$, $L = L \vee \cap\{\cup x: x \in \mathfrak{M}\} = \cap\{L \vee \cup x: x \in \mathfrak{M}\}$. From the maximality of the polar $\cup x$ it follows that $L \vee \cup x = G$ or $L \subseteq \cup x$. Consequently, $L = \cap\{\cup x: x \in \mathfrak{M}, L \subseteq \cup x\}$.

2 $\Rightarrow$ 1. From (2) it follows that $G$ is of kind $\alpha$, hence $G$ has a base by 1.4.

1.11 Proposition. Let $(\mathfrak{M}, \cup)$ be a standard regulator of $G$. Then the following conditions are equivalent.

(a) $(\mathfrak{M}, \cup)$ is similar to a regulator of type $\alpha$.

(b) The simplification of the regulator $(\mathfrak{M}, \cup)$ is of type $\alpha$.

(c) $\cup x$ is a (maximal) polar of $G$ for every $x \in \mathfrak{M}$.

Proof. Let $(\mathfrak{M}, \cup)$ be the simplification of $(\mathfrak{M}, \cup)$.

a $\Rightarrow$ b. If $(\mathfrak{M}, \cup)$ is similar to a regulator $(\mathfrak{M}_1, \cup_1)$ of type $\alpha$ and $\beta$ the corresponding mappings (see 0.1), then for every $x \in \mathfrak{M}$ \{0\} $\neq \alpha \cap\{\cup_1 \beta y: \beta y \in \mathfrak{M}_1, \beta y \neq \beta x\} = \cap\{\cup y: y \in \mathfrak{M}, y \neq \cup x\}$ because for the reduced regulator $(\mathfrak{M}_1, \cup_1)$ there holds $\beta y - \beta x = \cup y = \cup x$ ($x, y \in \mathfrak{M}$).

b $\Rightarrow$ c. By 1.7 $\cup \bar{x}$ is a maximal polar of $G$ for every $x \in \mathfrak{M}$. Hence $\cup x$ is a maximal polar of $G$ for every $x \in \mathfrak{M}$.

c $\Rightarrow$ a. If $\cup x$ is a polar of $G$ ($x \in \mathfrak{M}$), then $\cup x$ is a maximal polar ([5] 2.2 or [12] III 7.15). The simplification of $(\mathfrak{M}, \cup)$ is a regulator of type $\alpha$ by 1.7 and $(\mathfrak{M}, \cup)$ is similar to it.

1.12 Note. By [12] II 4.16, every $l$-group $G \neq \{0\}$ has a regulator. Moreover, for every regulator $(\mathfrak{M}_1, \cup_1)$ of $G$ there exists a reduced, completely regular regulator $(\mathfrak{M}_2, \cup_2)$ and a mapping $\varphi: \mathfrak{M}_1$ onto $\mathfrak{M}_2$ such that $\cup_1 x \supseteq \cup_2 \varphi(x)$ ($x \in \mathfrak{M}_1$). As $\varphi(x)$
(x ∈ ℜ), we define a minimal prime subgroup contained in ∪, r and for ∪, the identical mapping will be chosen. The regulator (ℜ, ∪) is evidently reduced and by [13] II 1.4 completely regular.

1.13 Theorem. A standard regulator (ℜ, ∪) of an l-group G is of type α iff the topological space (ℜ, G) is discrete.

Proof. If (ℜ, ∪) is of type α and x ∈ ℜ, then there exists 0 ≠ f ∈ ∩{y: y ∈ ℜ, y ≠ x}, thus ℜ ≠ Z(f) ⊇ ℜ\{x} and hence Z(f) = ℜ\{x}. Thus {x} is an open set.

If (ℜ, G) is a discrete space and x ∈ ℜ, then {y: y ∈ ℜ, y ≠ x} is a closed set, hence there exists f ∈ G such that x ∈ Z(f) and y ∈ Z(f) for y ≠ x. Thus 0 ≠ f ∈ ∩{y: y ∈ ℜ, y ≠ x}.

1.14 Proposition. Let (ℜ, ∪) be a standard regulator of an l-group G. Then G is of kind β iff the lattice ℜ(ℜ, G) has no atom.

Proof. The assertion follows from the fact that the existence of an atom of the lattice ℜ(ℜ, G) is equivalent to the existence of a dual atom of Γ(G) ([13] I 2.18), i.e. to the existence of a maximal polar of G.

1.15 Theorem. An l group G ≠ {0} is of kind β iff there exists a reduced regulator of type β of G.

(See [9] Satz 11).

Proof. Let G ≠ {0} be of kind β. There exists a reduced regulator of G and this is of type β by 1.6.

Conversely, let (ℜ, ∪) be a reduced regulator of type β of G and L a maximal polar of G. The set of all x ∈ ℜ with ∪x ∈ L has at least two elements. Otherwise, there holds ∩{y: y ∈ ℜ} ⊇ L' ≠ {0} or for some x ∈ ℜ, ∩{y: y ∈ ℜ\{x}} ⊇ L' ≠ {0}, a contradiction. Choose x, y ∈ ℜ, x ≠ y with ∪x ∩ y ∈ L. Since (ℜ, ∪) is reduced, there exist a, b ∈ G such that 0 < a ∈ ∪x \∪y, 0 < b ∈ ∪y ∪ x and a ∧ b = 0. Since L is a prime subgroup ([12] III 7.15 or [5] 2.2) there holds a ∈ L or b ∈ L ([5] 2.3 or [2] 2.4.1), thus a ∈ ∪y or b ∈ ∪x, a contradiction.

1.16 Corollary. A reduced regulator (ℜ, ∪) of an l-group G is of type β iff the lattice ℜ(ℜ, G) has no atom.

Proof. By 1.15 the condition may be replaced by the following one: G is of kind β. If this is the case, then by 1.6 (ℜ, ∪) is of type β. Conversely, if (ℜ, ∪) is reduced and of type β, G is of kind β by 1.15.

2.0 By 1.9 the role of maximal polars in the regulators of type α is described. In the following (sec. 3) we try to clarify the participation of maximal polars in reduced regulators of l-groups having a base, in other words, to what extent the reduced regulators of l-groups having a base “approximate” the regulators of type α. Sec. 2 has an auxiliary character

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2.1 Definition. ([2] 2.3.1) Let $J$ be a solid subgroup of an $l$-group $G$ and $G/J$ the set of left cosets of $G$ modulo $J$. Defining $a + J \geq b + J \iff$ there exists $f \in G$ such that $a + f \geq b$ ($a, b \in G$) we obtain a binary relation $\geq$, which is a distributive lattice ordering on $G/J$. If $(\mathfrak{M}, \cup)$ is a regulator of $G$, $x \in \mathfrak{M}$ and $f \in G$, $f(x)$ means the coset of $G/x$ containing $f(x)$. $f(0)$ will be denoted by $\cup x$, too.

2.2 Lemma. A regulator $(\mathfrak{M}, \cup)$ of an $l$-group $G \neq \{0\}$ is reduced iff for $x, y \in \mathfrak{M}$, $x \neq y$ there exists $f \in G$ such that $f(x) \cup x$ and $f(y) \cup y$.

Proof. Let regulator $(\mathfrak{M}, \cup)$ be reduced and $x, y \in \mathfrak{M}$, $x \neq y$. Then there exist $g \in \cup x \cup y$ and $h \in \cup y \cup x$. Denote $g = \lfloor |g| - |g| \wedge |h| \rfloor$, $h = \lfloor |h| - |g| \wedge |h| \rfloor$. Thus $0 < g, h \in \cup x \cup y$, $0 < h, g \in \cup y \cup x$ and $g, h \in \mathfrak{M}$. The element $f = -g, h$ fulfills the condition $f(x) = f + \cup x = -g, h + \cup x = h, g + \cup x \cup y$ ($g$ and $h$ commute) and $f(y) = f + \cup y = -h, g + \cup y = y, g + \cup y \cup y$.

