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Jaroslav Stuchlý

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THE BAYES ESTIMATOR OF THE VARIANCE COMPONENTS AND ITS ADMISSIBILITY

JAROSLAV STUCHLÝ

ABSTRACT. In the paper the necessary and sufficient conditions for the existence of the Bayes invariant quadratic unbiased estimator of the linear function of the variance components in the mixed linear model $\boldsymbol{t} = \boldsymbol{X\beta} + \varepsilon$, $\mathrm{E}(\boldsymbol{t}) = \boldsymbol{X\beta}$, $\mathrm{Var}(\boldsymbol{t}) = \sum_{i=1}^{p} \theta_i \boldsymbol{V}_i$, in the normal case, have been presented. Moreover, explicit expressions for this estimator have been found and the admissibility question has been considered.

Introduction

Let us consider a mixed linear model

$$(1) \qquad \boldsymbol{t} = \boldsymbol{X\beta} + \varepsilon, \quad \mathrm{E}(\boldsymbol{t}) = \boldsymbol{X\beta}, \quad \mathrm{Var}(\boldsymbol{t}) = \sum_{i=1}^{p} \theta_i \boldsymbol{U}_i = \boldsymbol{U}(\boldsymbol{\theta}),$$

where $\boldsymbol{t}$ is an N-dimensional, normally distributed random vector, $\boldsymbol{X}$ is a known $N \times m$ matrix of rank $\mathrm{r}(\boldsymbol{X}) = s$, $\boldsymbol{\beta} \in \mathbb{R}^m$ is an unknown vector, $\boldsymbol{U}_1, \ldots, \boldsymbol{U}_p$ are known symmetric matrices, and $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_p)'$ is a vector of unknown variance components, $\boldsymbol{\theta} \in \mathscr{T}$, where $\mathscr{T} = \{\boldsymbol{\theta}\colon \theta_1 > 0, \theta_2 \geqq 0, \ldots, \theta_p \geqq 0, \boldsymbol{U}(\boldsymbol{\theta})$ is a positive semidefinite matrix$\}$.

We shall look for the Bayes invariant quadratic unbiased estimator (BAIQUE) $\hat{\gamma}(\boldsymbol{t}) = \boldsymbol{t}'\boldsymbol{Bt}$ of the parametric function $\gamma = \boldsymbol{f}'\boldsymbol{\theta}$ ($\boldsymbol{B}$ is a symmetric N-matrix and $\boldsymbol{f} = (f_1, \ldots, f_p)'$), i.e. for an unbiased estimator which minimizes the Bayes risk function

$$\mathrm{r}(\hat{\gamma}) = \tfrac{1}{2} \int \mathrm{E}_\theta(\hat{\gamma} - \gamma)^2 \, \mathrm{dP}^\theta,$$

where P^θ is the a priori distribution for the vector parameter $\boldsymbol{\theta}$, and which is invariant with respect to the translation $\boldsymbol{t} \to \boldsymbol{t} + \boldsymbol{X\beta}$, i.e. which satisfies the condition

$$\hat{\gamma}(\boldsymbol{t}) = \hat{\gamma}(\boldsymbol{t} + \boldsymbol{X\beta})$$

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for all $\boldsymbol{\beta} \in \mathbb{R}^m$. The aim of the paper is to derive explicit expressions for the BAIQUE of the estimable function $\boldsymbol{f}'\boldsymbol{\theta}$, to find sufficient conditions for the uniqueness, and to investigate the admissibility of this estimator.

1. Preliminary considerations

As a starting point, let us transform the model (1) to the form

$$(2) \qquad \boldsymbol{y} = \boldsymbol{P}\boldsymbol{t}, \quad \mathrm{E}(\boldsymbol{y}) = \boldsymbol{0}, \quad \mathrm{Var}(\boldsymbol{y}) = \sum_{i=1}^{p} \theta_i \boldsymbol{V}_i = \boldsymbol{V}(\boldsymbol{\theta}),$$

where $\boldsymbol{P}$ is an $(N-s) \times N$ matrix satisfying $\boldsymbol{P}'\boldsymbol{P} = \boldsymbol{M} = \boldsymbol{I} - \boldsymbol{X}\boldsymbol{X}^+$ ($\boldsymbol{X}^+$ is the Moore-Penrose inverse of the matrix $\boldsymbol{X}$), $\boldsymbol{P}\boldsymbol{P}' = \boldsymbol{I}_n$, $n = N - s$, $\boldsymbol{V}_i = \boldsymbol{P}\boldsymbol{U}_i\boldsymbol{P}'$, $i = 1, \ldots, p$. The estimator $\hat{\gamma} = \boldsymbol{t}'\boldsymbol{B}\boldsymbol{t}$ is the BAIQUE for γ in the model (1) iff $\boldsymbol{B} = \boldsymbol{P}'\boldsymbol{A}\boldsymbol{P}$ and $\hat{\gamma} = \boldsymbol{y}'\boldsymbol{A}\boldsymbol{y}$ is the Bayes quadratic unbiased estimator (BAQUE) for γ under the model(2).

