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REGULATORS OF TYPE α OF LATTICE ORDERED GROUPS

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The purpose of the present paper is to investigate the lattice ordered groups (l-groups) having a base by using the algebraic and topological methods. (Note that in [9, 10, 12], the l-groups having a base are called l-groups of kind α; see Definition 1.2 and Lemma 1.4.) The algebraic examination is carried out by means of the so-called regulators, i.e. the indexed systems of prime subgroups having the zero meet and the topological examination by means of the topology induced on a regulator (structure space). For terminology and notations, cf. [13] I and [10]. A short review is also given in sec. 0 of the present paper. Other structure spaces were dealt with by S. J. Bernau [1]. His spaces are defined on the systems of all prime z-subgroups. Similar considerations will be included in another paper. Prime subgroups need not be z-subgroups, while minimal prime subgroups do it. The regulators of type α are formed by minimal prime subgroups and are equipped with a topology inherited from the hull-kernel topology defined in [1].

In the present paper it is proved that there exists (up to equivalence) at most one regulator of type α of an l-group, namely the set of all maximal polars (1.9). The existence of the regulator of type α characterizes the l-groups having a base (1.10). A topological characterization to a regulator of be of type α is given in 1.13 (the induced space is discrete). A topological characterization of l-groups having a base is given in 3.5 (the set of all isolated points is dense in $(\mathfrak{R}, G)$ provided that the standard regulator $(\mathfrak{R}, \cup)$* is similar to a reduced one) and in 2.4 (the union of all atoms of the lattice $\mathfrak{M}(\mathfrak{R}, G)$ is a dense subset of $(\mathfrak{R}, G)$ assuming only the standardness of $(\mathfrak{R}, \cup)$).

The similarity of a standard regulator $(\mathfrak{R}, \cup)$ to a regulator of type α is described by the relation $\mathfrak{N}(\mathfrak{R}, G) = \mathfrak{M}(\mathfrak{R}, G)$ (3.1). The property $\mathfrak{N}(\mathfrak{R}, G) = \mathfrak{M}(\mathfrak{R}, G)$ is then characterized by a number of equivalent conditions in 2.10 and 3.6. In Theorem 3.7, where the results of Theorem 3.6 are specified for a completely regular regulator $(\mathfrak{R}, \cup)$, this equality is described by conditions of various kinds.

* The symbol $\cup$ has the same meaning as the symbol $\bigcup$ in the preceeding papers [10], [13].

An algebraic condition reads: The set of all minimal prime subgroups J with $Z(J)\neq\emptyset$ is equal to the set of all maximal polars of G. A set condition: Every ultraantifilter x on $\Pi'(G)$ with $Z(\cup x)\neq\emptyset$ is principal. A topological condition: The space $(\mathfrak{R}, G)$ is locally connected. If the regulator $(\mathfrak{R}, \cup)$ is reduced, the above condition reads: The space $(\mathfrak{R}, G)$ is discrete. In sec. 4 conditions are studied under which the regulators $\mathfrak{R}_\Pi$ and $\mathfrak{R}_\Gamma$ are of type α or finite and of type α. The results are given in 4.5, 4.7 and 4.8.

0.1 A *regulator* $(\mathfrak{R}, \cup)$ of an l-group G is a set $\mathfrak{R}$ $(\neq\emptyset)$ and a mapping $\cup$: $\mathfrak{R}\to\mathcal{P}(G)$, the family of all prime subgroups of G such that $\cap\{\cup x: x\in\mathfrak{R}\}=\{0\}$. $(\mathfrak{R}, \cup)$ is called *standard* if $\cup x\neq G$ for every $x\in\mathfrak{R}$, *reduced* if $x, y\in\mathfrak{R}$, $x\neq y \Rightarrow \cup x\|\cup y$ and *completely regular* if it has the following property: $x\in\mathfrak{R}$, $f\in G$, $f\notin\cup x$ implies that there exists $g\in G$ such that $f\delta g$ and $g\notin\cup x$ ($f\delta g$ means $|f|\wedge |g|=0$). $(\mathfrak{R}, \cup)$ is said to be *finite* if the set $\mathfrak{R}$ is finite. Two special types of regulators (the *Π'-regulator* and *Γ-regulator*) are defined in 0.5.

Let $(\mathfrak{R}_i, \cup_i)$ be a regulator of an l-group G_i $(i=1, 2)$. The regulator $(\mathfrak{R}_2, \cup_2)$ is said to be *similar (equivalent)* to the regulator $(\mathfrak{R}_1, \cup_1)$ if there exists an l-isomorphism α of G_1 onto G_2 and a surjection (a bijection) β: $\mathfrak{R}_2$ onto $\mathfrak{R}_1$ such that $f\in\cup_1\beta x \equiv \alpha f\in\cup_2 x$ for every $f\in G_1$ and every $x\in\mathfrak{R}_2$ or equivalently $\alpha\cup_1\beta x=\cup_2 x$ for every $x\in\mathfrak{R}_2$. (The mapping β is continuous, open and closed (a homeomorphism) with respect to the induced topology defined in sec. 0.2 below, [13] II 4.2.)

An equivalence of $(\mathfrak{R}_1, \cup_1)$ and $(\mathfrak{R}_2, \cup_2)$ with $G_1=G_2$ $(=G)$ and $\alpha=\mathrm{id}_G$ is called an *equality*.

Let $(\mathfrak{R}, \cup)$ be a regulator of G. Take $x\in\mathfrak{R}$ and define $\bar{x}=\{y\in\mathfrak{R}: \cup x=\cup y\}$, $\bar{\mathfrak{R}}=\{\bar{x}: x\in\mathfrak{R}\}$ and $\bar{\cup}\bar{x}=\cup x$. Then $\bar{\cup}$ is a mapping of $\bar{\mathfrak{R}}$ into $\mathcal{P}(G)$ and $(\bar{\mathfrak{R}}, \bar{\cup})$ is a regulator similar to $(\mathfrak{R}, \cup)$, the so-called *simplification* of $(\mathfrak{R}, \cup)$.

0.2 For $f\in G$ define $Z(f)=\{x\in\mathfrak{R}: f\in\cup x\}$. If $(\mathfrak{R}, \cup)$ is a standard regulator of G, then $(G\neq\{0\}$ and) the set $\mathfrak{F}=\{Z(f): f\in G\}$ is a base of closed sets for a topology on the set $\mathfrak{R}$ ([13] I 1.2). This topology (in the sense of Bourbaki) is called the *topology induced on $\mathfrak{R}$ by the l-group G*. The corresponding topological space is denoted by $(\mathfrak{R}, G)$.

0.3 Let $(\mathfrak{R}, \cup)$ be a regulator of an l-group G. We define

$$\Psi(A)=\{f\in G: f\in\cup x \text{ for every } x\in A\} \quad (\emptyset\subsetneq A\subset\mathfrak{R}),$$
$$Z(P)=\{x\in\mathfrak{R}: f\in\cup x \text{ for every } f\in P\} \quad (\emptyset \subsetneq P\subset G).$$

If $A=\{x\}$ or $P=\{f\}$ is a singleton, we write $\Psi(x)$ or $Z(f)$ instead of $\Psi(\{x\})$ or $Z(\{f\})$, respectively. Ψ and Z are evidently dual isotone mappings between the sets exp $\mathfrak{R}$ and exp G ordered by inclusion. $\Psi(x)=\cup x$ and $Z(f)$ coincides with the notation in 0.2. We denote by $\mathfrak{R}(\mathfrak{R}, G)$ or $\mathfrak{M}(\mathfrak{R}, G)$ or $\mathfrak{C}(\mathfrak{R}, G)$ the system of all closed or regular closed or clopen sets of $(\mathfrak{R}, G)$, respectively.

0.4 The Boolean algebra of all polars of G is denoted by $\Gamma(G)$. Being $\emptyset \neq A \subseteq G$, we define $A' = \{g \in G: g\delta f$ for every $f \in A\}$. Then the complement of a polar K in $\Gamma(G)$ is K'. $\Pi'(G) := \{f' : f \in G\}$ or $\Pi(G) := \{f'' : f \in G\}$ is the system of all dual principal or principal polars of G, respectively. $\Pi'(G)$ and $\Pi(G)$ are sublattices of the lattice $\Gamma(G)$.

0.5 By an *ultraantifilter* on a $\vee$-semilattice Λ there is meant a maximal antifilter on Λ and an *antifilter* is a dual notion to that of a filter. The family of all ultraantifilters on Λ is denoted by $\mathfrak{U}(\Lambda)$. If Λ is a $\vee$-semilattice of subsets of G (e.g. $\Lambda = \Gamma(G)$ or $= \Pi'(G)$ or $= \Pi(G)$) and $x \in \mathfrak{U}(\Lambda)$, we define $\cup x = \cup\{K \in \Lambda: K \in x\}$. An ultraantifilter x is called *standard* if $\cup x \neq G$. If $G \neq \{0\}$, every $x \in \mathfrak{U}(\Pi'(G))$ is standard and every $x \in \mathfrak{U}(\Lambda)$, where $\Lambda = \Gamma(G)$ or $\Pi(G)$, is standard iff G has a weak unit. The set of all standard ultraantifilters on $\Gamma(G)$ is denoted by $\mathfrak{U}_s(\Gamma(G))$. Assuming $\Lambda = \Gamma(G)$ or $\Pi'(G)$ or $\Pi(G)$ and $x \in \mathfrak{U}(\Lambda)$, then $\cup x$ is a prime subgroup of G. $(\mathfrak{U}_s(\Gamma), \cup)$ and $(\mathfrak{U}(\Pi'), \cup)$ — briefly denoted by $\mathfrak{R}_\Gamma$ and $\mathfrak{R}_\Pi$, respectively, are standard regulators of G, the latter is reduced and completely regular. $\mathfrak{R}_\Gamma$ or $\mathfrak{R}_\Pi$ is called the *Γ-regulator* or the *Π'-regulator* of G, respectively.