Conversely, let the condition hold. Pick $x, y \in \mathfrak{M}$, $x \neq y$. By supposition, there exists $f \in G$ such that $f(x) \cup x$ and $f(y) \cup y$. Then $f^+(x) \cup x$ and $f^-(y) \cup y$ because $f^+ \geq f \geq f^-$. From the relation $f^+ \delta f^-$ and $f^+ \in \cup x$ it follows that $f^+ \in \cup x$ and similarly $f^- \in \cup y$. Finally, $f^+ \in \cup x \cup y$ and $f^- \in \cup y \cup x$, thus the regulator $(\mathfrak{M}, \cup)$ is reduced.

2.3 Proposition. a) A regulator of an $l$-group which is similar to a reduced regulator is standard.

b) If a reduced regulator is similar to a reduced regulator, then the similarity is an equivalence.

c) A regulator which is equivalent to a regulator of type $\alpha$ is itself of type $\alpha$.

d) A reduced regulator which is similar to a regulator of type $\alpha$ is itself of type $\alpha$.

Proof. Let $(\mathfrak{M}_i, \cup_i)$ be a regulator of an $l$-group $G$, $(i = 1, 2)$, let $(\mathfrak{M}_2, \cup_2)$ be similar to $(\mathfrak{M}_1, \cup_1)$, $\alpha$ and $\beta$ mappings from the definition of the similarity (0.1).

a) If $(\mathfrak{M}_1, \cup_1)$ is reduced and $\cup_2 x = G_2$ for some $x \in \mathfrak{M}_2$, then $\cup_1 \beta x = \alpha^{-1} \cup_2 x = G_1$, hence $(\mathfrak{M}_1, \cup_1)$ is not reduced, which is a contradiction.

b) If $(\mathfrak{M}_i, \cup_i)$ $(i = 1, 2)$ is reduced and $x, y \in \mathfrak{M}_2$, $x \neq y$, then $\cup_2 x \neq \cup_2 y$, hence $\cup_1 \beta x = \alpha^{-1} \cup_2 x \neq \alpha^{-1} \cup_2 y = \cup_1 \beta y$. Since $\cup_1$ is injective, we have $\beta x \neq \beta y$, thus the mapping $\beta$ is a bijection.

c) Let $(\mathfrak{M}_1, \cup_1)$ be of type $\alpha$ and let the similarity be an equivalence. Take $x \in \mathfrak{M}_2$. Then $\cap \{ \cup_2 y: y \in \mathfrak{M}_2, y \neq x \} = \cap \{ \cup_1 \beta y: y \in \mathfrak{M}, y \neq x \} = \cap \{ \cup_1 z: z \in \mathfrak{M}_1, z \neq \beta x \} \neq \{0\}$, hence $(\mathfrak{M}_2, \cup_2)$ is of type $\alpha$.

d) Let $(\mathfrak{M}_1, \cup_1)$ be of type $\alpha$ and $(\mathfrak{M}_2, \cup_2)$ reduced. Since both regulators are reduced, the similarity is an equivalence by b), and by c) $(\mathfrak{M}_2, \cup_2)$ is of type $\alpha$.

2.4 Theorem. An $l$-group $G$ has a base iff for a standard regulator $(\mathfrak{M}, \cup)$ the union of a subset $\mathfrak{A}$ of atoms of the lattice $\mathfrak{M}(\mathfrak{M}, G)$ is a dense subset of the space $(\mathfrak{M}, G)$.
Note. If the condition of Theorem is fulfilled, then \( \mathfrak{A} \) is the set of all atoms of the lattice \( \mathfrak{W}(\mathfrak{M}, G) \).

Proof. Let \( G \) have a base and let \((\mathfrak{M}, \cup)\) be a regulator of type \( \alpha \) of \( G \) (1.10). Then \( \{\cup x: x \in \mathfrak{M}\} \) is the set of all maximal polars (1.9(2)), \( \cap \{\cup x: x \in \mathfrak{M}\} = \{0\} \) (1.10) and hence \( \mathfrak{M} = Z(\cap \{\cup x: x \in \mathfrak{M}\}) = V_{\mathfrak{W}} \{Z(\cup x): x \in \mathfrak{M}\} \) = \( cl_{(\mathfrak{M}, G)}(\cup \{Z(\cup x): x \in \mathfrak{M}\}) \) and \( Z(\cup x) \) is an atom of the lattice \( \mathfrak{W}(\mathfrak{M}, G) \), [13] I 2.18 and 2.19.

Let \((\mathfrak{M}, \cup)\) be a standard regulator of \( G \). Let \( \mathfrak{A} = \{A_v: v \in \mathfrak{N}\} \) be a set of atoms of the lattice \( \mathfrak{W}(\mathfrak{M}, G) \) and \( \bigcup A_v \) a dense subset of the space \((\mathfrak{M}, G)\). Then \( cl_{(\mathfrak{M}, G)}(\bigcup A_v) = \mathfrak{M} \), whence \( V_{\mathfrak{W}} A_v = \mathfrak{M}, \{0\} = \Psi\left(\bigvee_{v \in \mathfrak{N}} A_v\right) = \bigcap_{v \in \mathfrak{N}} \Psi A_v \) and every \( \Psi A_v \) is a maximal polar of \( G \), and so the set \( \mathfrak{R} = \{\Psi A_v: v \in \mathfrak{N}\} \) together with the identical mapping is a regulator of \( G \) because maximal polars are prime subgroups, [12] III 7.15 or [5] 2.2. This regulator is of type \( \alpha \) by 1.7. By 1.9(2), \( \mathfrak{R} \) is the set of all maximal polars of \( G \), hence \( \mathfrak{A} \) is the set of all atoms of \( \mathfrak{W}(\mathfrak{M}, G) \).

2.5 Lemma. Let \((\mathfrak{M}, \cup)\) be a standard regulator of an l-group \( G \) and let \( A \subseteq \mathfrak{M} \).

Then \( y \in A \Leftrightarrow \cap \{\cup x: x \in A\} \subseteq u y \), especially \( y \in \hat{x} \Leftrightarrow u x \subseteq u y \).

Proof. We have: \( Z(f) \supseteq A \Leftrightarrow \{x: f \in \cup x\} \supseteq A \Leftrightarrow f \in \cap \{\cup x: x \in A\} \Leftrightarrow \cup y \supseteq \cap \{\cup x: x \in A\} \).

2.6 Proposition. Let \((\mathfrak{M}, \cup)\) be a standard regulator of an l-group \( G \) and \( x \in \mathfrak{M} \).

The following conditions are equivalent.

1. \( \cup x \) is a polar of \( G \).
2. \( \cup x \) is a maximal polar of \( G \).
3. \( \hat{x} \in \mathfrak{W}(\mathfrak{M}, G) \).
4. \( \hat{x} \) is an atom of the lattice \( \mathfrak{W}(\mathfrak{M}, G) \).

If the regulator \((\mathfrak{M}, \cup)\) is reduced, then the following condition is equivalent to the preceding ones.

5. \( x \) is an isolated point of the space \((\mathfrak{M}, G)\).

Proof. 1 \( \Rightarrow \) 3. If \( \cup x \in \Gamma(G) \), then \( \hat{x} = Z\Psi(x) = Z(\cup x) \in \mathfrak{W}(\mathfrak{M}, G) \) ([13] I 2.8 and 2.18).

3 \( \Rightarrow \) 4. If \( A \) is an open subset of \((\mathfrak{M}, G), \emptyset \neq A \subseteq \hat{x} \) and \( \emptyset \neq \hat{x} \), then \( x \in A \), hence the closed set \( \hat{x} \setminus A \) contains the point \( x \). Consequently \( \hat{x} \subseteq \hat{x} \setminus A \), a contradiction.