Following [1, Theorem 7a], $\boldsymbol{y}'\boldsymbol{A}\boldsymbol{y}$ is the BAQUE for $\boldsymbol{f}'\boldsymbol{\theta}$ under the model (2) iff

$$(3) \qquad \sum_{i,j=1}^{p} c_{ij} \boldsymbol{V}_i \boldsymbol{A} \boldsymbol{V}_j = \sum_{i=1}^{p} \lambda_i \boldsymbol{V}_i$$

holds, where

$$c_{ij} = \mathrm{E}(\theta_i \theta_j) = \int \theta_i \theta_j \, \mathrm{dP}^\theta, \quad i,j = 1, \ldots, p,$$

and $\lambda_1, \ldots, \lambda_p$ satisfy the unbiasedness conditions

$$(4) \qquad \mathrm{tr}(\boldsymbol{A}\boldsymbol{V}_i) = f_i, \quad i = 1, \ldots, p.$$

We characterize the admissibility with respect to the risk function

$$\mathrm{R}(\hat{\gamma}, \boldsymbol{\theta}) = \tfrac{1}{2} \mathrm{E}_\theta (\hat{\gamma} - \gamma)^2 = \mathrm{tr}[\boldsymbol{A}\boldsymbol{V}(\boldsymbol{\theta})\boldsymbol{A}\boldsymbol{V}(\boldsymbol{\theta})].$$

The quadratic estimator $\hat{\gamma}_1$ is better than the quadratic estimator $\hat{\gamma}_2$ iff $\mathrm{R}(\hat{\gamma}_1, \boldsymbol{\theta}) \leqq \mathrm{R}(\hat{\gamma}_2, \boldsymbol{\theta})$ for all $\boldsymbol{\theta} \in \mathscr{T}$ and $\mathrm{R}(\hat{\gamma}_1, \boldsymbol{\theta}_0) < \mathrm{R}(\hat{\gamma}_2, \boldsymbol{\theta}_0)$ at some point $\boldsymbol{\theta}_0 \in \mathscr{T}$.

The quadratic estimator $\hat{\gamma}$ is admissible among a subclass of quadratic estimators on $\mathscr{T}$ iff no other quadratic estimator in this subclass is better than $\hat{\gamma}$ on $\mathscr{T}$.

Let us denote by $\mathfrak{M}(\boldsymbol{A})$ the vector space generated by the columns of $\boldsymbol{A}$, by $\mathfrak{N}(\boldsymbol{A})$ the null space of $\boldsymbol{A}$, by $\boldsymbol{A} \otimes \boldsymbol{B}$ the Kronecker product of $\boldsymbol{A}$ and $\boldsymbol{B}$, and $\mathrm{vec}\,\boldsymbol{A} = (a_{11}, \ldots, a_{n1}, a_{12}, \ldots, a_{n2}, a_{13}, \ldots, a_{mn})'$, if $\boldsymbol{A}$ is an $n \times m$ matrix with elements a_{ij}, $i = 1, \ldots, n$, $j = 1, \ldots, m$. First let us prove some lemmas.

Lemma 1. $\mathfrak{M}(\boldsymbol{A}) \subset \mathfrak{M}(\boldsymbol{B}) \Leftrightarrow \{\boldsymbol{x}'\boldsymbol{B} = \boldsymbol{0} \Rightarrow \boldsymbol{x}'\boldsymbol{A} = \boldsymbol{0}\}$.

Proof. It is easily seen that the following statements are equivalent.

(a) $\mathfrak{M}(\boldsymbol{A}) \subset \mathfrak{M}(\boldsymbol{B})$;

(b) $\mathfrak{N}(\boldsymbol{B}') \subset \mathfrak{N}(\boldsymbol{A}')$;

(c) $\boldsymbol{B}'\boldsymbol{x} = \boldsymbol{0}$ implies $\boldsymbol{A}'\boldsymbol{x} = \boldsymbol{0}$;

(d) $\boldsymbol{x}'\boldsymbol{B} = \boldsymbol{0}$ implies $\boldsymbol{x}'\boldsymbol{A} = \boldsymbol{0}$.

Lemma 2. *Let $\mathfrak{M}(\boldsymbol{S}') \subset \mathfrak{M}(\boldsymbol{H})$ and $\boldsymbol{H}$ be a positive semidefinite matrix. Then $\mathfrak{M}(\boldsymbol{S}) = \mathfrak{M}(\boldsymbol{S}\boldsymbol{H}^-\boldsymbol{S}')$.*

Proof. There exists a matrix $\boldsymbol{D}$ so that $\boldsymbol{S}' = \boldsymbol{H}\boldsymbol{D}$. Hence

$$\boldsymbol{S}\boldsymbol{H}^-\boldsymbol{S}' = \boldsymbol{S}\boldsymbol{H}^-\boldsymbol{H}\boldsymbol{D} = \boldsymbol{S}\boldsymbol{D} = \boldsymbol{D}'\boldsymbol{S}'.$$

If $\boldsymbol{x}'\boldsymbol{S}\boldsymbol{H}^-\boldsymbol{S}' = \boldsymbol{0}$, then $\boldsymbol{x}'\boldsymbol{D}'\boldsymbol{S}' = \boldsymbol{x}'\boldsymbol{D}'\boldsymbol{H}\boldsymbol{D} = \boldsymbol{0}$ and $\boldsymbol{x}'\boldsymbol{D}'\boldsymbol{H} = \boldsymbol{x}'\boldsymbol{S} = \boldsymbol{0}$, i.e. $\mathfrak{M}(\boldsymbol{S}) \subset \mathfrak{M}(\boldsymbol{S}\boldsymbol{H}^-\boldsymbol{S}')$.
The other inclusion is obvious.