Put $\Lambda = \mathfrak{U}_s(\Gamma)$ or $=$ a $\vee$-semilattice, respectively. Then the set

$$\Sigma' = \{\mathfrak{U}f' : f \in G\} \quad \text{or} \quad \Sigma = \{\mathfrak{U}K: K \in \Lambda\},$$

where $\mathfrak{U}K = \{x \in \mathfrak{U}(\Lambda): K \in x\}$ $(K \in \Lambda)$, is a base of open sets for a topology on $\mathfrak{U}_s(\Gamma(G))$ or $\mathfrak{U}(\Lambda)$, respectively.

$$(\mathfrak{U}_s(\Gamma(G)), \Sigma') \quad \text{or} \quad (\mathfrak{U}(\Lambda), \Sigma),$$

respectively, is the notation of the corresponding space.

1.

1.1 Definition. A regulator $(\mathfrak{R}, \cup)$ of an l-group G is called a *regulator of type α (of type β)* if $\cap\{\cup y: y \in \mathfrak{R}, y \neq x\} \neq \{0\}$ $(=\{0\})$ for every $x \in \mathfrak{R}$. If $(\mathfrak{R}, \cup)$ is a regulator of type α of G, then $G \neq \{0\}$ and $(\mathfrak{R}, \cup)$ is clearly reduced (and hence standard).

1.2 Definition. An l-group G is said to be an l-group *of kind α (of kind β)* if an arbitrary polar of G different from G is contained in a maximal polar of G (if in G no maximal polar exists). A representable l-group is of kind α iff G has an irreducible representation (see the following Proposition 1.3 and [5] 3.11). In [9] p. 407, I called the corresponding realization a *realization of type α.*

By a maximal polar of G there is meant a dual atom of the lattice $\Gamma(G)$ of polars of G. Dually, a minimal polar of G is defined.

1.3 Lemma. *The set of all dual atoms of $\Gamma(G)$ is equal to the set of all dual atoms of $\Pi(G)$.*

Proof. $\subset$: A dual atom of $\Gamma(G)$ (= a maximal polar of G) is a dual principal polar because its disjoint complement, a minimal polar of G, is a principal polar.

$\supset$: If K is a dual atom of $\Pi'(G)$, then K' is a minimal polar of G. If not, there exists $a \in G$ such that a'' K', $\{0\} \neq a \neq K'$, hence a K, $G \neq a' \neq K$, a contradiction. Consequently, K is a maximal polar of G (a dual atom of $\Gamma(G)$)

1.4 Proposition. *An l group $G \neq \{0\}$ is of kind α iff G has a base.*

Proof follows from [5] Theorem 3 4

1.5 Lemma. *Let $(\mathfrak{R}, \cup)$ be a regulator of an l group G. If there exists $x \in \mathfrak{R}$ such that $M = \cap\{\cap y\colon y \in \mathfrak{R}, y \neq x\} \neq \{0\}$, then $\cup x = M'$ is a maximal polar of G.*

Proof. Denote J $\cup x$. Suppose $K \in \Gamma(G)$, $K \neq G$ and K M'. There holds $J \cap M = \{0\}$, hence M' _ J. Thus we have K M' J. If $K \nsubseteq J$, then K' J K, whence $K = G$ a contradiction. Consequently, K J M' and M' is a maximal polar of G since clearly $M' \neq G$

1.6 Corollary. *Every regulator of an l-group of kind β is of type β.*

1.7 Theorem. *A standard regulator $(\mathfrak{R}, \cup)$ of an l group G is of type α iff the mapping $\cup$ is injective and $\cup x$ a (maximal) polar of G for every $x \in \mathfrak{R}$.*

Proof. Every regulator $(\mathfrak{R}, \cup)$ of type α is reduced, thus the mapping $\cup$ is injective. By 1.5, $\cup x$ is a maximal polar of G for every $x \in \mathfrak{R}$.

Conversely, let the condition of Theorem be fulfilled, $x \in \mathfrak{R}$ and $M = \cap\{\cup y\colon y \in \mathfrak{R}, y \neq x\}$. By the definition of a prime subgroup $(\cup x)$ M holds. Since $\cup x \cap M = \{0\}$, we have $(\cup x)'$ M, hence $(\cup x)' = M$. Consequently $M \neq \{0\}$ and $(\mathfrak{R}, \cup)$ is of type α.

1.8 Note. An analogical assertion as in 1.6 for l groups of kind β is not true, in general, for l-groups having a base, namely there does not hold the following statement:

(∗) Every reduced regulator of an l-group having a base is of type α.

Indeed, the set of all minimal prime subgroups of an arbitrary l group $G \neq \{0\}$ is a reduced regulator ([13] II 1 5(1)). If G has a base and if there exists a minimal prime subgroup of G, which is not a maximal polar, then by 1.7 this regulator is not of type α. A characterization of l-groups whose every minimal prime subgroup is a (maximal) polar is given in 4 6.

1.9 Corollary. (1) *Let an l-group $G \neq \{0\}$ have a base. Then the set of all maximal polars of G together with the identical mapping is a regulator of type α of G.*

(2) *If $(\mathfrak{R}, \cup)$ is a regulator of type α of an l group G, then $\{\cup x\colon x \in \mathfrak{R}\}$ is the set of all maximal polars of G.*

Proof. (1) By 3.4 [5] the intersection of the set $\mathfrak{R}$ of all prime subgroups that are polars is zero. Each of these polars is maximal or equal to G, [12] III 7.15. Hence the set of all maximal polars of G together with the identical mapping is a standard regulator of G. This regulator is of type α by 1.7.

(2) By 1.7 $\cup x$ is a maximal polar of G for every $x \in \mathfrak{R}$. G has a base. In fact, for $G \neq L \in \Gamma(G)$, $L = L \vee_{\Gamma} \cap\{\cup x: x \in \mathfrak{R}\} = \cap\{L \vee_{\Gamma} \cup x: x \in \mathfrak{R}\}$. From the maximality of the polar $\cup x$, $L \vee_{\Gamma} \cup x = G$ or $L \subseteq \cup x$. The set of $x \in \mathfrak{R}$ with the property $L \subseteq \cup x$ is clearly nonempty, hence G is of kind α and by 1.4 G has a base. Now if $\{\cup x: x \in \mathfrak{R}\}$ is not the set of all maximal polars of G, then by (1), $\cap\{\cup x: x \in \mathfrak{R}\} \neq \{0\}$, a contradiction.

1.10 Theorem. *Let G be an l-group $\neq \{0\}$. Then the following conditions are equivalent.*

(1) *G has a base.*

(2) *Every polar is an intersection of maximal polars of G*

(3) *There exists a regulator of type α of G.*

Proof. $1 \Rightarrow 3$. By 1.9(1).

$3 \Rightarrow 2$. If $(\mathfrak{R}, \cup)$ is a regulator of type α, then $\cup x$ $(x \in \mathfrak{R})$ is a maximal polar of G by 1.7. Since $\cap\{\cup x: x \in \mathfrak{R}\} = \{0\}$ for an arbitrary $L \in \Gamma(G)$ $L = L \vee_{\Gamma} \cap\{\cup x: x \in \mathfrak{R}\} = \cap\{L \vee_{\Gamma} \cup x. x \in \mathfrak{R}\}$. From the maximality of the polar $\cup x$ it follows that $L \vee_{\Gamma} \cup x = G$ or $L \subseteq \cup x$. Consequently, $L = \cap\{\cup x: x \in \mathfrak{R}, L \subseteq \cup x\}$.

$2 \Rightarrow 1$. From (2) it follows that G is of kind α, hence G has a base by 1.4.

1.11 Proposition. *Let $(\mathfrak{R}, \cup)$ be a standard regulator of G. Then the following conditions are equivalent.*

(a) *$(\mathfrak{R}, \cup)$ is similar to a regulator of type α.*

(b) *The simplification of the regulator $(\mathfrak{R}, \cup)$ is of type α.*

(c) *$\cup x$ is a (maximal) polar of G for every $x \in \mathfrak{R}$.*

Proof. Let $(\bar{\mathfrak{R}}, \bar{\cup})$ be the simplification of $(\mathfrak{R}, \cup)$.

$a \Rightarrow b$. If $(\mathfrak{R}, \cup)$ is similar to a regulator $(\mathfrak{R}_1, \cup_1)$ of type α and α, β the corresponding mappings (see 0.1), then for every $x \in \mathfrak{R}$ $\{0\} \neq \alpha \cap\{\cup_1 \beta y: \beta y \in \mathfrak{R}_1, \beta y \neq \beta x\} = \cap\{\cup y. y \in \mathfrak{R}, \cup y \neq \cup x\} = \cap\{\bar{\cup} y: y \in \bar{\mathfrak{R}}, \bar{y} \neq \bar{x}\}$ because for the reduced regulator $(\mathfrak{R}_1, \cup_1)$ there holds $\beta y = \beta x = \cup y = \cup x$ $(x, y \in \mathfrak{R})$.

$b \Rightarrow c$. By 1.7 $\bar{\cup}\bar{x}$ is a maximal polar of G for every $x \in \bar{\mathfrak{R}}$. Hence $\cup x$ is a maximal polar of G for every $x \in \mathfrak{R}$.

$c \Rightarrow a$. If $\cup x$ is a polar of G $(x \in \mathfrak{R})$, then $\cup x$ is a maximal polar ([5] 2.2 or [12] III 7.15). The simplification of $(\mathfrak{R}, \cup)$ is a regulator of type α by 1.7 and $(\mathfrak{R}, \cup)$ is similar to it.

1.12 Note. By [12] II 4.16, every l group $G \neq \{0\}$ has a regulator. Moreover, for every regulator $(\mathfrak{R}_1, \cup_1)$ of G there exists a reduced, completely regular regulator $(\mathfrak{R}_2, \cup_2)$ and a mapping φ: $\mathfrak{R}_1$ onto $\mathfrak{R}_2$ such that $\cup_1 x \supseteq \cup_2 \varphi(x)$ $(x \in \mathfrak{R}_1)$. As $\varphi(x)$

($x \in \mathfrak{R}_1$), we define a minimal prime subgroup contained in $\cup_1 x$ and for $\cup_2$ the identical mapping will be chosen. The regulator ($\mathfrak{R}_2$, $\cup$) is evidently reduced and by [13] II 1.4 completely regular.