4 \( \Rightarrow \) 2. If \( \hat{x} \) is an atom of \( \mathfrak{W}(\mathfrak{M}, G) \), then \( \Psi(\hat{x}) = \Psi Z\Psi(x) = \Psi(x) = \cup x \) is a maximal polar of \( G \) ([13] I 2.4 and 2.8).

2 \( \Rightarrow \) 1 is evident.

If \((\mathfrak{M}, \cup)\) is reduced, then by 2.5, \( x = \hat{x} \) for every \( x \in \mathfrak{M} \), i.e. \((\mathfrak{M}, G)\) is a \( T_1 \)-space and we have there:

\( \hat{x} \in \mathfrak{W}(\mathfrak{M}, G) \Leftrightarrow x \) is an isolated point of \((\mathfrak{M}, G)\).
2.7 Definition. The atoms of the lattice of closed sets of a topological space $P$ will be called trivial closed sets of $P$. Analogously for open or clopen sets.

Some simple lemmas concerning the preceding notions follow.

2.8 Lemma. a) If the trivial open sets of $P$ form a partition on $P$ (say $S$), then the trivial closed sets of $P$ form a partition on $P$ (say $R$) and $R - S$ holds.

b) If $T$ is a trivial closed set of $P$ and $\text{Int } T \neq \emptyset$, then $T$ is a trivial open set of $P$.

Proof. a) The blocks $T$ of $S$ are closed sets. If some $T$ is not trivial closed, there exists a closed set $V \subseteq T$ such that $\emptyset \neq V \neq T$. Then $X = (P \setminus V) \cap T$ is an open set, $X \subseteq T$, $\emptyset \neq X \neq T$ and $T$ is not trivial open.

b) If $\emptyset \neq A \subseteq T$ and $A$ is open, then either $T \setminus A = \emptyset$ or $T \setminus A$ is a proper closed subset of $T$, hence $T = A$, i.e. $T$ is a trivial open set.

2.9 Lemma. Let $P$ be a topological space, $A, B \subseteq P$ and $A = P \setminus B$. Then there holds

$$\overline{\text{Int } B} = B = \text{Int } A = A.$$ 

i.e. the complement of a regular closed set is a regular open set and conversely.

Proof. Suppose $\overline{\text{Int } B} = B$. Then

$$A = P \setminus B \Rightarrow A = P \setminus B \Rightarrow P \setminus A = P \setminus (P \setminus B) = \text{Int } B \Rightarrow P \setminus A = \text{Int } A.$$ 

Suppose $\text{Int } A = A$. Then

$$P \setminus B = A = \text{Int } \overline{A} = P \setminus (P \setminus A) \Rightarrow B = P \setminus A = P \setminus (P \setminus B) = \overline{\text{Int } B} \Rightarrow B = \text{Int } B.$$

2.10 Proposition. Let $P$ be a topological space. The following conditions are equivalent.

1. a) $P$ contains a base for closed sets formed by open sets.
   b) $P$ is a locally connected space.
2. Every base for closed sets of the space $P$ is formed by open sets.
3. Trivial open sets form a partition on $P$.
4. Every closed set of $P$ is open ($\equiv$ every open set of $P$ is closed).
5. $A \subseteq P$, $A$ open in $P$ $\Rightarrow \text{Int } \overline{A} = A$ (i.e. open sets of $P$ are regular open).
6. $\mathcal{M}(P) = \mathcal{R}(P)$ (i.e. closed sets of $P$ are regular closed)
7. $x$ is an open set of $P$ for every $x \in P$
8. $x \in \mathcal{M}(P)$ for every $x \in P$.
   If $P$ is a $T_1$-space, then evidently the preceding conditions are equivalent to the following one.
9. $P$ is a discrete space.
Proof. 1⇒3. From [12] IV 9.2 it follows that 1a ⇒ every block T of the partition on P, the blocks of which are maximal connected sets, is a trivial closed set.

From 1b it follows that T is an open set ([7] I, Ex. Ua) and by 2.8(b), T is a trivial open set.

3⇒4. By 2.8(a), every nonempty closed set is a union of blocks of the partition which is formed by the trivial open sets, hence it is open.

4⇒5. The closure \( \bar{A} \) of every set \( A \subseteq P \) is open, hence Int \( \bar{A} = \bar{A} \). If \( A \) is open, then it is closed by supposition, hence \( \bar{A} = A \). Thus Int \( \bar{A} = A \) for every open set \( A \) of \( P \).

5⇒6. \( B \in \mathfrak{M}(P) \Rightarrow P \setminus B = A \) is open ⇒ Int \( \bar{A} = A \) ⇒ \( B \in \mathfrak{M}(P) \) (2.9).

6⇒8. \( x \in P \Rightarrow \bar{x} \in \mathfrak{M}(P) \Rightarrow x \in \mathfrak{M}(P) \).

8⇒7. Choose \( x \in P \). If Int \( \bar{x} \neq \bar{x} \), then there exists \( y \in \bar{x} \setminus \text{Int} \bar{x} \). Since the set \( \bar{x} \setminus \text{Int} \bar{x} \) is closed, there holds Int \( \bar{y} \subseteq \bar{y} \subseteq \bar{x} \setminus \text{Int} \bar{x} \) and hence Int \( \bar{y} \cap \text{Int} \bar{x} = \emptyset \). From the relation \( \bar{y} \subseteq \bar{x} \) we obtain Int \( \bar{y} \subseteq \text{Int} \bar{x} \), whence Int \( \bar{y} = \emptyset \). But this contradicts the relations \( \emptyset \neq \bar{y} = \text{Int} \bar{y} = \emptyset \). Finally, Int \( \bar{x} = \bar{x} \), and \( \bar{x} \) is an open set.

7⇒2 is evident.

2⇒1. 1a holds evidently. We prove 1b. Every closed set is open because the system of all closed sets is a base for closed sets. By 2.8(b) every trivial closed set of \( P \) is a trivial open set. By [12] IV 9.2 the partition \( R, \) every block of which is a maximal connected set of \( P, \) is equal to the set of all trivial closed sets of \( P. \) Now it follows immediately that the space \( P \) is locally connected. Indeed, the maximal connected sets are trivial open and hence form a base for open sets, [7] I, Ex. Ub.

3.1 Theorem. A standard regulator \( (\mathfrak{R}, \cup) \) of an \( l \)-group \( G \) is similar to a regulator of type \( \alpha \) iff \( \mathfrak{M}(\mathfrak{R}, G) = \mathfrak{R}(\mathfrak{R}, G) \). If the condition is fulfilled, then \( G \) has a base.

Proof. Let a regulator \( (\mathfrak{R}, \cup) \) of \( G \) be similar to a regulator of type \( \alpha \). By 1.9(2) and 0.1 \( \cup x \) is a maximal polar of \( G \) for every \( x \in \mathfrak{R} \); by 2.6 \( \bar{x} \in \mathfrak{M}(\mathfrak{R}, G) \) and by 2.10 \( \mathfrak{K}(\mathfrak{R}, G) = \mathfrak{R}(\mathfrak{R}, G) \).

Conversely, suppose \( \mathfrak{M}(\mathfrak{R}, G) = \mathfrak{R}(\mathfrak{R}, G) \). This equality implies \( \Gamma(G) = \Omega(\mathfrak{R}, G) \), [13] I 2.12 and 2.18, and so \( \Gamma(G) = \Omega(\mathfrak{R}, G) \supseteq \{ \Psi(x): x \in \mathfrak{R} \} = \{ \cup x: x \in \mathfrak{R} \} \). Hence \( \cup x \) is a polar for every \( x \in \mathfrak{R} \). By 1.11 \( (\mathfrak{R}, \cup) \) is similar to a regulator of type \( \alpha \). By the same theorem the simplification of \( (\mathfrak{R}, \cup) \) is of type \( \alpha \) and by 1.10 \( G \) has a base.