Lemma 3. *Let $\boldsymbol{A}$, $\boldsymbol{B}$ be positive semidefinite matrices of order n. Then $\boldsymbol{A} \otimes \boldsymbol{B}$ is a positive semidefinite matrix of order n^2.*

For proof see [5].

Lemma 4. *Let $\boldsymbol{P}$ be an $n \times N$ matrix. Let $\boldsymbol{A}$ be a symmetric and $\boldsymbol{B}$ a positive semidefinite matrix, both of order N. If $\mathfrak{M}(\boldsymbol{A}) \supset \mathfrak{M}(\boldsymbol{B})$, then $\mathfrak{M}(\boldsymbol{P}\boldsymbol{A}\boldsymbol{P}') \supset \mathfrak{M}(\boldsymbol{P}\boldsymbol{B}\boldsymbol{P}')$.*

Proof. There exists a matrix $\boldsymbol{Q}$ so that $\boldsymbol{B} = \boldsymbol{A}\boldsymbol{Q}$. Then $\boldsymbol{P}\boldsymbol{B} = \boldsymbol{P}\boldsymbol{A}\boldsymbol{Q}$ and $\mathfrak{M}(\boldsymbol{P}\boldsymbol{B}) \subset \mathfrak{M}(\boldsymbol{P}\boldsymbol{A}) = \mathfrak{M}(\boldsymbol{P}\boldsymbol{A}\boldsymbol{P}')$. Hence $\mathfrak{M}(\boldsymbol{P}\boldsymbol{B}\boldsymbol{P}') \subset \mathfrak{M}(\boldsymbol{P}\boldsymbol{A}\boldsymbol{P}')$.

Lemma 5. *Let $\boldsymbol{M} = \boldsymbol{P}'\boldsymbol{P}$, $\boldsymbol{P}\boldsymbol{P}' = \boldsymbol{I}$ and let $\boldsymbol{A}$ be a positive semidefinite matrix. Then $\mathfrak{M}(\boldsymbol{P}\boldsymbol{B}) \subset \mathfrak{M}(\boldsymbol{P}\boldsymbol{A}\boldsymbol{P}')$ iff $\mathfrak{M}(\boldsymbol{M}\boldsymbol{B}) \subset \mathfrak{M}(\boldsymbol{M}\boldsymbol{A}\boldsymbol{M})$.*

Proof. $\mathfrak{M}(\boldsymbol{P}\boldsymbol{B}) \subset \mathfrak{M}(\boldsymbol{P}\boldsymbol{A}\boldsymbol{P}') = \mathfrak{M}(\boldsymbol{P}\boldsymbol{A})$ iff there exists a matrix $\boldsymbol{Q}$ so that $\boldsymbol{P}\boldsymbol{B} = \boldsymbol{P}\boldsymbol{A}\boldsymbol{Q}$, i.e. $\boldsymbol{M}\boldsymbol{B} = \boldsymbol{M}\boldsymbol{A}\boldsymbol{Q}$, which means that $\mathfrak{M}(\boldsymbol{M}\boldsymbol{B}) \subset \mathfrak{M}(\boldsymbol{M}\boldsymbol{A}) = \mathfrak{M}(\boldsymbol{M}\boldsymbol{A}\boldsymbol{M})$.

2. Main results

Theorem 1. a) *The* BAQUE *for the parametric function $\gamma = \boldsymbol{f}'\boldsymbol{\theta}$ in the model* (2) *exists iff*

$$\boldsymbol{f} \in \mathfrak{M}(\boldsymbol{S}), \tag{5}$$

where $\boldsymbol{S} = (\operatorname{vec} \boldsymbol{V}_1, \ldots, \operatorname{vec} \boldsymbol{V}_p)'$.

b) *If the matrix $\boldsymbol{H} = \sum_{i,j=1}^{p} c_{ij}(\boldsymbol{V}_i \otimes \boldsymbol{V}_j)$ is regular, then the* BAQUE *is uniquely given by*

$$\hat{\gamma} = \boldsymbol{f}'(\boldsymbol{S}\boldsymbol{H}^{-1}\boldsymbol{S}')^+ \boldsymbol{S}\boldsymbol{H}^{-1} \operatorname{vec}(\boldsymbol{y}\boldsymbol{y}') \tag{6}$$

and is therefore admissible.

c) *If*

$$\mathfrak{M}(\boldsymbol{S}') \subset \mathfrak{M}(\boldsymbol{H}), \tag{7}$$

then the BAQUE *is given by*

$$\hat{\gamma} = [\boldsymbol{f}'(\boldsymbol{S}\boldsymbol{H}^+\boldsymbol{S}')^+\,\boldsymbol{S}\boldsymbol{H}^+ + \boldsymbol{x}'(\boldsymbol{I} - \boldsymbol{H}\boldsymbol{H}^+)]\ \mathrm{vec}(\boldsymbol{y}\boldsymbol{y}'), \tag{8}$$

where $\boldsymbol{x}$ *is an arbitrary* n^2 — *vector.*

d) *If the condition* (7) *is not fulfilled, then the* BAQUE *is given by*

$$\hat{\gamma} = \{\boldsymbol{f}'[\boldsymbol{S}(\boldsymbol{H} + \boldsymbol{S}'\boldsymbol{S})^+\,\boldsymbol{S}']^+\,\boldsymbol{S}(\boldsymbol{H} + \boldsymbol{S}'\boldsymbol{S})^+ + \boldsymbol{x}'[\boldsymbol{I} - (\boldsymbol{H} + \boldsymbol{S}'\boldsymbol{S})(\boldsymbol{H} + \boldsymbol{S}'\boldsymbol{S})^+\} \times$$

$$\times\ \mathrm{vec}(\boldsymbol{y}\boldsymbol{y}'), \tag{9}$$

where $\boldsymbol{x}$ *is an arbitrary* n^2 — *vector.*

e) *The sufficient conditions for the admissibility of the* BAQUE *is*

$$\mathfrak{M}(\boldsymbol{V}_i \otimes \boldsymbol{V}_j) \subset \mathfrak{M}(\boldsymbol{H}), \quad \textit{for all} \quad i, j = 1, \ldots, p. \tag{10}$$