1.13 Theorem. *A standard regulator* ($\mathfrak{R}$, $\cup$) *of an l-group G is of type α iff the topological space* ($\mathfrak{R}$, G) *is discrete.*

Proof. If ($\mathfrak{R}$, $\cup$) is of type α and $x \in \mathfrak{R}$, then there exists $0 \neq f \in \cap\{\cup y\colon y \in \mathfrak{R}, y \neq x\}$, thus $\mathfrak{R} \neq Z(f) \supseteq \mathfrak{R}\setminus\{x\}$ and hence $Z(f) = \mathfrak{R}\setminus\{x\}$. Thus $\{x\}$ is an open set.

If ($\mathfrak{R}$, G) is a discrete space and $x \in \mathfrak{R}$, then $\{y\colon y \in \mathfrak{R}, y \neq x\}$ is a closed set, hence there exists $f \in G$ such that $x \in Z(f)$ and $y \in Z(f)$ for $y \neq x$. Thus $0 \neq f \in \cap\{\cup y\colon y \in \mathfrak{R}, y \neq x\}$.

1.14 Proposition. *Let* ($\mathfrak{R}$, $\cup$) *be a standard regulator of an l-group G. Then G is of kind β iff the lattice* $\mathfrak{M}(\mathfrak{R}, G)$ *has no atom.*

Proof. The assertion follows from the fact that the existence of an atom of the lattice $\mathfrak{M}(\mathfrak{R}, G)$ is equivalent to the existence of a dual atom of $\Gamma(G)$ ([13] I 2.18), i.e. to the existence of a maximal polar of G.

1.15 Theorem. *An l group $G \neq \{0\}$ is of kind β iff there exists a reduced regulator of type β of G.*

(See [9] Satz 11).

Proof. Let $G \neq \{0\}$ be of kind β. There exists a reduced regulator of G and this is of type β by 1.6.

Conversely, let ($\mathfrak{R}$, $\cup$) be a reduced regulator of type β of G and L a maximal polar of G. The set of all $x \in \mathfrak{R}$ with $\cup x \supseteq L$ has at least two elements. Otherwise, there holds $\cap\{\cup y\colon y \in \mathfrak{R}\} \supseteq L' \neq \{0\}$ or for some $x \in \mathfrak{R}$, $\cap\{\cup y\colon y \in \mathfrak{R}\setminus\{x\}\} \supset L' \neq \{0\}$, a contradiction. Choose $x, y \in \mathfrak{R}$, $x \neq y$ with $\cup x \cap \cup y \supseteq L$. Since ($\mathfrak{R}$, $\cup$) is reduced, there exist a, $b \in G$ such that $0 < a \in \cup x\setminus\cup y$, $0 < b \in \cup y\setminus\cup x$ and $a \wedge b = 0$. Since L is a prime subgroup ([12] III 7.15 or [5] 2.2) there holds $a \in L$ or $b \in L$ ([5] 2.3 or [2] 2.4.1), thus $a \in \cup y$ or $b \in \cup x$, a contradiction.

1.16 Corollary. *A reduced regulator* ($\mathfrak{R}$, $\cup$) *of an l-group G is of type β iff the lattice* $\mathfrak{M}(\mathfrak{R}, G)$ *has no atom.*

Proof. By 1.15 the condition may be replaced by the following one: G is of kind β. If this is the case, then by 1.6 ($\mathfrak{R}$, $\cup$) is of type β. Conversely, if ($\mathfrak{R}$ $\cup$) is reduced and of type β, G is of kind β by 1.15

2.

2.0 By 1.9 the role of maximal polars in the regulators of type α is described. In the following (sec. 3) we try to clarify the participation of maximal polars in reduced regulators of l-groups having a base, in other words, to what extent the reduced regulators of l-groups having a base "approximate" the regulators of type α. Sec. 2 has an auxiliary character

2.1 Definition. ([2] 2.3.1) Let J be a solid subgroup of an l-group G and G/J the set of left cosets of G modulo J. Defining $a+J \geq b+J \equiv$ there exists $f \in G$ such that $a+f \geq b$ $(a, b \in G)$ we obtain a binary relation $\geq$, which is a distributive lattice ordering on G/J. If $(\mathfrak{R}, \cup)$ is a regulator of G, $x \in \mathfrak{R}$ and $f \in G$, $f(x)$ means the coset of $G/\cup x$ containing f. $f(0)$ will be denoted by $\cup x$, too.

2.2 Lemma. *A regulator* $(\mathfrak{R}, \cup)$ *of an* l*-group* $G \neq \{0\}$ *is reduced iff for* $x, y \in \mathfrak{R}$, $x \neq y$ *there exists* $f \in G$ *such that* $f(x) > \cup x$ *and* $f(y) < \cup y$.

Proof. Let regulator $(\mathfrak{R}, \cup)$ be reduced and $x, y \in \mathfrak{R}$, $x \neq y$. Then there exist $g \in \cup x \backslash \cup y$ and $h \in \cup y \backslash \cup x$. Denote $g_1 = |g| - |g| \wedge |h|$, $h_1 = |h| - |g| \wedge |h|$. Thus $0 < g_1 \in \cup x \backslash \cup y$, $0 < h_1 \in \cup y \backslash \cup x$ and $g_1 \delta h_1$. The element $f = -g_1 + h_1$ fulfils the condition $f(x) = f + \cup x = -g_1 + h_1 + \cup x = h_1 + \cup x > \cup x$ (g_1 and h_1 commute) and $f(y) = f + \cup y = -g_1 + h_1 + \cup y = -g_1 + \cup y < \cup y$.

Conversely, let the condition hold. Pick $x, y \in \mathfrak{R}$, $x \neq y$. By supposition, there exists $f \in G$ such that $f(x) > \cup x$ and $f(y) < \cup y$. Then $f^+(x) > \cup x$ and $f^-(y) < \cup y$ because $f^+ \geq f \geq f^-$. From the relation $f^+ \delta f^-$ and $f^+ \bar{\in} \cup x$ it follows that $f^- \in \cup x$ and similarly $f^+ \in \cup y$. Finally, $f^- \in \cup x \backslash \cup y$ and $f^+ \in \cup y \backslash \cup x$, thus the regulator $(\mathfrak{R}, \cup)$ is reduced.

2.3 Proposition. a) *A regulator of an* l*-group which is similar to a reduced regulator is standard.*

b) *If a reduced regulator is similar to a reduced regulator, then the similarity is an equivalence.*

c) *A regulator which is equivalent to a regulator of type* α *is itself of type* α.

d) *A reduced regulator which is similar to a regulator of type* α *is itself of type* α.

Proof. Let $(\mathfrak{R}_i, \cup_i)$ be a regulator of an l-group G_i $(i = 1, 2)$, let $(\mathfrak{R}_2, \cup_2)$ be similar to $(\mathfrak{R}_1, \cup_1)$, α and β mappings from the definition of the similarity (0.1).

a) If $(\mathfrak{R}_1, \cup_1)$ is reduced and $\cup_2 x = G_2$ for some $x \in \mathfrak{R}_2$, then $\cup_1 \beta x = \alpha^{-1} \cup_2 x = G_1$, hence $(\mathfrak{R}_1, \cup_1)$ is not reduced, which is a contradiction.

b) If $(\mathfrak{R}_i, \cup_i)$ $(i = 1, 2)$ is reduced and $x, y \in \mathfrak{R}_2$, $x \neq y$, then $\cup_2 x \neq \cup_2 y$, hence $\cup_1 \beta x = \alpha^{-1} \cup_2 x \neq \alpha^{-1} \cup_2 y = \cup_1 \beta y$. Since $\cup_1$ is injective, we have $\beta x \neq \beta y$, thus the mapping β is a bijection.

c) Let $(\mathfrak{R}_1, \cup_1)$ be of type α and let the similarity be an equivalence. Take $x \in \mathfrak{R}_2$. Then $\cap\{\cup_2 y\colon y \in \mathfrak{R}_2, y \neq x\} = \cap\{\alpha \cup_1 \beta y\colon y \in \mathfrak{R}, y \neq x\} = \alpha \cap \{\cup_1 z\colon z \in \mathfrak{R}_1, z \neq \beta x\} \neq \{0\}$, hence $(\mathfrak{R}_2, \cup_2)$ is of type α.

d) Let $(\mathfrak{R}_1, \cup_1)$ be of type α and $(\mathfrak{R}_2, \cup_2)$ reduced. Since both regulators are reduced, the similarity is an equivalence by b), and by c) $(\mathfrak{R}_2, \cup_2)$ is of type α.

2.4 Theorem. *An* l*-group* G *has a base iff for a standard regulator* $(\mathfrak{R}, \cup)$ *the union of a subset* $\mathfrak{A}$ *of atoms of the lattice* $\mathfrak{M}(\mathfrak{R}, G)$ *is a dense subset of the space* $(\mathfrak{R}, G)$.

Note. If the condition of Theorem is fulfilled, then $\mathfrak{A}$ is the set of all atoms of the lattice $\mathfrak{M}(\mathfrak{R}, G)$.

Proof. Let G have a base and let $(\mathfrak{R}, \cup)$ be a regulator of type α of G (1.10). Then $\{\cup x\colon x \in \mathfrak{R}\}$ is the set of all maximal polars (1.9(2)), $\cap\{\cup x\colon x \in \mathfrak{R}\} = \{0\}$ (1.10) and hence $\mathfrak{R} = Z(\cap\{\cup x\colon x \in \mathfrak{R}\}) = \vee_{\mathfrak{M}}\{Z(\cup x)\colon x \in \mathfrak{R}\} = cl_{(\mathfrak{R}, G)}(\cup\{Z(\cup x)\colon x \in \mathfrak{R}\})$ and $Z(\cup x)$ is an atom of the lattice $\mathfrak{M}(\mathfrak{R}, G)$, [13] I 2.18 and 2.19.