3.2 Proposition. Let a regulator \( (\mathfrak{R}, \cup) \) of \( G \neq \{0\} \) be similar to a reduced regulator, \( \alpha \) and \( \beta \) the corresponding mappings (see 0.1) and \( R \) the partition induced by \( \beta \) on \( \mathfrak{R} \). Then the blocks of \( R \) are trivial closed sets of the space \( (\mathfrak{R}, G) \).
Conversely, if trivial closed sets of \((\mathcal{M}, G)\) form a partition on \(\mathcal{M}\), then \((\mathcal{M}, \cup)\) is similar to a reduced regulator and the simplification of \((\mathcal{M}, \cup)\) is a reduced regulator.

Proof. Let a regulator \((\mathcal{M}, \cup)\) of \(G\neq\{0\}\) be similar to a reduced regulator \((\mathcal{M}_1, \cup_1)\) and \(T = \beta^{-1}y\) for some \(y \in \mathcal{M}_1\). If \(T\) is not trivial closed, there exist \(x_1, x_2 \in T\) and \(f \in G\) such that \(x_1 \in Z(f)\) and \(x_2 \in Z(f)\). Thus \(f \in \cup_1 \beta x_1 \cup_1 \beta x_2\). This set is empty because \(\beta x_1 = \beta x_2 = y\), a contradiction.

Conversely, let trivial closed sets of \((\mathcal{M}, G)\) form a partition on \(\mathcal{M}\), say \(R\). Let \(\cup_1\) be a mapping of \(R\) into \(\mathfrak{A}(G)\) such that \(\cup_1 \tilde{x} = \cup x_\alpha\) for every \(\tilde{x} \in R\) and for a fixed \(x_\alpha \in \tilde{x}\). Then \((R, \cup_1)\) is a regulator of \(G\). Indeed, choose \(f \in G\) and \(x \in R\) with \(f \in \cup_1 \tilde{x}\) and pick \(y \in \tilde{x}\). Then \(f \in \cup x_\alpha\), i.e. \(x_\alpha \in Z(f)\), whence \(y \in Z(f)\) because \(\tilde{x}\) is trivial closed. We have got \(\cup x_\alpha \subseteq \cup y\). Consequently \(\cap\{\cup_1 \tilde{x} : \tilde{x} \in R\} \subseteq \cap\{\cup x : x \in \mathcal{M}\} = \{0\}\). \((R, \cup_1)\) is reduced. In fact, suppose \(\tilde{x}, \tilde{y} \in R\) and \(\cup_1 \tilde{x} \supseteq \cup_1 \tilde{y}\). Then \(\cup x_\alpha \supseteq \cup y_\alpha\) and by 2.5 \(x_\alpha \in cl_{(\mathcal{M}, \alpha)}(\{y_\alpha\}) = \tilde{y}\). Hence \(\tilde{x} = cl_{(\mathcal{M}, \alpha)} x_\alpha = \tilde{y}\). Finally, \((R, \cup_1)\) is clearly the simplification of \((\mathcal{M}, \cup)\).

### 3.3 Corollary
Let \((\mathcal{M}, \cup)\) be a standard regulator of \(G\). Then the following conditions are equivalent.

1. \((\mathcal{M}, \cup)\) is similar to a reduced regulator.
2. The simplification of \((\mathcal{M}, \cup)\) is a reduced regulator.
3. The blocks of the equivalence relation \(R\) on \(\mathcal{M}\), defined by the rule \(xRy \equiv \cup x = \cup y\), are trivial closed sets of \((\mathcal{M}, \cup)\).

Proof. 1 \(\Rightarrow\) 2 by 3.2.

2 \(\Rightarrow\) 3. \((\mathcal{M}, \cup)\) is similar to its simplification, thus we have 3 by 3.2.

3 \(\Rightarrow\) 1 by 3.2.

### 3.4 Lemma
Let \((\mathcal{M}, \cup)\) be a regulator of \(G\neq\{0\}\) similar to a reduced regulator, \(\alpha\) and \(\beta\) the corresponding mappings and \(R\) the partition induced by \(\beta\) on \(\mathcal{M}\).

a) If \(B\) is an open set of \((\mathcal{M}, G)\), then \(B\) contains every trivial closed set which it meets.

b) If \(A\) is an atom of the lattice \(\mathfrak{A}(\mathcal{M}, G)\), then \(A\) is a trivial clopen set of \((\mathcal{M}, G)\) and if \(T \in R\) and \(T \cap A \neq \emptyset\), then \(T = A\).

Proof. By 3.2 trivial closed sets of \((\mathcal{M}, G)\) are blocks of the partition \(R\).

a) \(T \in R, \emptyset \neq T \cap B \neq T \Rightarrow \emptyset \neq T \cap B \subseteq T, T \cap B\) closed \(\Rightarrow T \cap B = T \Rightarrow T \cap B = \emptyset\), a contradiction.

b) Choose \(T, V \in R, T \neq V\). By 2.2 and 3.3 for arbitrary \(x \in T\) and \(y \in V\) there exists \(f \in G\) such that \(f(x) > \cup x\) and \(f(y) < \cup y\). Then \(f^*(-x) > \cup x, f^*(-y) < \cup y\) and so \(x \in \mathcal{M}\setminus Z(f^*)\) and \(y \in \mathcal{M}\setminus Z(f^*)\). Since \(\hat{f}^*\delta^*_f\), we have \((\mathcal{M}\setminus Z(f^*)) \cap (\mathcal{M}\setminus Z(f^*)) = \emptyset\). [13] 1 2.15. We have proved the existence of disjoint open neighbourhoods \(C\) and \(D\) of the points \(x\) and \(y\), respectively, \(C = \mathcal{M}\setminus Z(f^*)\) and \(D = \mathcal{M}\setminus Z(f^*)\). Let \(A\) be an atom of the lattice \(\mathfrak{A}(\mathcal{M}, G)\) such that \(x, y \in \text{Int } A\) (= \(B\)). The set \(A\setminus C\) is closed and \(\emptyset \neq D \cap B \subseteq A\setminus C\) holds. Then \(\emptyset \neq cl_{(\mathcal{M}, \cup)}(D \cap B) \subseteq A\setminus C\), a
contradiction. It follows that $B$ meets only one block of the partition $R$, say $T$. Thus $T \supseteq B$ and by a) $T = B$. Thus $T = \text{cl}_{(\mathfrak{R}, \mathfrak{G})} T = \text{cl}_{(\mathfrak{R}, \mathfrak{G})} B = A$ and $A$ is a trivial clopen set of $(\mathfrak{R}, \mathfrak{G})$.

3.5 Theorem. Let $(\mathfrak{R}, \mathfrak{G})$ be a regulator of an $I$-group $G \neq \{0\}$ similar to a reduced regulator. Then the following conditions are equivalent.

1. $G$ has a base.
2. The union $\mathfrak{S}$ of all atoms of the lattice $\mathfrak{W}(\mathfrak{R}, \mathfrak{G})$ is a dense subset of $(\mathfrak{R}, \mathfrak{G})$.
3. There exists a dense (open) subspace $\mathfrak{S}$ of the space $(\mathfrak{R}, \mathfrak{G})$ such that $\mathfrak{R}(\mathfrak{S}) = \mathfrak{W}(\mathfrak{S})$.

If $(\mathfrak{R}, \mathfrak{G})$ is reduced, the following condition is equivalent to the preceding ones.

4. The set of all isolated points of the space $(\mathfrak{R}, \mathfrak{G})$ is a dense subset of $(\mathfrak{R}, \mathfrak{G})$.

Note. If condition 2 is true, then the set $\mathfrak{S}$ from 2 has the property of the set $\mathfrak{S}$ from condition 3.

Proof. Let $\{A_a\}$ be the system of all atoms of the lattice $\mathfrak{W}(\mathfrak{R}, \mathfrak{G})$. By [13] I 2.18 $\{\Psi(A_a)\}$ is the system of all maximal polars of $G$.