Proof. Let us rewrite the equations (3), (4) to the form

$$\sum_{i,j=1}^{p} c_{ij}(\boldsymbol{V}_i \otimes \boldsymbol{V}_j)\ \mathrm{vec}\,\boldsymbol{A} = \sum_{i=1}^{p} \lambda_i \boldsymbol{V}_i,$$

$$(\mathrm{vec}\,\boldsymbol{V}_i)'\ \mathrm{vec}\,\boldsymbol{A} = f_i, \quad i = 1, \ldots, p.$$

Using the previous notation and $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_p)'$ we have

$$\boldsymbol{H}\ \mathrm{vec}\,\boldsymbol{A} = \boldsymbol{S}'\boldsymbol{\lambda}, \tag{3a}$$

$$\boldsymbol{S}\ \mathrm{vec}\,\boldsymbol{A} = \boldsymbol{f}. \tag{4a}$$

Under the assumption (7), all the solutions of the equation (3a) are

$$\mathrm{vec}\,\boldsymbol{A} = \boldsymbol{H}^+\boldsymbol{S}'\boldsymbol{\lambda} + (\boldsymbol{I} - \boldsymbol{H}^+\boldsymbol{H})\,\boldsymbol{x}, \tag{11}$$

where $\boldsymbol{x}$ is an arbitrary n^2 — vector. Substituting in (4a) we get

$$\boldsymbol{S}\boldsymbol{H}^+\boldsymbol{S}'\boldsymbol{\lambda} + \boldsymbol{S}(\boldsymbol{I} - \boldsymbol{H}^+\boldsymbol{H})\,\boldsymbol{x} = \boldsymbol{f}.$$

According to the condition (7), the last expression vanishes. Our equation

$$\boldsymbol{S}\boldsymbol{H}^+\boldsymbol{S}'\boldsymbol{\lambda} = \boldsymbol{f}$$

has a solution iff (5) holds. Then

$$\boldsymbol{\lambda} = (\boldsymbol{S}\boldsymbol{H}^+\boldsymbol{S}')^+\,\boldsymbol{f} + [\boldsymbol{I} - (\boldsymbol{S}\boldsymbol{H}^+\boldsymbol{S}')^+\,(\boldsymbol{S}\boldsymbol{H}^+\boldsymbol{S}')]\,\boldsymbol{y},$$

where $\boldsymbol{y}$ is an arbitrary n^2 — vector. Substituting in (11) we get

$$\operatorname{vec} \boldsymbol{A} = \boldsymbol{H}^+ \boldsymbol{S}'(\boldsymbol{SH}^+ \boldsymbol{S}')^+ \boldsymbol{f} + \boldsymbol{H}^+ \boldsymbol{S}'[\boldsymbol{I} - (\boldsymbol{SH}^+ \boldsymbol{S}')^+ (\boldsymbol{SH}^+ \boldsymbol{S}')] \boldsymbol{y} + (\boldsymbol{I} - \boldsymbol{H}^+ \boldsymbol{H}) \boldsymbol{x}.$$

Since

$$\mathfrak{M}(\boldsymbol{SH}^+ \boldsymbol{S}') \subset \mathfrak{M}(\boldsymbol{S}),$$

there is

$$\operatorname{vec} \boldsymbol{A} = \boldsymbol{H}^+ \mathrm{S}'(\boldsymbol{SH}^+ \boldsymbol{S}')^+ \boldsymbol{f} + (\boldsymbol{I} - \boldsymbol{H}^+ \boldsymbol{H}) \boldsymbol{x}.$$

If we use

$$\hat{\gamma} = \boldsymbol{y}' \boldsymbol{A} \boldsymbol{y} = \operatorname{tr}(\boldsymbol{A} \boldsymbol{y} \boldsymbol{y}') = (\operatorname{vec} \boldsymbol{A})' \operatorname{vec}(\boldsymbol{y} \boldsymbol{y}'),$$

we get (8).

If the condition (7) is not fulfilled, we can rewrite the system (3a)—(4a) to the form

$$(3b) \qquad (\boldsymbol{H} + \boldsymbol{S}' \boldsymbol{S}) \operatorname{vec} \boldsymbol{A} = \boldsymbol{S}'(\lambda + \boldsymbol{f}),$$

$$(4a) \qquad \boldsymbol{S} \operatorname{vec} \boldsymbol{A} = \boldsymbol{f}.$$

Since the matrix $\boldsymbol{H}$ is positive semidefinite, we have now

$$\mathfrak{M}(\boldsymbol{S}') \subset \mathfrak{M}(\boldsymbol{H} + \boldsymbol{S}' \boldsymbol{S}),$$

and in the same way we get that BAQUE exists iff

$$\boldsymbol{f} \in \mathfrak{M}[\boldsymbol{S}(\boldsymbol{H} + \boldsymbol{S}' \boldsymbol{S})^+ \boldsymbol{S}'] = \mathfrak{M}(\boldsymbol{S})$$

(see Lemma 2) and has the form (8) (cf. [2, p. 15—16]).