Let $(\mathfrak{R}, \cup)$ be a standard regulator of G. Let $\mathfrak{A} = \{A_\nu\colon \nu \in N\}$ be a set of atoms of the lattice $\mathfrak{M}(\mathfrak{R}, G)$ and $\bigcup_{\nu \in N} A_\nu$ a dense subset of the space $(\mathfrak{R}, G)$. Then $\mathrm{cl}_{(\mathfrak{R}, G)}\left(\bigcup_{\nu \in N} A_\nu\right) = \mathfrak{R}$, whence $\bigvee_{\nu \in N} {}_{\mathfrak{M}} A_\nu = \mathfrak{R}$, $\{0\} = \Psi\left(\bigvee_{\nu \in N} {}_{\mathfrak{M}} A_\nu\right) = \bigcap_{\nu \in N} \Psi A_\nu$ and every ΨA_ν is a maximal polar of G, and so the set $\mathfrak{R}_1 = \{\Psi A_\nu\colon \nu \in N\}$ together with the identical mapping is a regulator of G because maximal polars are prime subgroups, [12] III 7.15 or [5] 2.2. This regulator is of type α by 1.7. By 1.9(2), $\mathfrak{R}_1$ is the set of all maximal polars of G, hence $\mathfrak{A}$ is the set of all atoms of $\mathfrak{M}(\mathfrak{R}, G)$.

2.5 Lemma. *Let $(\mathfrak{R}, \cup)$ be a standard regulator of an l-group G and let $A \subseteq \mathfrak{R}$. Then $y \in \bar{A} \Leftrightarrow \cap\{\cup x\colon x \in A\} \subseteq \cup y$, especially $y \in \bar{x} \Leftrightarrow \cup x \subseteq \cup y$.*

Proof. We have: $Z(f) \supseteq A \Leftrightarrow \{x\colon f \in \cup x\} \supseteq A \Leftrightarrow f \in \cap\{\cup x\colon x \in A\}$. Hence $y \in \bar{A} \Leftrightarrow y \in Z(f)$ for every $f \in G$ such that $Z(f) \supseteq A \Leftrightarrow f \in \cup y$ for every $f \in \cap\{\cup x\colon x \in A\} \Leftrightarrow \cup y \supseteq \cap\{\cup x\colon x \in A\}$.

2.6 Proposition. *Let $(\mathfrak{R}, \cup)$ be a standard regulator of an l-group G and $x \in \mathfrak{R}$. The following conditions are equivalent.*

1. *$\cup x$ is a polar of G.*
2. *$\cup x$ is a maximal polar of G.*
3. *$\bar{x} \in \mathfrak{M}(\mathfrak{R}, G)$.*
4. *$\bar{x}$ is an atom of the lattice $\mathfrak{M}(\mathfrak{R}, G)$.*

If the regulator $(\mathfrak{R}, \cup)$ is reduced, then the following condition is equivalent to the preceding ones.

5. *x is an isolated point of the space $(\mathfrak{R}, G)$.*

Proof. 1⇒3. If $\cup x \in \Gamma(G)$, then $\bar{x} = Z\Psi(x) = Z(\cup x) \in \mathfrak{M}(\mathfrak{R}, G)$ ([13] I 2.8 and 2.18).

3⇒4. If A is an open subset of $(\mathfrak{R}, G)$, $\emptyset \neq A \subseteq \bar{x}$ and $\bar{A} \neq \bar{x}$, then $x \bar{\in} A$, hence the closed set $\bar{x} \setminus A$ contains the point x. Consequently $\bar{x} \subseteq \bar{x} \setminus A$, a contradiction.

4⇒2. If $\bar{x}$ is an atom of $\mathfrak{M}(\mathfrak{R}, G)$, then $\Psi(\bar{x}) = \Psi Z \Psi(x) = \Psi(x) = \cup x$ is a maximal polar of G ([13] I 2.4 and 2.8).

2⇒1 is evident.

If $(\mathfrak{R}, \cup)$ is reduced, then by 2.5, $x = \bar{x}$ for every $x \in \mathfrak{R}$, i.e. $(\mathfrak{R}, G)$ is a T_1-space and we have there:

$\bar{x} \in \mathfrak{M}(\mathfrak{R}, G) \Leftrightarrow x$ is an isolated point of $(\mathfrak{R}, G)$.

2.7 Definition. The atoms of the lattice of closed sets of a topological space P will be called *trivial closed sets* of P. Analogously for open or clopen sets.

Some simple lemmas concerning the preceding notions follow.

2.8 Lemma. a) *If the trivial open sets of P form a partition on P (say S), then the trivial closed sets of P form a partition on P (say R) and $R - S$ holds.*

b) *If T is a trivial closed set of P and* $\text{Int } T \neq \emptyset$, *then T is a trivial open set of P.*

Proof. a) The blocks T of S are closed sets. If some T is not trivial closed, there exists a closed set $V \subseteq T$ such that $\emptyset \neq V \neq T$. Then $X = (P \setminus V) \cap T$ is an open set, $X \subset T$, $\emptyset \neq X \neq T$ and T is not trivial open.

b) If $\emptyset \neq A \subseteq T$ and A is open, then either $T \setminus A = \emptyset$ or $T \setminus A$ is a proper closed subset of T, hence $T = A$, i.e. T is a trivial open set.

2.9 Lemma. *Let P be a topological space, A, $B \subseteq P$ and $A = P \setminus B$. Then there holds*

$$\overline{\text{Int } B} = B = \text{Int } \bar{A} = A,$$

i.e. the complement of a regular closed set is a regular open set and conversely.

Proof. Suppose $\overline{\text{Int } B} = B$. Then

$$A = P \setminus B \Rightarrow A = \overline{P \setminus B} \Rightarrow P \setminus A = P \setminus \overline{P \setminus B} = \text{Int } B \Rightarrow \overline{P \setminus A} =$$
$$= \overline{\text{Int } B} = B \Rightarrow A = P \setminus B = P \setminus \overline{P \setminus A} - \text{Int } \bar{A} \Rightarrow A = \text{Int } \bar{A}.$$

Suppose $\text{Int } A = A$. Then

$$P \setminus B = A = \text{Int } \bar{A} = P \setminus \overline{P \setminus \bar{A}} \Rightarrow B = \overline{P \setminus \bar{A}} = \overline{P \setminus \overline{P \setminus B}} -$$
$$= \overline{\text{Int } B} \Rightarrow B = \overline{\text{Int } B}$$

2.10 Proposition. *Let P be a topological space. The following conditions are equivalent.*

1. a) *P contains a base for closed sets formed by open sets.*
   b) *P is a locally connected space.*
2. *Every base for closed sets of the space P is formed by open sets.*
3. *Trivial open sets form a partition on P.*
4. *Every closed set of P is open ($\equiv$ every open set of P is closed).*
5. *$A \subseteq P$, A open in $P \Rightarrow \text{Int } \bar{A} = A$ (i.e open sets of P are regular open).*
6. *$\mathfrak{M}(P) = \mathfrak{R}(P)$ (i.e. closed sets of P are regular closed)*
7. *$\bar{x}$ is an open set of P for every $x \in P$*
8. *$\bar{x} \in \mathfrak{M}(P)$ for every $x \in P$.*

   If P is a T_1-space, then evidently the preceding conditions are equivalent to the following one.
9. *P is a discrete space.*

Proof. 1⇒3. From [12] IV 9.2 it follows that 1a ⇒ every block T of the partition on P, the blocks of which are maximal connected sets, is a trivial closed set.

From 1b it follows that T is an open set ([7] I, Ex. Ua) and by 2.8(b), T is a trivial open set.

3⇒4. By 2.8(a), every nonempty closed set is a union of blocks of the partition which is formed by the trivial open sets, hence it is open.

4⇒5. The closure $\bar{A}$ of every set $A \subseteq P$ is open, hence Int $\bar{A} = \bar{A}$. If A is open, then it is closed by supposition, hence $\bar{A} = A$. Thus Int $\bar{A} = A$ for every open set A of P.

5⇒6. $B \in \mathfrak{N}(P) \Rightarrow P \backslash B = A$ is open ⇒ Int $\bar{A} = A$ ⇒ $B \in \mathfrak{M}(P)$ (2.9).

6⇒8. $x \in P \Rightarrow \bar{x} \in \mathfrak{N}(P) \Rightarrow \bar{x} \in \mathfrak{M}(P)$.

8⇒7. Choose $x \in P$. If Int $\bar{x} \neq \bar{x}$, then there exists $y \in \bar{x} \backslash \text{Int } \bar{x}$. Since the set $\bar{x} \backslash \text{Int } \bar{x}$ is closed, there holds Int $\bar{y} \subseteq \bar{y} \subseteq \bar{x} \backslash \text{Int } \bar{x}$ and hence Int $\bar{y} \cap \text{Int } \bar{x} = \emptyset$. From the relation $\bar{y} \subseteq \bar{x}$ we obtain Int $\bar{y} \subseteq \text{Int } \bar{x}$, whence Int $\bar{y} = \emptyset$. But this contradicts the relations $\emptyset \neq \bar{y} = \overline{\text{Int } \bar{y}} = \emptyset$. Finally, Int $\bar{x} = \bar{x}$, and $\bar{x}$ is an open set.

7⇒2 is evident.

2⇒1. 1a holds evidently. We prove 1b. Every closed set is open because the system of all closed sets is a base for closed sets. By 2.8(b) every trivial closed set of P is a trivial open set. By [12] IV 9.2 the partition R_s, every block of which is a maximal connected set of P, is equal to the set of all trivial closed sets of P. Now it follows immediately that the space P is locally connected. Indeed, the maximal connected sets are trivial open and hence form a base for open sets, [7] I, Ex. Ub.

3.