1$\Rightarrow$2. By 1.9 or [5] Theorem 3.4, $\bigcap_a \Psi(A_a) = \{0\}$. It follows that $\mathfrak{R} = \bigvee_a A_a = \text{cl}_{(\mathfrak{R}, \mathfrak{G})} \bigcup A_a$, [13] I 2.19. Hence the union $\mathfrak{S}$ of all atoms of the lattice $\mathfrak{W}(\mathfrak{R}, \mathfrak{G})$ is a dense subset of $(\mathfrak{R}, \mathfrak{G})$.

2$\Rightarrow$3. The union $\mathfrak{S}$ of all atoms of the lattice $\mathfrak{W}(\mathfrak{R}, \mathfrak{G})$ is open by 3.4(b) and by the supposition a dense subset of $(\mathfrak{R}, \mathfrak{G})$. Let $R$ be the partition induced by the mapping $\beta$ defining the similarity of $(\mathfrak{R}, \mathfrak{G})$. By 3.4(b) every $T \in R$ which meets $\mathfrak{S}$ is a trivial clopen set of $(\mathfrak{R}, \mathfrak{G})$. Hence if $A \in \mathfrak{W}(\mathfrak{R}, \mathfrak{G})$ meets $\mathfrak{S}$, then $A \cap \mathfrak{S}$ is an open subset of $(\mathfrak{R}, \mathfrak{G})$ and a closed subset of the subspace $\mathfrak{S}$. It follows that $A \cap \mathfrak{S} = \mathfrak{S} \cap \text{cl}_{(\mathfrak{R}, \mathfrak{G})}(A \cap \mathfrak{S}) = \text{cl}_{\mathfrak{S}}(A \cap \mathfrak{S}) \in \mathfrak{W}(\mathfrak{S})$, hence $\mathfrak{R}(\mathfrak{S}) = \mathfrak{W}(\mathfrak{S})$.

3$\Rightarrow$1. Let $\mathfrak{S}$ be a dense subspace of $(\mathfrak{R}, \mathfrak{G})$ such that $\mathfrak{R}(\mathfrak{S}) = \mathfrak{W}(\mathfrak{S})$. Then $(\mathfrak{S}, \mathfrak{U})$, where $\mathfrak{U} = \mathfrak{U}|_{\mathfrak{S}}$ is a standard regulator of $G$, [13] II 4.9.

It is evident that $\mathfrak{S} \cap Z_{(\mathfrak{R}, \mathfrak{U})}(f) = Z_{(\mathfrak{S}, \mathfrak{U})}(f)$, hence the identical mapping of $\mathfrak{S}$ is a homeomorphism of the space $(\mathfrak{S}, \mathfrak{G})$ onto the subspace $\mathfrak{S}$ of $(\mathfrak{R}, \mathfrak{G})$. Consequently, $\mathfrak{R}(\mathfrak{S}) = \mathfrak{W}(\mathfrak{S}, \mathfrak{G})$ and $\mathfrak{W}(\mathfrak{S}) = \mathfrak{W}(\mathfrak{S}, \mathfrak{G})$. By 3.1 $G$ has a base.

1$\Rightarrow$4. As in 1$\Rightarrow$2, $\mathfrak{R} = \text{cl}_{(\mathfrak{R}, \mathfrak{G})} \bigcup A_a$, where $A_a$ are atoms of the lattice $\mathfrak{W}(\mathfrak{R}, \mathfrak{G})$.

By 3.4(b) every $A_a$ is a trivial clopen set and is equal to a block of the partition $R$ induced on $\mathfrak{R}$ by the mapping $\beta$ defining the similarity of $(\mathfrak{R}, \mathfrak{G})$. Since $(\mathfrak{R}, \mathfrak{G})$ is reduced, the similarity is an equivalence (2.3(b)) and hence $\beta$ is one-to-one. Therefore, every $A_a$ is an isolated point of $(\mathfrak{R}, \mathfrak{G})$.

4$\Rightarrow$2 is evident.

In the following Theorem, the results of Theorem 2.6, 2.10 and 3.1 will be summarized.
3.6 Theorem. Let \((\mathcal{R}, \cup)\) be a standard regulator of an \(l\)-group \(G\). The following conditions are equivalent.

\begin{enumerate}[
  \text{a)}
\end{enumerate}

- The \(l\)-group \(G\) has a base, the regulator \((\mathcal{R}, \cup)\) is completely regular and the union of all atoms of the lattice \(\mathcal{M}(\mathcal{R}, G)\) is a closed set of \((\mathcal{R}, G)\).

\begin{enumerate}[
  \text{b)}
\end{enumerate}

- Any condition of Theorem 2.6 fulfilled for every \(x \in \mathcal{R}\).

\begin{enumerate}[
  \text{c)}
\end{enumerate}

- Any condition of Theorem 2.10 for \(P = (\mathcal{R}, G)\).

\begin{enumerate}[
  \text{d)}
\end{enumerate}

- Any condition of Theorem 3.1.

Moreover, if \((\mathcal{R}, \cup)\) is reduced, then the following conditions are equivalent to the preceding ones.

\begin{enumerate}[
  \text{e)}
\end{enumerate}

- The regulator \((\mathcal{R}, \cup)\) is of type \(a\).

\begin{enumerate}[
  \text{f)}
\end{enumerate}

- The space \((\mathcal{R}, G)\) is discrete.

\textbf{Proof.} \(b \Rightarrow c\) because 2.6(3) = 2.10(8).

\(c \Rightarrow d\) because both Theorems have the condition \(\mathcal{M}(\mathcal{R}, G) = \mathcal{M}(\mathcal{R}, G)\) in common.

\(c \land d \Rightarrow a\). From \(c\) (2.10(2)) it follows that \((\mathcal{R}, \cup)\) is completely regular ([13] II 1.5). The remaining two conditions follow from \(d\) \((G\) has a base\) and \(c\) (2.10(4)).

\(a \Rightarrow c\) (2.10(1)). We shall prove that every point \(x \in \mathcal{R}\) has a fundamental system of connected neighbourhoods ([4] I § 11.6, Df. 4). Thus it will be shown that the space \((\mathcal{R}, G)\) is locally connected which is the condition 1(b) of 2.10. If \(B\) is a neighbourhood of the point \(x\), then there exists \(f \in G\) such that \(x \in \mathcal{R} \setminus Z(f) \supset B\). Since \(G\) has a base, the meet of all maximal polars \(g'_a\) \((a \in A)\) is equal to zero,

\[\bigcap_{a \in A} g'_a = \{0\}\] ([13] I 2.18 and 2.19). Since \(Z(g_a)\) is a clopen set ([13] II 1.4), the set \(Z(g'_a) = \mathcal{R} \setminus Z(g_a)\) is clopen as well. Since \(\{Z(g'_a): a \in A\}\) is the family of all atoms of the lattice \(\mathcal{M}(\mathcal{R}, G)\) ([13] I 2.18) and \(\bigcup_{a \in A} Z(g'_a)\) is closed by supposition, then \(\mathcal{R} = \bigcup_{a \in A} Z(g'_a)\).

Thus there exists \(a_0 \in A\) such that \(x \in Z(g'_a)\). \(Z(g'_a)\) is a connected neighbourhood of the point \(x\) because it is clopen and an atom of \(\mathcal{M}(\mathcal{R}, G)\). Now \(Z(f')\), \(Z(g'_a) \in \mathcal{M}(\mathcal{R}, G)\), the set \(Z(g'_a)\) is an atom of the lattice \(\mathcal{M}(\mathcal{R}, G)\) and intersects \(Z(f')\) (in \(x\), since \(x \in \mathcal{R} \setminus Z(f) = Z(f')\)), hence \(Z(f') \supseteq Z(g'_a)\). Consequently \(B \supseteq \mathcal{R} \setminus Z(f) = Z(f') \supseteq Z(g'_a)\). We have proved that an arbitrary neighbourhood of the point \(x\) contains a connected neighborhood of \(x\). Thus the space \((\mathcal{R}, G)\) is locally connected. Finally, 2.10(1a) follows from the complete regularity of \((\mathcal{R}, \cup)\) ([13] II 1.4).