If the matrix $\boldsymbol{H}$ is regular, then the equations (3)—(4) have the only solution $\operatorname{vec} \boldsymbol{A} = \boldsymbol{H}^{-1} \boldsymbol{S}'(\boldsymbol{SH}^{-1} \boldsymbol{S}')^+ \boldsymbol{f}$ and the BAQUE has the form (6).

If the matrix $\boldsymbol{H}$ is singular, then the BAQUE is not uniquely given. To investigate the admissibility, let us express the risk function in the form

$$\mathrm{R}(\hat{\gamma}, \boldsymbol{\theta}) = \sum_{i,j=1}^{p} \theta_i \theta_j \operatorname{tr}(\boldsymbol{V}_i \boldsymbol{A} \boldsymbol{V}_j \boldsymbol{A}) =$$

$$= \sum_{i,j=1}^{p} \theta_i \theta_j [\operatorname{vec}(\boldsymbol{V}_j \boldsymbol{A} \boldsymbol{V}_i)]' \operatorname{vec} \boldsymbol{A} = \left[\sum_{i,j=1}^{p} \theta_i \theta_j (\boldsymbol{V}_i \otimes \boldsymbol{V}_j) \operatorname{vec} \boldsymbol{A} \right]' \operatorname{vec} \boldsymbol{A}.$$

Under the assumption (7) we get

$$(12) \quad \mathrm{R}(\hat{\gamma}, \boldsymbol{\theta}) = \left\{ \sum_{i,j=1}^{p} \theta_i \theta_j (\boldsymbol{V}_i \otimes \boldsymbol{V}_j) [\boldsymbol{H}^+ \boldsymbol{S}'(\boldsymbol{SH}^+ \boldsymbol{S}')^+ \boldsymbol{f} + (\boldsymbol{I} - \boldsymbol{H}^+ \boldsymbol{H}) \boldsymbol{x}] \right\}' \operatorname{vec} \boldsymbol{A}.$$

If the conditions (10) are fulfilled, then

$$\mathrm{R}(\hat{\gamma}, \boldsymbol{\theta}) = \sum_{i,j=1}^{p} \theta_i \theta_j (\boldsymbol{V}_i \otimes \boldsymbol{V}_j) [\boldsymbol{H}^+ \boldsymbol{S}'(\boldsymbol{SH}^+ \boldsymbol{S}')^+ \boldsymbol{f}] \operatorname{vec} \boldsymbol{A}.$$

Let us denote $\operatorname{vec} \boldsymbol{A}_0 = \boldsymbol{H}^+ \boldsymbol{S}'(\boldsymbol{S}\boldsymbol{H}^+ \boldsymbol{S}')^+ \boldsymbol{f}$. Then

$$\begin{aligned} R(\hat{\gamma}, \boldsymbol{\theta}) &= \sum_{i,j=1}^{p} \theta_i \theta_j [(\boldsymbol{V}_i \otimes \boldsymbol{V}_j) \operatorname{vec} \boldsymbol{A}_0]' \operatorname{vec} \boldsymbol{A} = \\ &= \sum_{i,j=1}^{p} \theta_i \theta_j [\operatorname{vec}(\boldsymbol{V}_j \boldsymbol{A}_0 \boldsymbol{V}_i)]' \operatorname{vec} \boldsymbol{A} = \\ &= \sum_{i,j=1}^{p} \theta_i \theta_j \operatorname{tr}(\boldsymbol{V}_j \boldsymbol{A} \boldsymbol{V}_i \boldsymbol{A}_0). \end{aligned}$$

By a repeated application of this method we shall show that

$$R(\hat{\gamma}, \boldsymbol{\theta}) = \sum_{i,j=1}^{p} \theta_i \theta_j \operatorname{tr}(\boldsymbol{V}_i \boldsymbol{A}_0 \boldsymbol{V}_j \boldsymbol{A}_0),$$

i.e. the risk function does not depend on the choice of the vector $\boldsymbol{x}$. Therefore all the BAQUE are admissible if the condition (10) holds. If the condition (7) is not fulfilled, then the matrix $\boldsymbol{H}$ in (12) must be changed to $\boldsymbol{H} + \boldsymbol{S}'\boldsymbol{S}$. Then $\mathfrak{M}(\boldsymbol{H}) \subset$ $\subset \mathfrak{M}(\boldsymbol{H} + \boldsymbol{S}'\boldsymbol{S})$ and we get by assumption (10) the same conclusion.

Remark 1. The conditions (10) are fulfilled and the given estimates (8), (9) are admissible in the case that $\boldsymbol{V}_1, \ldots, \boldsymbol{V}_p$ are positive semidefinite matrices and $c_{ii} \neq 0$ for $i = 1, \ldots, p$. The matrix $\boldsymbol{C} = (c_{ij})$ is obviously positive semidefinite. If $c_{ii} = 0$, then $c_{ik} = c_{ki} = \mathrm{E}(\theta_i \theta_k) = 0$, $k = 1, \ldots, p$. Since $\theta_i \geqq 0$ we have $\mathrm{P}(\theta_i = 0) = 1$. Therefore we can solve this situation by reducing the number of the variance components.