3.1 Theorem. *A standard regulator* $(\mathfrak{R}, \cup)$ *of an* l*-group* G *is similar to a regulator of type* α *iff* $\mathfrak{N}(\mathfrak{R}, G) = \mathfrak{M}(\mathfrak{R}, G)$. *If the condition is fulfilled, then* G *has a base.*

Proof. Let a regulator $(\mathfrak{R}, \cup)$ of G be similar to a regulator of type α. By 1.9(2) and 0.1 $\cup x$ is a maximal polar of G for every $x \in \mathfrak{R}$; by 2.6 $\bar{x} \in \mathfrak{M}(\mathfrak{R}, G)$ and by 2.10 $\mathfrak{M}(\mathfrak{R}, G) = \mathfrak{N}(\mathfrak{R}, G)$.

Conversely, suppose $\mathfrak{M}(\mathfrak{R}, G) = \mathfrak{N}(\mathfrak{R}, G)$. This equality implies $\Gamma(G) = \Omega(\mathfrak{R}, G)$, [13] I 2.12 and 2.18, and so $\Gamma(G) = \Omega(\mathfrak{R}, G) \supseteq \{\Psi(x) : x \in \mathfrak{R}\} = \{\cup x : x \in \mathfrak{R}\}$. Hence $\cup x$ is a polar for every $x \in \mathfrak{R}$. By 1.11 $(\mathfrak{R}, \cup)$ is similar to a regulator of type α. By the same theorem the simplification of $(\mathfrak{R}, \cup)$ is of type α and by 1.10 G has a base.

3.2 Proposition. *Let a regulator* $(\mathfrak{R}, \cup)$ *of* $G \neq \{0\}$ *be similar to a reduced regulator,* α *and* β *the corresponding mappings* (see 0.1) *and* R *the partition induced by* β *on* $\mathfrak{R}$. *Then the blocks of* R *are trivial closed sets of the space* $(\mathfrak{R}, G)$.

Conversely, if trivial closed sets of $(\mathfrak{R}, G)$ *form a partition on* $\mathfrak{R}$, *then* $(\mathfrak{R}, \cup)$ *is similar to a reduced regulator and the simplification of* $(\mathfrak{R}, \cup)$ *is a reduced regulator.*

Proof. Let a regulator $(\mathfrak{R}, \cup)$ of $G \neq \{0\}$ be similar to a reduced regulator $(\mathfrak{R}_1, \cup_1)$ and $T = \beta^{-1}y$ for some $y \in \mathfrak{R}_1$. If T is not trivial closed, there exist $x_1, x_2 \in T$ and $f \in G$ such that $x_1 \in Z(f)$ and $x_2 \bar{\in} Z(f)$. Thus $f \in \cup x_1 \setminus \cup x_2$ and hence $\alpha^{-1} f \in \cup_1 \beta x_1 \setminus \cup_1 \beta x_2$. This set is empty because $\beta x_1 = \beta x_2 = y$, a contradiction.

Conversely, let trivial closed sets of $(\mathfrak{R}, G)$ form a partition on $\mathfrak{R}$, say R. Let $\cup_1$ be a mapping of R into $\mathscr{P}(G)$ such that $\cup_1 \tilde{x} = \cup x_G$ for every $\tilde{x} \in R$ and for a fixed $x_G \in \tilde{x}$. Then $(R, \cup_1)$ is a regulator of G. Indeed, choose $f \in G$ and $\tilde{x} \in R$ with $f \in \cup_1 \tilde{x}$ and pick $y \in \tilde{x}$. Then $f \in \cup x_G$, i.e. $x_G \in Z_{\mathfrak{R}}(f)$, whence $y \in Z_{\mathfrak{R}}(f)$ because $\tilde{x}$ is trivial closed. We have got $\cup x_G \subseteq \cup y$. Consequently $\cap\{\cup_1 \tilde{x}\colon \tilde{x} \in R\} \subseteq \cap\{\cup x\colon x \in \mathfrak{R}\} = \{0\}$. $(R, \cup_1)$ is reduced. In fact, suppose $\tilde{x}, \tilde{y} \in R$ and $\cup_1 \tilde{x} \supseteq \cup_1 \tilde{y}$. Then $\cup x_G \supseteq \cup y_G$ and by 2.5 $x_G \in \mathrm{cl}_{(\mathfrak{R}, G)}\{y_G\} = \tilde{y}$. Hence $\tilde{x} = \mathrm{cl}_{(\mathfrak{R}, G)} x_G = \tilde{y}$. Finally, $(R, \cup_1)$ is clearly the simplification of $(\mathfrak{R}, \cup)$.

3.3 Corollary. *Let* $(\mathfrak{R}, \cup)$ *be a standard regulator of G. Then the following conditions are equivalent.*

1. $(\mathfrak{R}, \cup)$ *is similar to a reduced regulator.*
2. *The simplification of* $(\mathfrak{R}, \cup)$ *is a reduced regulator.*
3. *The blocks of the equivalence relation R on* $\mathfrak{R}$, *defined by the rule* $xRy \equiv \cup x = \cup y$, *are trivial closed sets of* $(\mathfrak{R}, \cup)$.

Proof. $1 \Rightarrow 2$ by 3.2.

$2 \Rightarrow 3$. $(\mathfrak{R}, \cup)$ is similar to its simplification, thus we have 3 by 3.2.

$3 \Rightarrow 1$ by 3.2.

3.4 Lemma. *Let* $(\mathfrak{R}, \cup)$ *be a regulator of* $G \neq \{0\}$ *similar to a reduced regulator,* α *and* β *the corresponding mappings and R the partition induced by* β *on* $\mathfrak{R}$.

a) *If B is an open set of* $(\mathfrak{R}, G)$, *then B contains every trivial closed set which it meets.*

b) *If A is an atom of the lattice* $\mathfrak{M}(\mathfrak{R}, G)$, *then A is a trivial clopen set of* $(\mathfrak{R}, G)$ *and if* $T \in R$ *and* $T \cap A \neq \emptyset$, *then* $T = A$.

Proof. By 3.2 trivial closed sets of $(\mathfrak{R}, G)$ are blocks of the partition R.

a) $T \in R, \emptyset \neq T \cap B \not\supseteq T \Rightarrow \emptyset \neq T \setminus B \subseteq T$, $T \setminus B$ closed $\Rightarrow T \setminus B = T \Rightarrow T \cap B = \emptyset$, a contradiction.

b) Choose $T, V \in R$, $T \neq V$. By 2.2 and 3.3 for arbitrary $x \in T$ and $y \in V$ there exists $f \in G$ such that $f(x) > \cup x$ and $f(y) < \cup y$. Then $f^+(x) > \cup x$, $f^-(y) < \cup y$ and so $x \in \mathfrak{R} \setminus Z(f^+)$ and $y \in \mathfrak{R} \setminus Z(f^-)$. Since $f^+ \delta f^-$, we have $(\mathfrak{R} \setminus Z(f^+)) \cap (\mathfrak{R} \setminus Z(f^-)) = \emptyset$, [13] I 2.15. We have proved the existence of disjoint open neighbourhoods C and D of the points x and y, respectively, $C = \mathfrak{R} \setminus Z(f^+)$ and $D = \mathfrak{R} \setminus Z(f^-)$. Let A be an atom of the lattice $\mathfrak{M}(\mathfrak{R}, G)$ such that $x, y \in \mathrm{Int}\, A$ $(= B)$. The set $A \setminus C$ is closed and $\emptyset \neq D \cap B \subseteq A \setminus C$ holds. Then $\emptyset \neq \mathrm{cl}_{(\mathfrak{R}, G)}(D \cap B) \subseteq A \setminus C \subsetneqq A$, a contradiction. It follows that B meets only one block of the partition R, say T. Thus $T \supseteq B$ and by a) $T = B$. Thus $T = \mathrm{cl}_{(\mathfrak{R}, G)} T = \mathrm{cl}_{(\mathfrak{R}, G)} B = A$ and A is a trivial clopen set of $(\mathfrak{R}, G)$.

3.5 Theorem. *Let $(\mathfrak{R}, \cup)$ be a regulator of an l-group $G \neq \{0\}$ similar to a reduced regulator. Then the following conditions are equivalent.*

1. *G has a base.*
2. *The union $\mathfrak{S}$ of all atoms of the lattice $\mathfrak{M}(\mathfrak{R}, G)$ is a dense subset of $(\mathfrak{R}, G)$.*
3. *There exists a dense (open) subspace $\mathfrak{S}$ of the space $(\mathfrak{R}, G)$ such that $\mathfrak{N}(\mathfrak{S}) = \mathfrak{M}(\mathfrak{S})$.*

If $(\mathfrak{R}, \cup)$ is reduced, the following condition is equivalent to the preceding ones.

4. *The set of all isolated points of the space $(\mathfrak{R}, G)$ is a dense subset of $(\mathfrak{R}, G)$.*

Note. If condition 2 is true, then the set $\mathfrak{S}$ from 2 has the property of the set $\mathfrak{S}$ from condition 3.

Proof. Let $\{A_\alpha\}$ be the system of all atoms of the lattice $\mathfrak{M}(\mathfrak{R}, G)$. By [13] I 2.18 $\{\Psi(A_\alpha)\}$ is the system of all maximal polars of G.

1$\Rightarrow$2. By 1.9 or [5] Theorem 3.4, $\bigcap_\alpha \Psi(A_\alpha) = \{0\}$. It follows that $\mathfrak{R} = \bigvee_\alpha{}_{\mathfrak{M}} A_\alpha = \mathrm{cl}_{(\mathfrak{R}, G)} \bigcup_\alpha A_\alpha$, [13] I 2 19. Hence the union $\mathfrak{S}$ of all atoms of the lattice $\mathfrak{M}(\mathfrak{R}, G)$ is a dense subset of $(\mathfrak{R}, G)$.