\(e \Rightarrow d\) is evident.

\(d \Rightarrow e\) by 2.3(d).

\(e \Leftrightarrow f\) by 1.13.

3.7 Theorem. Let \((\mathcal{R}, \cup)\) be a completely regular regulator of an \(l\)-group \(G\). The following conditions are equivalent.
a) The connected components of the space $(\mathcal{M}, G)$ are open.
b) The space $(\mathcal{M}, G)$ is locally connected.
c) If $J$ is a minimal prime subgroup of $G$ and $Z(J) \neq \emptyset$, then $J$ is a (maximal) polar of $G$.
d) If $x \in \Pi(\Pi'(G))$ and $Z(\cup x) \neq \emptyset$, then $x$ is a principal antifilter on $\Pi'(G)$.
e) Any condition of Theorem 3.6.

Note. In a topological space the conditions a) and b) are not equivalent in general. There holds $b \Rightarrow a$, see [4] I § 11, 6, Prop. 11.

Proof. a $\Rightarrow$ e (2.10(7)). By [4] I § 11, Ex. 12 the condition a) is equivalent to the following one: For an arbitrary $x \in \mathcal{M}$ the meet $\hat{x}$ of all clopen sets containing $x$ is an open set. By the definition of the closure $\bar{x}$ of $\{x\}$ there holds $\bar{x} \supset x$. By supposition the basic sets $Z(f)$ ($f \in G$) containing $x$ are clopen ([13] II 1.4), hence their meet (equal to $x$) contains $\hat{x}$. Thus we have $x = x$ and so $x$ is an open set.

e $\Rightarrow$ b is evident.

b $\Rightarrow$ a by [4] I § 11, 6 Prop. 11.
d $\Rightarrow$ c. Choose $J \in \mathcal{M}(G)$ with $Z(J) \neq \emptyset$. There holds $J = \cup x$ for some $x \in \Pi(\Pi'(G))$ (remember that $\cup x = \cup \{a : a \in x\}$), see [2] 3.4.15 Since $x$ is a principal antifilter, it is generated by a maximal element of the lattice $\Pi'(G)$ say $a'$, hence by a maximal polar of $G$ (1.3). Thus $\cup x = J$ is a maximal polar of $G$.
c $\Rightarrow$ e. $\cup x$ is a minimal prime subgroup for every $x \in \mathcal{M}$ ([13] II 1.4). Since the set $Z(\cup x)$ contains $x$, it is nonempty, and so by c) $\cup x$ is a polar of $G$. By 1 11 ($\mathcal{M}, \cup$) is similar to a regulator of type $\alpha$ (which is one of the conditions of 3.1).
e $\Rightarrow$ d. Choose $x \in \Pi(\Pi'(G))$ with $Z(\cup x) \neq \emptyset$. Then $\cup x$ is a minimal prime subgroup of $G$ and since $Z(\cup x) \neq \emptyset$, there holds $\cup x = \cup y$ for some $y \in \mathcal{M}$ ([13] II 1 4). By supposition $\cup y$ is a maximal polar of $G$ (2.6(2)) Consequently, $\cup y = a$ for some $a \in G$, thus $x$ is a principal antifilter on $\Pi'(G)$ generated by the dual principal polar $a'$.

3.8 Theorem. Let $(\mathcal{M}, \cup)$ be a regulator of type $\alpha$ of an l-group $G$ and $(\mathcal{M}, \cup)$ a regulator of $G$ similar to a reduced regulator. Let $\mathcal{E}$ be the union of all atoms of the lattice $\mathcal{M}(\mathcal{M}, G)$. Then there exists a continuous, open and closed mapping $\sigma$ of the subspace $\mathcal{E}$ of the space $(\mathcal{M}, G)$ onto the space $(\mathcal{M}, G)$. If the regulator $(\mathcal{M}, \cup)$ is reduced, $\sigma$ is a homeomorphism.

Proof. The regulator $(\mathcal{M}, \cup)$ ($i = 1$, 2) is standard. Define a binary relation $\sigma$ between the sets $\mathcal{E}$ and $\mathcal{M}_i$ as follows: $\sigma^{-1}(x) = Z_{\mathcal{M}_i}(\cup_i x)$ for every $x \in \mathcal{M}_i$. We shall show that $\sigma$ is a mapping of $\mathcal{E}$ onto $\mathcal{M}_i$. Since $\cup_i x$ is a maximal polar of $G$ (1.9(2)), $Z_{\mathcal{M}_i}(\cup_i x)$ is an atom of the lattice $\mathcal{M}(\mathcal{M}_i, G)$ ([13] I 2.18). Hence it is a subset of $\mathcal{E}$. For different elements $x, y \in \mathcal{M}_i$ the sets $\sigma^{-1}(x)$ and $\sigma^{-1}(y)$ are different because the mapping $Z_{\mathcal{M}_i}(\cdot) : \mathcal{M}(\mathcal{M}_i, G) \rightarrow \mathcal{M}(\mathcal{M}_i, G)$ is one-to-one. Hence $\sigma$ is a mapping of a subset of $\mathcal{E}$ onto $\mathcal{M}_i$. Pick an arbitrary atom $A$ of the lattice $\mathcal{M}(\mathcal{M}_i, G)$. Then $\mathcal{M}_i(A)$ is a maximal polar of $G$ and $Z_{\mathcal{M}_i}(A)$ is an atom of the lattice
Since the space \((\mathfrak{M}_1, G)\) is discrete by 1.13, this is a singleton, say \(\{x\}\). Hence
\[
\sigma^{-1}(x) = Z_{\mathfrak{M}_2}(\cup, x) = Z_{\mathfrak{M}_2}, \Psi_{\mathfrak{M}_2}, Z_{\mathfrak{M}_1}, \Psi_{\mathfrak{M}_1}(A) = Z_{\mathfrak{M}_1}, \Psi_{\mathfrak{M}_1}(A) = A,
\]
[13] 1.2.4. Thus it is proved that \(\sigma\) is a mapping of the set \(\mathcal{S}\) onto \(\mathfrak{M}_1\). Since the space \((\mathfrak{M}_2, G)\) is discrete, \(\sigma\) is an open and closed mapping of the subspace \(\mathcal{S}\) of the space \((\mathfrak{M}_2, G)\) onto the space \((\mathfrak{M}_1, G)\). \(\sigma\) is continuous. In fact, as we know, the set \(\sigma^{-1}(x) = Z_{\mathfrak{M}_2}(\cup, x)\) is an atom of the lattice \(\mathfrak{M}(\mathfrak{M}_2, G)\), consequently by 3.4(b) it is a trivial clopen set of the space \((\mathfrak{M}_2, G)\).

If the regulator \((\mathfrak{M}_2, \cup)\) is reduced, then atoms of the lattice \(\mathfrak{M}(\mathfrak{M}_2, G)\) are singletons, hence the mapping \(\sigma\) is one-to-one. In this case, \(\mathcal{S}\) is the set of all isolated points of \((\mathfrak{M}_2, G)\), hence \(\sigma\) is a homeomorphism.

4.

4.1 Lemma. Let \(\Lambda\) be a \(\lor\)-semilattice with the greatest element 1. An ultraantifilter on \(\Lambda\) is a principal antifilter iff it is generated by a dual atom of \(\Lambda\).

The proof is straightforward.

4.2 Lemma. Let \(\Lambda\) be a \(\lor\)-semilattice with the greatest element 1. If an ultraantifilter \(x\) on \(\Lambda\) is a principal antifilter, then \(x\) is an isolated point of the topological space \((\overline{\Lambda}(\Lambda), \Sigma)\).

Proof. If an ultraantifilter \(x\) on \(\Lambda\) is a principal antifilter and \(L\) its generator, then \(L\) is a dual atom of \(\Lambda\) (4.1), thus \(\overline{\Lambda}(x) = \{x\}\) and hence \(x\) is an isolated point of \((\overline{\Lambda}(\Lambda), \Sigma)\).