Remark 2. If $\boldsymbol{C} = \boldsymbol{R}'\boldsymbol{R}$, where $\boldsymbol{R}$ is an upper triangular matrix of order p, then we can write $\boldsymbol{H} = \sum_{k=1}^{p} (\boldsymbol{W}_k \otimes \boldsymbol{W}_k)$, where

$$\boldsymbol{W}_k = \sum_{j=k}^{p} r_{kj} \boldsymbol{V}_j, \quad c_{ij} = \sum_{k=1}^{q} r_{ki} r_{kj}, \quad q = \min(i, j), \quad i, j = 1, \ldots, p.$$

This form of notation was used in [5] and [4]. In [5] this problem is solved in the case $p = 3$, $\boldsymbol{V}_1$, $\boldsymbol{V}_2$ are positive semidefinite matrices, $\boldsymbol{V}_3 = \boldsymbol{I}$ and $\boldsymbol{H}$ is a regular matrix. In [4] we found the solution for the case $p = 2$, $\boldsymbol{V}_1$, $\boldsymbol{V}_2$ being positive semidefinite matrices and $\mathfrak{M}(\boldsymbol{W}_2) \subset \mathfrak{M}(\boldsymbol{W}_1)$. The BAQUE of the parametric function $\boldsymbol{f}'\boldsymbol{\theta} = f_1\theta_1 + f_2\theta_2$ exists if $\boldsymbol{f} \in \mathfrak{M}(\boldsymbol{R})$, where $\boldsymbol{R} = (\operatorname{tr}(\boldsymbol{V}_i \boldsymbol{M}_j))$, $\boldsymbol{M}_j =$ $= \frac{1}{2}(\boldsymbol{W}_1^+ \boldsymbol{V}_j \boldsymbol{K}^+ + \boldsymbol{K}^+ \boldsymbol{V}_j \boldsymbol{W}_1^+)$, $j = 1, 2$, $\boldsymbol{K} = \boldsymbol{W}_1 + \boldsymbol{W}_2 \boldsymbol{W}_1^+ \boldsymbol{W}_2$. If the matrix $\boldsymbol{W}_1$ is regular, then the BAQUE is uniquely given by $\hat{\gamma} = \boldsymbol{y}'\boldsymbol{A}\boldsymbol{y}$, $\boldsymbol{A} = \frac{1}{2}[\boldsymbol{W}_1^{-1}(\lambda_1 \boldsymbol{V}_1 +$ $+ \lambda_2 \boldsymbol{V}_2)(\boldsymbol{W}_1 + \boldsymbol{W}_2 \boldsymbol{W}_1^{-1} \boldsymbol{W}_2)^{-1} + (\boldsymbol{W}_1 + \boldsymbol{W}_2 \boldsymbol{W}_1^{-1} \boldsymbol{W}_2)^{-1}(\lambda_1 \boldsymbol{V}_1 + \lambda_2 \boldsymbol{V}_2) \boldsymbol{W}_1^{-1}]$, where $\boldsymbol{\lambda} = (\lambda_1, \lambda_2)'$ satisfies the condition $\boldsymbol{R}\boldsymbol{\lambda} = \boldsymbol{f}$.

If the matrix $\boldsymbol{W}_1$ is singular, then the BAQUE is $\hat{\gamma} = \boldsymbol{y}'\boldsymbol{A}\boldsymbol{y}$, $\boldsymbol{A} = \frac{1}{2}\boldsymbol{W}^-(\lambda_1 \boldsymbol{V}_1 + \lambda_2 \boldsymbol{V}_2)(\boldsymbol{W}_1 + \boldsymbol{W}_2 \boldsymbol{W}_1^- \boldsymbol{W}_2)^- + \frac{1}{2}(\boldsymbol{W}_1 + \boldsymbol{W}_2 \boldsymbol{W}_1^- \boldsymbol{W}_1)^-(\lambda_1 \boldsymbol{V}_1 + \lambda_2 \boldsymbol{V}_2)\boldsymbol{W}_1^- + \boldsymbol{N}\boldsymbol{Z}'_{12}\boldsymbol{Q}' + \boldsymbol{Q}\boldsymbol{Z}_{12}\boldsymbol{N}' + \boldsymbol{N}\boldsymbol{Z}_{22}\boldsymbol{N}'$, where λ satisfies the condition $\boldsymbol{R}\lambda = \boldsymbol{f}$, $\boldsymbol{Z}_{12}$, $\boldsymbol{Z}_{22}$ are arbitrary matrices, and the matrices $\boldsymbol{N}$, $\boldsymbol{Q}$ are given by the conditions $\boldsymbol{N}'\boldsymbol{W}_1 = \boldsymbol{0}$, $\boldsymbol{W}_1^- = \boldsymbol{Q}\boldsymbol{Q}'$, respectively. We see that the BAQUE is not uniquely given, but since the risk function $\mathrm{R}(\hat{\gamma}, \boldsymbol{\theta})$ does not depend on the choice of the matrices $\boldsymbol{Z}_{12}$, $\boldsymbol{Z}_{22}$ and is invariant with respect to the g-inverses, this estimator is admissible.

Now we rewrite Theorem 1 for the model (1) as follows.