2$\Rightarrow$3. The union $\mathfrak{S}$ of all atoms of the lattice $\mathfrak{M}(\mathfrak{R}, G)$ is open by 3.4(b) and by the supposition a dense subset of $(\mathfrak{R}, G)$. Let R be the partition induced by the mapping β defining the similarity of $(\mathfrak{R}, \cup)$. By 3.4(b) every $T \in R$ which meets $\mathfrak{S}$ is a trivial clopen set of $(\mathfrak{R}, G)$. Hence if $A \in \mathfrak{N}(\mathfrak{R}, G)$ meets $\mathfrak{S}$, then $A \cap \mathfrak{S}$ is an open subset of $(\mathfrak{R}, G)$ and a closed subset of the subspace $\mathfrak{S}$. It follows that $A \cap \mathfrak{S} = \mathfrak{S} \cap \mathrm{cl}_{(\mathfrak{R}, G)}(A \cap \mathfrak{S}) = \mathrm{cl}_{\mathfrak{S}}(A \cap \mathfrak{S}) \in \mathfrak{M}(\mathfrak{S})$, hence $\mathfrak{N}(\mathfrak{S}) = \mathfrak{M}(\mathfrak{S})$.

3$\Rightarrow$1. Let $\mathfrak{S}$ be a dense subspace of $(\mathfrak{R}, G)$ such that $\mathfrak{N}(\mathfrak{S}) = \mathfrak{M}(\mathfrak{S})$. Then $(\mathfrak{S}, \cup_1)$, where $\cup_1 = \cup|_{\mathfrak{S}}$ is a standard regulator of G, [13] II 4.9.

It is evident that $\mathfrak{S} \cap Z_{(\mathfrak{R}, \cup)}(f) = Z_{(\mathfrak{S}, \cup_1)}(f)$, hence the identical mapping of $\mathfrak{S}$ is a homeomorphism of the space $(\mathfrak{S}, G)$ onto the subspace $\mathfrak{S}$ of $(\mathfrak{R}, G)$. Consequently, $\mathfrak{N}(\mathfrak{S}) = \mathfrak{N}(\mathfrak{S}, G)$ and $\mathfrak{M}(\mathfrak{S}) = \mathfrak{M}(\mathfrak{S}, G)$. By 3.1 G has a base.

1$\Rightarrow$4. As in 1$\Rightarrow$2, $\mathfrak{R} = \mathrm{cl}_{(\mathfrak{R}, G)} \bigcup_\alpha A_\alpha$, where A_α are atoms of the lattice $\mathfrak{M}(\mathfrak{R}, G)$. By 3.4(b) every A_α is a trivial clopen set and is equal to a block of the partition R induced on $\mathfrak{R}$ by the mapping β defining the similarity of $(\mathfrak{R}, \cup)$. Since $(\mathfrak{R}, \cup)$ is reduced, the similarity is an equivalence (2.3(b)) and hence β is one-to-one. Therefore, every A_α is an isolated point of $(\mathfrak{R}, G)$.

4$\Rightarrow$2 is evident.

In the following Theorem, the results of Theorem 2.6, 2.10 and 3.1 will be summarized.

3.6 Theorem. *Let $(\mathfrak{R}, \cup)$ be a standard regulator of an l-group G. The following conditions are equivalent.*

a) *The l-group G has a base, the regulator $(\mathfrak{R}, \cup)$ is completely regular and the union of all atoms of the lattice $\mathfrak{M}(\mathfrak{R}, G)$ is a closed set of $(\mathfrak{R}, G)$.*

b) *Any condition of Theorem 2.6 fulfilled for every $x \in \mathfrak{R}$.*

c) *Any condition of Theorem 2.10 for $P = (\mathfrak{R}, G)$.*

d) *Any condition of Theorem 3.1.*

Moreover, if $(\mathfrak{R}, \cup)$ is reduced, then the following conditions are equivalent to the preceding ones.

e) *The regulator $(\mathfrak{R}, \cup)$ is of type α.*

f) *The space $(\mathfrak{R}, G)$ is discrete.*

Proof. b≡c because 2.6(3)=2.10(8).

c≡d because both Theorems have the condition $\mathfrak{M}(\mathfrak{R}, G) = \mathfrak{N}(\mathfrak{R}, G)$ in common.

c∧d ⇒ a. From c) (2 10(2)) it follows that $(\mathfrak{R}, \cup)$ is completely regular ([13] II 1.5). The remaining two conditions follow from d) (G has a base) and c) (2.10(4)).

a ⇒ c (2.10(1)). We shall prove that every point $x \in \mathfrak{R}$ has a fundamental system of connected neighbourhoods ([4] I § 11,6, Df. 4). Thus it will be shown that the space $(\mathfrak{R}, G)$ is locally connected which is the condition 1(b) of 2.10. If B is a neighbourhood of the point x, then there exists $f \in G$ such that $x \in \mathfrak{R} \backslash Z(f) \_ B$. Since G has a base, the meet of all maximal polars g'_α $(\alpha \in A)$ is equal to zero, $\bigcap_{\alpha \in A} g'_\alpha = \{0\}$ (1.9). It follows that $\mathfrak{R} = Z(0) = \bigvee_{\alpha \in A}{}_{\mathfrak{M}} Z(g'_\alpha) = \mathrm{cl}_{(\mathfrak{R}, G)} \bigcup_{\alpha \in A} Z(g'_\alpha)$ ([13] I 2.18 and 2.19). Since $Z(g_\alpha)$ is a clopen set ([13] II 1.4), the set $Z(g'_\alpha) = \mathfrak{R} \backslash Z(g_\alpha)$ is clopen as well. Since $\{Z(g'_\alpha)\colon \alpha \in A\}$ is the family of all atoms of the lattice $\mathfrak{M}(\mathfrak{R}, G)$ ([13] I 2.18) and $\bigcup_{\alpha \in A} Z(g'_\alpha)$ is closed by supposition, then $\mathfrak{R} = \bigcup_{\alpha \in A} Z(g'_\alpha)$. Thus there exists $\alpha_0 \in A$ such that $x \in Z(g'_{\alpha_0})$. $Z(g'_{\alpha_0})$ is a connected neighbourhood of the point x because it is clopen and an atom of $\mathfrak{M}(\mathfrak{R}, G)$. Now $Z(f')$, $Z(g'_\alpha) \in \mathfrak{M}(\mathfrak{R}, G)$, the set $Z(g'_\alpha)$ is an atom of the lattice $\mathfrak{M}(\mathfrak{R}, G)$ and intersects $Z(f')$ (in x, since $x \in \mathfrak{R} \backslash Z(f) = Z(f')$), hence $Z(f') \supseteq Z(g'_\alpha)$. Consequently $B \supseteq \mathfrak{R} \backslash Z(f) = Z(f') \supseteq Z(g'_\alpha)$. We have proved that an arbitrary neighbourhood of the point x contains a connected neighboruhood of x Thus the space $(\mathfrak{R}, G)$ is locally connected. Finally, 2.10(1a) follows from the complete regularity of $(\mathfrak{R}, \cup)$ ([13] II 1.4).

e ⇒ d is evident.

d ⇒ e by 2.3(d).

e ⇔ f by 1.13.

3.7 Theorem. *Let $(\mathfrak{R}, \cup_1)$ be a completely regular regulator of an l-group G. The following conditions are equivalent.*

a) *The connected components of the space* $(\mathfrak{R}, G)$ *are open.*
b) *The space* $(\mathfrak{R}, G)$ *is locally connected.*
c) *If J is a minimal prime subgroup of G and* $Z(J) \neq \emptyset$, *then J is a (maximal) polar of G.*
d) *If* $x \in \mathfrak{U}(\Pi'(G))$ *and* $Z(\cup x) \neq \emptyset$, *then x is a principal antifilter on* $\Pi'(G)$.
e) *Any condition of Theorem 3.6.*

Note. In a topological space the conditions a) and b) are not equivalent in general. There holds b $\Rightarrow$ a, see [4] I § 11, 6, Prop. 11).

Proof. a $\Rightarrow$ e (2.10(7)). By [4] I § 11, Ex. 12 the condition a) is equivalent to the following one: For an arbitrary $x \in \mathfrak{R}$ the meet $\tilde{x}$ of all clopen sets containing x is an open set. By the definition of the closure $\bar{x}$ of $\{x\}$ there holds $\tilde{x} \supset \bar{x}$. By supposition the basic sets $Z(f)$ $(f \in G)$ containing x are clopen ([13] II 1.4), hence their meet (equal to $\bar{x}$) contains $\tilde{x}$. Thus we have $\bar{x} = \tilde{x}$ and so $\bar{x}$ is an open set.

e $\Rightarrow$ b is evident.

b $\Rightarrow$ a by [4] I § 11, 6 Prop. 11.

d $\Rightarrow$ c. Choose $J \in m\mathcal{P}(G)$ with $Z(J) \neq \emptyset$. There holds $J = \cup x$ for some $x \in \mathfrak{U}(\Pi'(G))$ (remember that $\cup x = \cup\{a : a \in x\}$), see [2] 3.4.15. Since x is a principal antifilter, it is generated by a maximal element of the lattice $\Pi'(G)$ say a', hence by a maximal polar of G (1.3). Thus $\cup x = J$ is a maximal polar of G.

c $\Rightarrow$ e. $\cup_1 x$ is a minimal prime subgroup for every $x \in \mathfrak{R}$ ([13] II 1.4). Since the set $Z(\cup_1 x)$ contains x, it is nonempty, and so by c) $\cup_1 x$ is a polar of G. By 1.11 $(\mathfrak{R}, \cup_1)$ is similar to a regulator of type α (which is one of the conditions of 3.1).

e $\Rightarrow$ d. Choose $x \in \mathfrak{U}(\Pi'(G))$ with $Z(\cup x) \neq \emptyset$. Then $\cup x$ is a minimal prime subgroup of G and since $Z(\cup x) \neq \emptyset$, there holds $\cup x = \cup_1 y$ for some $y \in \mathfrak{R}$ ([13] II 1.4). By supposition $\cup_1 y$ is a maximal polar of G (2.6(2)). Consequently, $\cup_1 y = a'$ for some $a \in G$, thus x is a principal antifilter on $\Pi'(G)$ generated by the dual principal polar a'.