The converse assertion is true only if a supplementary condition is fulfilled.

4.3 Lemma. Let \(\Lambda\) be a sublattice of a Boolean algebra \(\Theta\) with the following properties:

a) The greatest element 1 of \(\Theta\) belongs to \(\Lambda\).

b) To an arbitrary element \(I \in \Theta\), \(I \neq 1\), there exists \(J \in \Lambda\), \(J \neq 1\) with \(J \geq I\).

If an ultraantifilter \(x\) on \(\Lambda\) is an isolated point of the topological space \((\overline{\Lambda}(\Lambda), \Sigma)\), then \(x\) is a principal antifilter on \(\Lambda\).

Proof. If \(x\) is an isolated point of the space \((\overline{\Lambda}(\Lambda), \Sigma)\), then \(\overline{\Lambda}(K) = \{x\}\) for some \(K \in x\). If \(x\) is not principal, then \(K\) is no dual atom of \(\Lambda\) (4.1). Hence there exists \(L \in \Lambda\) with \(L \supseteq K\), \(1 \neq L \neq K\). For the complement \(L'\) of \(L\) in the algebra \(\Theta\) there holds \(1 \neq L' \lor K\), because \(1 = L' \lor K \Rightarrow L = L \land (L' \lor K) = (L \land L') \lor (L \land K) = L \land K = K\), a contradiction. By supposition to the element \(L' \lor K \in \Theta\) there exists \(J \in \Lambda\), \(J \neq 1\) such that \(J \supseteq L' \lor K\). The elements \(L\) or \(J\) generate different ultraantifilters \(y\) or \(z\) on \(\Lambda\) containing \(K\), respectively, because \(1 = L \lor (L' \lor K) \leq L \lor J\). Therefore, \(y, z \in \overline{\Lambda}(K), y \neq x\) or \(z \neq x\), which contradicts the supposition. Thus \(x\) is a principal antifilter on \(\Lambda\).
4.4 Corollary. Let \( G \neq \{0\} \) be an \( l \)-group. Then \( x \in \mathcal{U}(\Pi(\Gamma(G))) \) is an isolated point of the topological space \( (\mathcal{U}(\Pi'), \Sigma) \) iff \( x \) is a principal antifilter on \( \Pi(G) \). An analogous statement holds for \( (\mathcal{U}(\Gamma(G)), \Sigma) \).

4.5 Theorem. Let \( G \neq \{0\} \) be an \( l \)-group. Then the following conditions are equivalent.

1. Minimal prime subgroups of \( G \) are maximal polars of \( G \).
2. Ultraantifilters on \( \Pi(G) \) are principal antifilters.
3. An analogical statement holds for \( (\mathcal{U}(\Pi(G)), \mathbb{Z}) \).

Note: The space \( (\mathcal{M}_n(G), \Gamma(G)) \) can be substituted by the space \( (\mathcal{U}(\Pi(G)), \Sigma) \). [13]

Proof. The \( \Pi \)-regulator is completely regular ([13], II, 1.5). We denote this \( \Pi' \)-regulator by the symbol \( (\mathcal{M}_n, \cup) \) to have the same notation as in 3.7. Here \( \mathcal{M}_n \) is the family of all minimal prime subgroups of \( G \) and \( \cup \) is the identical mapping of \( \mathcal{M}_n \). Now the condition 3.7(c) is equivalent to the condition 4.5(1) because \( \forall x \in \mathcal{M}_n(J) \) there holds \( Z_n(J) = Z_n(\cup J) = Z_n(\cup \{a \in \Pi(G) : a \in \mathcal{M}_n(J)\}) \). Hence, \( Z_n(J) \neq \emptyset \). In the first case, \( J \) denotes a subset of \( G \), in the other cases \( J \) is an element of \( \mathcal{M}_n \). By the same argument, \( Z_n(\cup x) \neq \emptyset \) holds for every \( x \in \mathcal{U}(\Pi(G)) \), since \( \cup x = \cup \{a \in \Pi(G) : a \in \mathcal{M}_n(J), a' \in x\} \) is a minimal prime subgroup of \( G \). Therefore, the conditions 3.7(d) and 4.5(2) are equivalent. This completes the proof of the Theorem.

4.6 Recall that an antifilter \( x \) on a lattice \( \Lambda \) with the greatest element 1 is called prime if there holds: \( K, L \in \Lambda \) and \( K \land L \in x \Rightarrow K \in x \) or \( L \in x \) (or equivalently: \( K \in \Lambda, i = 1, 2, \ldots, n \), \( n \) natural, \( \forall K \in \Lambda \Rightarrow K \in x \) for some \( i = 1, 2, \ldots, n \)).

It is well known that an ultraantifilter on a distributive lattice with the greatest element is a prime antifilter and that, conversely, a prime antifilter on a Boolean algebra is an ultraantifilter.

4.7 Theorem. Let \( G \neq \{0\} \) be an \( l \)-group. Then the following conditions are equivalent.

1. \( G \) has only a finite number of polars.
2. \( G \) has a base and only finitely many maximal polars.
3. \( \Pi(G) = \Pi'(G) \) and minimal subgroups of \( G \) are (maximal) polars of \( G \).
4. There exist only finitely many minimal prime subgroups of \( G \).
5. There exist only finitely many ultraantifilters on \( \Pi'(G) \).
6. \( (\mathcal{U}(\Pi'), \Sigma) \) is a finite discrete space.
7. \( (\mathcal{M}_n, \Gamma(G)) \) is a finite discrete space.
8. \( \mathcal{M}_n \) is a finite regulator of type \( \alpha \).
9. The regulator \( \mathcal{M}_n \) is finite.

Proof. 2 \( \Rightarrow \) 1 follows from 1.10 and 1.4.

2 \( \land \) 1 \( \Rightarrow \) 3 \( \land \) 6. Take \( x \in \mathcal{U}(\Pi') \) and \( a' \in x \). \( a' \) is the meet of a finite number of
maximal polars (by 1.10 and 1.4), hence $x$ contains at least one of them (4.6), say $b'$ (maximal polars are dual principal ones, 1.3). It follows that $a' \subseteq b'$. Since an antifilter on $\Pi'$ can contain at most one maximal polar, every ultraantifilter on $\Pi'(G)$ is principal. By 4.5 minimal prime subgroups are maximal polars and by 4.4 the space $(\Pi'(\Pi'), \Sigma)$ is discrete (hence 6). Since this space is finite, it is compact and by [13] I 1.9 $\Pi(G) = \Pi'(G)$.

3 $\Rightarrow$ 5. By [12] III 7.2 and 7.15 $\cup x = \cup \{a \in \Pi'(G): a' \in x\}$ is a maximal polar for every $x \in \Pi(\Pi')$. By 1.3 $\cup x$ is a dual principal polar and by [12] III 7.10 $\cup x \in x$, thus $x$ is a principal antifilter on $\Pi'(G)$. By 4.4 the space $(\Pi(\Pi'), \Sigma)$ is discrete. It is compact by [13] I 1.9, hence $\Pi(\Pi')$ is a finite set.

5 $\Rightarrow$ 4 follows from [12] 7.2 or [2] 3.4.15 (since $m \mathcal{P}(G) = \{\cup x: x \in \Pi(\Pi'(G))\}$).