Corollary 1. a) *The* BAIQUE *for the parametric function* $\gamma = \boldsymbol{f}'\boldsymbol{\theta}$ *in the model* (1) *exists iff*

(5a) $$\boldsymbol{f} \in \mathfrak{M}(\boldsymbol{R}\bar{\boldsymbol{M}}),$$

where $\boldsymbol{R} = (\mathrm{vec}\, \boldsymbol{U}_1, \ldots, \mathrm{vec}\, \boldsymbol{U}_p)'$, $\boldsymbol{M} = \boldsymbol{P}'\boldsymbol{P}$, $\bar{\boldsymbol{M}} = \boldsymbol{M} \otimes \boldsymbol{M}$.

b) *If the matrix* $\boldsymbol{G} = \sum_{i,j=1}^{p} c_{ij}(\boldsymbol{U}_i \otimes \boldsymbol{U}_j)$ *fulfils the condition* $\mathfrak{M}(\boldsymbol{G}) \supset \mathfrak{M}(\bar{\boldsymbol{M}})$, *then the BAIQUE is uniquelly given by*

(6a) $$\hat{\gamma} = \boldsymbol{f}'[\boldsymbol{R}(\bar{\boldsymbol{M}}\boldsymbol{G}\bar{\boldsymbol{M}})^+ \boldsymbol{R}']^+ \boldsymbol{R}(\bar{\boldsymbol{M}}\boldsymbol{G}\bar{\boldsymbol{M}})^+ \,\mathrm{vec}(\boldsymbol{t}\boldsymbol{t}'),$$

and is therefore admissible.

c) *If*

(7a) $$\mathfrak{M}(\bar{\boldsymbol{M}}\boldsymbol{R}') \subset \mathfrak{M}(\bar{\boldsymbol{M}}\boldsymbol{G}\bar{\boldsymbol{M}}),$$

then the BAIQUE is given by

(8a) $$\hat{\gamma} = \{\boldsymbol{f}'[\boldsymbol{R}(\bar{\boldsymbol{M}}\boldsymbol{G}\bar{\boldsymbol{M}})^+ \boldsymbol{R}']^+ \boldsymbol{R}(\bar{\boldsymbol{M}}\boldsymbol{G}\bar{\boldsymbol{M}})^+ + \boldsymbol{y}'[\boldsymbol{I} - \boldsymbol{G}(\bar{\boldsymbol{M}}\boldsymbol{G}\bar{\boldsymbol{M}})^+]\} \,\mathrm{vec}(\boldsymbol{t}\boldsymbol{t}'),$$

where $\boldsymbol{y}$ *is an arbitrary* N^2*-vector.*

d) *If the condition* (7a) *is not fulfilled, then the BAIQUE is given by*

(9a) $$\hat{\gamma} = \{\boldsymbol{f}'[\boldsymbol{R}(\bar{\boldsymbol{M}}\boldsymbol{G}\bar{\boldsymbol{M}} + \bar{\boldsymbol{M}}\boldsymbol{R}'\boldsymbol{R}\bar{\boldsymbol{M}})^+ \boldsymbol{R}']^+ \boldsymbol{R}(\bar{\boldsymbol{M}}(\boldsymbol{G} + \boldsymbol{R}'\boldsymbol{R})\bar{\boldsymbol{M}}]^+ + \boldsymbol{y}'[\boldsymbol{I} - (\boldsymbol{G} + \boldsymbol{R}'\boldsymbol{R})(\bar{\boldsymbol{M}}(\boldsymbol{G} + \boldsymbol{R}'\boldsymbol{R})\bar{\boldsymbol{M}})^+]\} \,\mathrm{vec}(\boldsymbol{t}\boldsymbol{t}'),$$

where $\boldsymbol{y}$ *is an arbitrary* N^2*-vector.*

e) *The sufficient conditions for the admissibility of the BAIQUE is*

(10a) $$\mathfrak{M}(\boldsymbol{M}\boldsymbol{U}_i\boldsymbol{M} \otimes \boldsymbol{M}\boldsymbol{U}_j\boldsymbol{M}) \subset \mathfrak{M}(\bar{\boldsymbol{M}}\boldsymbol{G}\bar{\boldsymbol{M}}), \quad i,j = 1, \ldots, p.$$

Proof. Since the BAQUE $\boldsymbol{y}'\boldsymbol{A}\boldsymbol{y}$ in the model (2) is simultaneously the BAIQUE $\boldsymbol{t}'\boldsymbol{B}\boldsymbol{t}$ in the model (1) and $\boldsymbol{B} = \boldsymbol{P}'\boldsymbol{A}\boldsymbol{P}$, we can write $\mathrm{vec}\,\boldsymbol{B} = (\boldsymbol{P}' \otimes \boldsymbol{P}')\,\mathrm{vec}\,\boldsymbol{A}$. Substituting $\boldsymbol{V}_i = \boldsymbol{P}\boldsymbol{U}_i\boldsymbol{P}'$, $i = 1, \ldots, p$, $\boldsymbol{S}' = (\mathrm{vec}(\boldsymbol{P}\boldsymbol{U}_1\boldsymbol{P}'), \ldots,$