3.8 Theorem. *Let* $(\mathfrak{R}_1, \cup_1)$ *be a regulator of type* α *of an l-group G and* $(\mathfrak{R}_2, \cup_2)$ *a regulator of G similar to a reduced regulator. Let* $\mathfrak{S}$ *be the union of all atoms of the lattice* $\mathfrak{M}(\mathfrak{R}_2, G)$. *Then there exists a continuous, open and closed mapping* σ *of the subspace* $\mathfrak{S}$ *of the space* $(\mathfrak{R}_2, G)$ *onto the space* $(\mathfrak{R}_1, G)$. *If the regulator* $(\mathfrak{R}_2, \cup_2)$ *is reduced,* σ *is a homeomorphism.*

Proof. The regulator $(\mathfrak{R}_i, \cup_i)$ $(i = 1, 2)$ is standard. Define a binary relation σ between the sets $\mathfrak{S}$ and $\mathfrak{R}_1$ as follows: $\sigma^{-1}(x) = Z_{\mathfrak{R}_2}(\cup_1 x)$ for every $x \in \mathfrak{R}_1$. We shall show that σ is a mapping of $\mathfrak{S}$ onto $\mathfrak{R}_1$. Since $\cup_1 x$ is a maximal polar of G (1.9(2)), $Z_{\mathfrak{R}_2}(\cup_1 x)$ is an atom of the lattice $\mathfrak{M}(\mathfrak{R}_2, G)$ ([13] I 2.18). Hence it is a subset of $\mathfrak{S}$. For different elements $x, y \in \mathfrak{R}_1$ the sets $\sigma^{-1}(x)$ and $\sigma^{-1}(y)$ are different because the mapping $Z_{\mathfrak{R}_2}: \Gamma(G) \rightarrow \mathfrak{M}(\mathfrak{R}_2, G)$ is one-to-one. Hence σ is a mapping of a subset of $\mathfrak{S}$ onto $\mathfrak{R}_1$. Pick an arbitrary atom A of the lattice $\mathfrak{M}(\mathfrak{R}_2, G)$. Then $\Psi_{\mathfrak{R}_2}(A)$ is a maximal polar of G and $Z_{\mathfrak{R}_1}\Psi_{\mathfrak{R}_2}(A)$ is an atom of the lattice

$\mathfrak{M}(\mathfrak{R}_1, G)$. Since the space $(\mathfrak{R}_1, G)$ is discrete by 1.13, this is a singleton, say $\{x\}$. Hence

$$\sigma^{-1}(x) = Z_{\mathfrak{R}_2}(\cup_1 x) = Z_{\mathfrak{R}_2}\Psi_{\mathfrak{R}_1}Z_{\mathfrak{R}_1}\Psi_{\mathfrak{R}}(A) = Z_{\mathfrak{R}_1}\Psi_{\mathfrak{R}_1}(A) = A,$$

[13] I 2.4. Thus it is proved that σ is a mapping of the set $\mathfrak{S}$ onto $\mathfrak{R}_1$. Since the space $(\mathfrak{R}_1, G)$ is discrete, σ is an open and closed mapping of the subspace $\mathfrak{S}$ of the space $(\mathfrak{R}_2, G)$ onto the space $(\mathfrak{R}_1, G) \cdot \sigma$ is continuous. In fact, as we know, the set $\sigma^{-1}(x) = Z_{\mathfrak{R}_2}(\cup_1 x)$ is an atom of the lattice $\mathfrak{M}(\mathfrak{R}_2, G)$, consequently by 3.4(b) it is a trivial clopen set of the space $(\mathfrak{R}_2, G)$.

If the regulator $(\mathfrak{R}_2, \cup_2)$ is reduced, then atoms of the lattice $\mathfrak{M}(\mathfrak{R}_2, G)$ are singletons, hence the mapping σ is one-to one. In this case, $\mathfrak{S}$ is the set of all isolated points of $(\mathfrak{R}_2, G)$, hence σ is a homeomorphism.

4.

4.1 Lemma. *Let Λ be a $\vee$-semilattice with the greatest element* 1. *An ultraantifilter on Λ is a principal antifilter iff it is generated by a dual atom of Λ.*

The proof is straightforward.

4.2 Lemma. *Let Λ be a $\vee$-semilattice with the greatest element* 1. *If an ultraantifilter x on Λ is a principal antifilter, then x is an isolated point of the topological space* $(\mathfrak{U}(\Lambda), \Sigma)$.

Proof. If an ultraantifilter x on Λ is a principal antifilter and L its generator, then L is a dual atom of Λ (4.1), thus $\mathfrak{U}L = \{x\}$ and hence x is an isolated point of $(\mathfrak{U}(\Lambda), \Sigma)$.

The converse assertion is true only if a supplementary condition is fulfilled.

4.3 Lemma. *Let Λ be a sublattice of a Boolean algebra Θ with the following properties:*

a) *The greatest element* 1 *of Θ belongs to Λ.*

b) *To an arbitrary element $I \in \Theta$, $I \neq 1$, there exists $J \in \Lambda$, $J \neq 1$ with $J \geqslant I$.*

If an ultraantifilter x on Λ is an isolated point of the topological space $(\mathfrak{U}(\Lambda), \Sigma)$, *then x is a principal antifilter on Λ.*

Proof. If x is an isolated point of the space $(\mathfrak{U}(\Lambda), \Sigma)$, then $\mathfrak{U}K = \{x\}$ for some $K \in x$. If x is not principal, then K is no dual atom of Λ (4.1). Hence there exists $L \in \Lambda$ with $L \geqslant K$, $1 \neq L \neq K$. For the complement L' of L in the algebra Θ there holds $1 \neq L' \vee K$, because $1 = L' \vee K \Rightarrow L = L \wedge (L' \vee K) = (L \wedge L') \vee (L \wedge K) = L \wedge K = K$, a contradiction. By supposition to the element $L' \vee K \in \Theta$ there exists $J \in \Lambda$, $J \neq 1$ such that $J \geqslant L' \vee K$. The elements L or J generate different ultraantifilters y or z on Λ containing K, respectively, because $1 = L \vee (L' \vee K) \leqslant L \vee J$. Therefore, $y, z \in \mathfrak{U}K$, $y \neq x$ or $z \neq x$, which contradicts the supposition. Thus x is a principal antifilter on Λ.

4.4 Corollary. *Let $G \neq \{0\}$ be an l-group. Then $x \in \mathfrak{U}(\Pi'(G))$ is an isolated point of the topological space $(\mathfrak{U}(\Pi'), \Sigma)$ iff x is a principal antifilter on $\Pi'(G)$. An analogical statement holds for $(\mathfrak{U}(\Gamma(G)), \Sigma)$.*

4.5 Theorem. *Let $G \neq \{0\}$ be an l-group. Then the following conditions are equivalent.*

1. *Minimal prime subgroups of G are maximal polars of G.*
2. *Ultraantifilters on $\Pi'(G)$ are principal antifilters.*
3. *Any condition of Theorem 3.7 for $(\mathfrak{P}, \cup) - \mathfrak{R}_{\Pi}$, the Π'-regulator.*

Note. The space $(\mathfrak{R}_\Pi, G)$ can be substituted by the space $(\mathfrak{U}(\Pi'(G)), \Sigma)$, [13] I 1.7.

Proof. The Π'-regulator is completely regular ([13] II 1.5). We denote the Π'-regulator by the symbol $(\mathfrak{R}_\Pi, \cup)$ to have the same notation as in 3.7. Here $\mathfrak{R}_\Pi$ is the family of all minimal prime subgroups of G and $\cup$ is the identical mapping of $\mathfrak{R}_\Pi$. Now the condition 3.7(c) is equivalent to the condition 4.5(1) because for an arbitrary minimal prime subgroup J (= an element of $\mathfrak{R}_\Pi$) there holds $Z_{\mathfrak{R}}(J) = Z_{\mathfrak{R}}(\cup_i J) = Z_{\mathfrak{R}_\Pi}(\Psi_{\mathfrak{R}_\Pi} J) = \{J\}$, hence $Z_{\mathfrak{R}_\Pi}(J) \neq \emptyset$ (In the first case J denotes a subset of G, in the other cases J is an element of $\mathfrak{R}_\Pi$). By the same argument $Z_{\mathfrak{R}_\Pi}(\cup x) \neq \emptyset$ holds for every $x \in \mathfrak{U}(\Pi'(G))$, since $\cup x = \cup\{a \in \Pi'(G) : a' \in x\}$ is a minimal prime subgroup of G. Therefore, the conditions 3.7(d) and 4.5(2) are equivalent. This completes the proof of the Theorem.

4.6 Recall that an antifilter x on a lattice Λ with the greatest element 1 is called *prime* if there holds: $K, L \in \Lambda$, $K \wedge L \in x \Rightarrow K \in x$ or $L \in x$ (or equivalently: $K_i \in \Lambda$, $i = 1, 2, \ldots, n$, n natural, $\wedge K_i \in x \Rightarrow K_i \in x$ for some $i = 1, 2, \ldots, n$).

It is well known that an ultraantifilter on a distributive lattice with the greatest element is a prime antifilter and that, conversely, a prime antifilter on a Boolean algebra is an ultraantifilter.

4.7 Theorem. *Let $G \neq \{0\}$ be an l-group. Then the following conditions are equivalent.*

1. *G has only a finite number of polars.*
2. *G has a base and only finitely many maximal polars.*
3. *$\Pi(G) = \Pi'(G)$ and minimal subgroups of G are (maximal) polars of G.*
4. *There exist only finitely many minimal prime subgroups of G.*
5. *There exist only finitely many ultraantifilters on $\Pi'(G)$.*
6. *$(\mathfrak{U}(\Pi'), \Sigma)$ is a finite discrete space.*
7. *$(\mathfrak{R}_\Pi, G)$ is a finite discrete space.*
8. *$\mathfrak{R}_\Pi$ is a finite regulator of type α.*
9. *The regulator $\mathfrak{R}_\Pi$ is finite.*

Proof. $2 \Leftrightarrow 1$ follows from 1.10 and 1.4.