4 $\Rightarrow$ 2. Maximal polars are minimal prime subgroups ([12] III 7.15 or [5] 2.2), thus $G$ contains only finitely many maximal polars. We shall show that every polar $K \neq G$ is contained in a maximal polar. Let $L$ be a dual principal polar $\neq K$ (such a polar exists since for $0 \neq c \in K'$ there holds $G \neq c' \supseteq K$) and let $x \in \Pi(\Pi')$ be generated by $L$. For every $y \in \Pi(\Pi')$, $y \neq x$ there exists $a_x \in G'$ with $a_x \in \cup x$ and $a_x \in \cup y$ because $\cup x$ and $\cup y$ as different minimal prime subgroups are incomparable. The infimum $b$ of these (finitely many) elements $a_x$ belongs to the meet of all $\cup y$ ($y \neq x$) and does not belong to $\cup x$ ([12] III 6.3 or [5] 1.7). Therefore $b' \in x$ and thus $b' \subseteq \cup x$ ([12] III 7.10 or [2] 3.4.1). Since $\cup x \cap \{\cup y: y \neq x\} = \{0\}$, $\cup x \cap b$ and hence $\cup x \subseteq b'$. Finally $b' = \cup x$ and $b'$ is the greatest element of $x$. $b'$ is a dual atom of the lattice $\Pi'(G)$ by 4.1, thus a dual atom of $\Pi(G)$ (by 1.3) and $b' \supseteq L \supseteq K$ holds. Hence $G$ has a base by 1.4.

6 $\Leftrightarrow$ 7 follows from [13] I 1.7.

7 $\Leftrightarrow$ 8 follows from 1.13.

8 $\Rightarrow$ 9 is evident.

9 $\Rightarrow$ 4 is evident since $\mathcal{M}_{\Pi}$ is the family of all minimal prime subgroups of $G$.

4.8 Theorem. Let $G \neq \{0\}$ be an $l$-group. Then the following conditions are equivalent.

1. $\mathcal{M}(\mathcal{M}_{\Pi}, G) = \mathcal{M}(\mathcal{M}_{\Pi}, G)$.

2. $\mathcal{M}(\Pi, (\Pi'), \Sigma') = \mathcal{M}(\Pi, (\Pi'), \Sigma')$.

3. Trivial open sets of the space $(\mathcal{M}_{\Pi}, G)$ form a finite partition on $\mathcal{M}_{\Pi}$.

4. $\mathcal{M}_{\Pi}$ is a finite regulator of type $\alpha$.

5. Any of the conditions of Theorem 4.7.

6. Any of the conditions of Theorem 4.5 together with the finiteness of the space $(\mathcal{M}_{\Pi}, G)$.

If one of the above conditions is true, then $G$ has a base and a weak unit, the space $(\mathcal{M}_{\Pi}, G)$ is compact and both regulators $\mathcal{M}_{\Pi}$ and $\mathcal{M}_{\Pi}$ are of type $\alpha$ and are equal.

1 ⇒ 3. Condition 1 implies the condition (4b), Theorem 4.1 of [10], thus G has a weak unit. Hence \( \Pi(\Gamma) = \Pi_r(\Gamma) \) ([12] V 12 6). Consequently, the space \((\Pi_r(\Gamma), \Sigma')\) is compact since the space \((\Pi(\Gamma), \Sigma)\) is compact ([11] Theorem 3) and the topology \(\Sigma'\) is weaker than \(\Sigma\) (see also [6] 3.3 or [10] 4.2). Then the space \((\mathfrak{M}_r, G)\) is compact, too ([13] I 1.7). By 2.10 trivial open sets form a partition on \(\mathcal{M}_r\). From the compactness it follows that this partition is finite.

3 ⇒ 5. (4.7(1)). The elements of the partition on trivial open sets of the space \((\mathfrak{M}_r, G)\) are atoms of the lattice \(\mathfrak{M}(\mathfrak{M}_r, G)\) and every element of \(\mathfrak{M}(\mathfrak{M}_r, G)\) is the union of finitely many atoms. Hence the lattice \(\mathfrak{M}(\mathfrak{M}_r, G)\) as well as the isomorphic lattice \(\mathcal{G}(G)\) of all polars is finite ([13] I 2.18).

5 ⇒ 1. If \(G\) has only finitely many polars (4.7(1)), then \(\mathcal{G}(G) = \Pi'(G)\). In fact, for \(K \in \Gamma\), \(K' = \vee_r \{a'' : a \in (K')^+\} = \vee_r \{a'' : a \in (K')^+, i = 1, 2, \ldots, n\}\), and so \(K = \bigcap a' - \left( \bigvee_{i=1}^n a \right) \in \Pi'(G)\). This means \(\mathfrak{M}_r, \mathfrak{M}_n,\) and so the space \((\mathfrak{M}_r, G) = (\mathfrak{M}_n, G)\) is discrete by 4.7 and \(\mathfrak{M}(\mathfrak{M}_r, G) = \mathfrak{M}(\mathfrak{M}_n, G)\) by 4.5.

5 ⇒ 6. Clearly, 4.7(7) − 4.5(3) \((the\ finiteness\ of\ the\ space\ (\mathfrak{M}_n, G)\) because 4.5(3) \(\equiv 3.7(e) - 3.6(f)\).

4 ⇒ 1. By 1.13 the space \((\mathfrak{M}_r, G)\) is finite and discrete, whence 1.

1 ⇒ 4. By [10] 4.1 (4b ⇒ 1a) there holds \(\mathfrak{M}_r = \mathfrak{M}_n\). Since 1 ⇒ 5, \((\mathfrak{M}_r, G)\) is a finite discrete space. Then by 1.13 we have 4.

REFERENCES

В работе изучены $l$-группы с базой при помощи алгебраических и топологических методов. Алгебраическим исследованиям служит так называемый регулятор $(\mathfrak{R}, \cup)$, то есть множество $\mathfrak{R} \neq \emptyset$ и отображение $\cup$ множества $\mathfrak{R}$ в систему простых подгрупп в $G$ такое, что $\{\cup x : x \in \mathfrak{R}\}$ имеет нулевое пересечение. Топологические исследования сопровождаются с помощью топологии, индуцированной на $\mathfrak{R}$ (структурное пространство). $(\mathfrak{R}, \cup)$ — регулятор типа $\alpha$, если $\cap\{\cup x : x \in \mathfrak{R}, x \neq y\} \neq \{0\}$ для всех $y \in \mathfrak{R}$. Доказывается, что существует (с точностью до эквиваленции) только один регулятор типа $\alpha l$-группы $G$, а именно множество $\mathfrak{R}$ всех минимальных простых подгрупп с отображением $\cup = \text{id}_{\mathfrak{R}}$ (1.9). Существование регулятора типа $\alpha$ характеризует $l$-группы с базой (1.10). Топологическая характеристика $l$-групп, обладающих базой, дана в 3.5 и 2.4. Подобие стандартного регулятора $(\mathfrak{R}, G)$ с регулятором типа $\alpha$ описано соотношением $\mathfrak{R}(\mathfrak{R}, G) = \mathfrak{M}(\mathfrak{R}, G)$ (3.1) (здесь мы используем обозначения, введенные в 0.1–0.3; см. также [10], [13]). Свойство $\mathfrak{R}(\mathfrak{R}, G) = \mathfrak{M}(\mathfrak{R}, G)$ характеризовано несколькими эквивалентными условиями в 2.10 и 3.6. В теореме 3.7, в которой результаты теоремы 3.6 специализируются на вполне регулярные регуляторы $(\mathfrak{R}, \cup)$, это равенство характеризовано следующими условиями: 1. Множество всех минимальных простых подгрупп $J$, обладающих свойством $Z(J) \neq \emptyset$, равно множеству всех максимальных поляр в $G$; 2. Всякий ультраантифильтр $x$ на $\Gamma'(G)$ с $Z(\cup x) \neq \emptyset$ — главный; 3. Пространство $(\mathfrak{R}, G)$ локально связно. Если регулятор $(\mathfrak{R}, \cup)$ редуцирован, то предыдущее условие выражается: Пространство $(\mathfrak{R}, \cup)$ дискретно. В абз.4 изучены условия, при которых $\Gamma$-регулятор или $\Pi'$-регулятор будет регулятором типа $\alpha$ и конечный типа $\alpha$. Результаты находятся в теоремах 4.5, 4.7 и 4.8.