$2 \wedge 1 \Rightarrow 3 \wedge 6$. Take $x \in \mathfrak{U}(\Pi')$ and $a' \in x$. a' is the meet of a finite number of maximal polars (by 1.10 and 1.4), hence x contains at least one of them (4.6), say b' (maximal polars are dual principal ones, 1.3). It follows that $a' \subseteq b'$. Since an antifilter on Π' can contain at most one maximal polar, every ultraantifilter on $\Pi'(G)$ is principal. By 4.5 minimal prime subgroups are maximal polars and by 4.4 the space $(\mathfrak{U}(\Pi'), \Sigma)$ is discrete (hence 6). Since this space is finite, it is compact and by [13] I 1.9 $\Pi(G) = \Pi'(G)$.

3⇒5. By [12] III 7.2 and 7.15 $\cup x = \cup\{a \in \Pi'(G): a' \in x\}$ is a maximal polar for every $x \in \mathfrak{U}(\Pi')$. By 1.3 $\cup x$ is a dual principal polar and by [12] III 7.10 $\cup x \in x$, thus x is a principal antifilter on $\Pi'(G)$. By 4.4 the space $(\mathfrak{U}(\Pi'), \Sigma)$ is discrete. It is compact by [13] I 1.9, hence $\mathfrak{U}(\Pi')$ is a finite set.

5⇒4 follows from [12] 7.2 or [2] 3.4.15 (since $m\mathscr{P}(G) = \{\cup x: x \in \mathfrak{U}(\Pi'(G))\}$).

4⇒2. Maximal polars are minimal prime subgroups ([12] III 7.15 or [5] 2.2), thus G contains only finitely many maximal polars. We shall show that every polar $K \neq G$ is contained in a maximal polar. Let L be a dual principal polar $\neq$ containing K (such a polar exists since for $0 \neq c \in K'$ there holds $G \neq c' \supseteq K$) and let $x \in \mathfrak{U}(\Pi')$ be generated by L. For every $y \in \mathfrak{U}(\Pi')$, $y \neq x$ there exists $a_y \in G^+$ with $a_y \bar{\in} \cup x$ and $a_y \in \cup y$ because $\cup x$ and $\cup y$ as different minimal prime subgroups are incomparable. The infimum b of these (finitely many) elements a_y belongs to the meet of all $\cup y$ $(y \neq x)$ and does not belong to $\cup x$ ([12] III 6.3 or [5] 1.7). Therefore $b' \in x$ and thus $b' \subseteq \cup x$ ([12] III 7.10 or [2] 3.4.1). Since $\cup x \cap \cap\{\cup y: y \neq x\} = \{0\}$, $\cup x \delta b$ and hence $\cup x \subseteq b'$. Finally $b' = \cup x$ and b' is the greatest element of x. b' is a dual atom of the lattice $\Pi'(G)$ by 4.1, thus a dual atom of $\Gamma(G)$ (by 1.3) and $b' \supseteq L \supseteq K$ holds. Hence G has a base by 1.4.

6⇔7 follows from [13] I 1.7.

7⇔8 follows from 1.13.

8⇒9 is evident.

9⇒4 is evident since $\mathfrak{R}_\Pi$ is the family of all minimal prime subgroups of G.

4.8 Theorem. *Let $G \neq \{0\}$ be an l-group. Then the following conditions are equivalent.*

1. $\mathfrak{R}(\mathfrak{R}_\Gamma, G) = \mathfrak{M}(\mathfrak{R}_\Gamma, G)$.
2. $\mathfrak{R}(\mathfrak{U}_s(\Gamma), \Sigma') = \mathfrak{M}(\mathfrak{U}_s(\Gamma), \Sigma')$.
3. *Trivial open sets of the space $(\mathfrak{R}_\Gamma, G)$ form a finite partition on $\mathfrak{R}_\Gamma$.*
4. *$\mathfrak{R}_\Gamma$ is a finite regulator of type α.*
5. *Any of the conditions of Theorem 4.7.*
6. *Any of the conditions of Theorem 4.5 together with the finiteness of the space $(\mathfrak{R}_\Pi, G)$.*

If one of the above conditions is true, then G has a base and a weak unit, the space $(\mathfrak{R}_\Gamma, G)$ is compact and both regulators $\mathfrak{R}_\Gamma$ and $\mathfrak{R}_\Pi$ are of type α and are equal.

Proof $1 \Leftrightarrow 2$ by [13] I 1.7.

$1 \Rightarrow 3$. Condition 1 implies the condition (4b), Theorem 4.1 of [10], thus G has a weak unit. Hence $\mathfrak{U}(\Gamma) = \mathfrak{U}_c(\Gamma)$ ([12] V 12 6). Consequently, the space $(\mathfrak{U}_c(\Gamma), \Sigma')$ is compact since the space $(\mathfrak{U}(\Gamma), \Sigma)$ is compact ([11] Theorem 3) and the topology Σ' is weaker than Σ (see also [6] 3.3 or [10] 4.2). Then the space $(\mathfrak{R}_\Gamma, G)$ is compact, too ([13] I 1.7). By 2.10 trivial open sets form a partition on $\mathfrak{R}_\Gamma$. From the compactness it follows that this partition is finite.

$3 \Rightarrow 5$. (4.7(1)). The elements of the partition on trivial open sets of the space $(\mathfrak{R}_\Gamma, G)$ are atoms of the lattice $\mathfrak{M}(\mathfrak{R}_\Gamma, G)$ and every element of $\mathfrak{M}(\mathfrak{R}_\Gamma, G)$ is the union of finitely many atoms. Hence the lattice $\mathfrak{M}(\mathfrak{R}_\Gamma, G)$ as well as the isomorphic lattice $\Gamma(G)$ of all polars is finite ([13] I 2.18).

$5 \Rightarrow 1$. If G has only finitely many polars (4.7(1)), then $\Gamma(G) = \Pi'(G)$. In fact, for $K \in \Gamma$, $K' = \vee_\Gamma\{a'': a \in (K')^+\} = \vee_\Gamma\{a'': a_i \in (K')^+, i = 1, 2, \ldots, n\}$, and so $K = \bigcap_{i=1}^{n} a' = \left(\bigvee_{i=1}^{n} {}_G a\right)' \in \Pi'(G)$. This means $\mathfrak{R}_\Gamma = \mathfrak{R}_\Pi$, and so the space $(\mathfrak{R}_\Gamma, G) = (\mathfrak{R}_\Pi, G)$ is discrete by 4 7 and $\mathfrak{N}(\mathfrak{R}_\Gamma, G) = \mathfrak{M}(\mathfrak{R}_\Gamma, G)$ by 4.5.

$5 \Leftrightarrow 6$. Clearly, $4.7(7) - 4.5(3) \wedge$(the finiteness of the space $(\mathfrak{R}_\Pi, G)$) because $4.5(3) \equiv 3.7(e) - 3.6(f)$.

$4 \Rightarrow 1$. By 1.13 the space $(\mathfrak{R}_\Gamma, G)$ is finite and discrete, whence 1.

$1 \Rightarrow 4$. By [10] 4.1 ($4b \Rightarrow 1a$) there holds $\mathfrak{R}_\Gamma = \mathfrak{R}_\Pi$. Since $1 \Rightarrow 5$, $(\mathfrak{R}_\Gamma, G)$ is a finite discrete space. Then by 1.13 we have 4.

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РЕГУЛЯТОРЫ ТИПА α СТРУКТУРНО УПОРЯДОЧЕННЫХ ГРУПП

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Резюме

В работе изучены l-группы с базой при помощи алгебраических и топологических методов. Алгебраическим исследованиям служит так называемый регулятор $(\mathfrak{R}, \cup)$, то есть множество $\mathfrak{R} \neq \emptyset$ и отображение $\cup$ множества $\mathfrak{R}$ в систему простых подгрупп в G такое, что $\{\cup x\colon x \in \mathfrak{R}\}$ имеет нулевое пересечение. Топологические исследования сопровождаются с помощью топологии, индуцированной на $\mathfrak{R}$ (структурное пространство). $(\mathfrak{R}, \cup)$ — регулятор типа α, если $\cap\{\cup x\colon x \in \mathfrak{R}, x \neq y\} \neq \{0\}$ для всех $y \in \mathfrak{R}$. Доказывается, что существует (с точностью до эквиваленции) только один регулятор типа α l-группы G, а именно множество $\mathcal{P}$ всех минимальных простых подгрупп с отображением $\cup = \mathrm{id}_{\mathcal{P}}$ (1.9). Существование регулятора типа α харак теризует l-группы с базой (1.10). Топологическая характеризация l-групп, обладающих базой, дана в 3.5 и 2.4. Подобие стандартного регулятора $(\mathfrak{R}, G)$ с регулятором типа α описано соотношением $\mathfrak{N}(\mathfrak{R}, G) = \mathfrak{M}(\mathfrak{R}, G)$ (3.1) (здесь мы используем обозначения, введенные в 0.1–0.3; см. тоже [10], [13]). Свойство $\mathfrak{N}(\mathfrak{R}, G) = \mathfrak{M}(\mathfrak{R}, G)$ характеризовано несколькими эквивалентными условиями в 2.10 и 3.6. В теореме 3.7, в которой результаты теоремы 3.6 специализируются на вполне регулярные регуляторы $(\mathfrak{R}, \cup)$, это равенство характеризовано следующими условиями: 1. Множество всех минимальных простых подгрупп J, обладающих свойством $Z(J) \neq \emptyset$, равно множеству всех максимальных поляр в G; 2. Всякий ультраантифильтр x на $\Pi'(G)$ с $Z(\cup x) \neq \emptyset$ — главный; 3. Пространство $(\mathfrak{R}, G)$ локально связно. Если регулятор $(\mathfrak{R}, \cup)$ редуцирован, то предыдущее условие выражается: Пространство $(\mathfrak{R}, \cup)$ дискретно. В абз. 4 изучены условия, при которых Γ-регулятор или Π'-регулятор будет регулятором типа α и конечный типа α. Результаты находятся в теоремах 4.5, 4.7 и 4.8.