$\mathrm{vec}(\boldsymbol{P}\boldsymbol{U}_p\boldsymbol{P}')) = (\boldsymbol{P}\otimes \boldsymbol{P})\,\boldsymbol{R}'$, $\boldsymbol{R}' = (\mathrm{vec}\,\boldsymbol{U}_1, \ldots, \mathrm{vec}\,\boldsymbol{U}_p)'$, $\boldsymbol{H} = \sum_{i,j=1}^{p} c_{ij}(\boldsymbol{P}\boldsymbol{U}_i\boldsymbol{P}') \otimes \otimes (\boldsymbol{P}\boldsymbol{U}_j\boldsymbol{P}') = (\boldsymbol{P}\otimes \boldsymbol{P})\,\boldsymbol{G}(\boldsymbol{P}'\otimes \boldsymbol{P}')$, $\boldsymbol{G} = \sum_{i,j=1}^{p} c_{ij}\boldsymbol{U}_i \otimes \boldsymbol{U}_j$, we get the statements of the Corollary 1 from the corresponding statements of the Theorem 1. From (5) it follows that (5a) holds, since $\lambda'\boldsymbol{R}(\boldsymbol{P}'\otimes\boldsymbol{P}') = \boldsymbol{0}$ iff $\lambda'\boldsymbol{R}(\boldsymbol{M}\otimes\boldsymbol{M}) = \boldsymbol{0}$. If $\mathfrak{M}(\boldsymbol{G}) \supset \mathfrak{M}(\bar{\boldsymbol{M}})$, then by Lemma 4

$$\mathfrak{M}(\boldsymbol{H}) = \mathfrak{M}[(\boldsymbol{P}\otimes\boldsymbol{P})\,\boldsymbol{G}(\boldsymbol{P}'\otimes\boldsymbol{P}')] \supset \mathfrak{M}[(\boldsymbol{P}\otimes\boldsymbol{P})(\boldsymbol{M}\otimes\boldsymbol{M})(\boldsymbol{P}'\otimes\boldsymbol{P}')] = \mathfrak{M}(\boldsymbol{I}),$$

i.e. the matrix $\boldsymbol{H}$ is regular. Since for every square matrix $\boldsymbol{B}$ of order n^2 the following formula holds

$$(\boldsymbol{P}'\otimes\boldsymbol{P}')[(\boldsymbol{P}\otimes\boldsymbol{P})\,\boldsymbol{B}(\boldsymbol{P}'\otimes\boldsymbol{P}')]^{+}(\boldsymbol{P}\otimes\boldsymbol{P}) = [(\boldsymbol{M}\otimes\boldsymbol{M})\,\boldsymbol{B}(\boldsymbol{M}\otimes\boldsymbol{M})]^{+},$$

we obtain

$$\boldsymbol{S}\boldsymbol{H}^{+}\boldsymbol{S} = \boldsymbol{R}(\boldsymbol{P}'\otimes\boldsymbol{P}')[(\boldsymbol{P}\otimes\boldsymbol{P})\,\boldsymbol{G}(\boldsymbol{P}'\otimes\boldsymbol{P}')]^{+}(\boldsymbol{P}\otimes\boldsymbol{P})\,\boldsymbol{R}' = \boldsymbol{R}(\bar{\boldsymbol{M}}\boldsymbol{G}\bar{\boldsymbol{M}})^{+}\,\boldsymbol{R}',$$

and the BAIQUE is

$$\hat{\gamma} = \boldsymbol{f}'(\boldsymbol{R}(\boldsymbol{P}'\otimes\boldsymbol{P}')((\boldsymbol{P}\otimes\boldsymbol{P})\,\boldsymbol{G}(\boldsymbol{P}'\otimes\boldsymbol{P}'))^{-1}(\boldsymbol{P}\otimes\boldsymbol{P})\,\boldsymbol{R}']^{+}\,\boldsymbol{R}(\boldsymbol{P}'\otimes\boldsymbol{P}')\times$$
$$\times[(\boldsymbol{P}\otimes\boldsymbol{P})\,\boldsymbol{G}(\boldsymbol{P}'\otimes\boldsymbol{P}')]^{-1}\,\mathrm{vec}(\boldsymbol{P}\boldsymbol{t}\boldsymbol{t}'\boldsymbol{P}') =$$
$$= \boldsymbol{f}'[\boldsymbol{R}(\bar{\boldsymbol{M}}\boldsymbol{G}\bar{\boldsymbol{M}})^{+}\,\boldsymbol{R}']^{+}\,\boldsymbol{R}(\bar{\boldsymbol{M}}\boldsymbol{G}\bar{\boldsymbol{M}})^{+}\,\mathrm{vec}(\boldsymbol{t}\boldsymbol{t}').$$

The condition (7) has the form

$$\mathfrak{M}[(\boldsymbol{P}\otimes\boldsymbol{P})\,\boldsymbol{R}'] \subset \mathfrak{M}[(\boldsymbol{P}\otimes\boldsymbol{P})\,\boldsymbol{G}(\boldsymbol{P}'\otimes\boldsymbol{P}')].$$

Hence we get (7a) using Lemma 5.
In the same way we get (8a), (9a). The condition (10) has the form

$$\mathfrak{M}[(\boldsymbol{P}\otimes\boldsymbol{P})(\boldsymbol{U}_i\otimes\boldsymbol{U}_j)(\boldsymbol{P}'\otimes\boldsymbol{P}')] \subset \mathfrak{M}[(\boldsymbol{P}\otimes\boldsymbol{P})\,\boldsymbol{G}(\boldsymbol{P}'\otimes\boldsymbol{P}')],$$

which can be rewritten in the form (10a) (see Lemma 4).

In the analysis of variance we meet often the case that the matrix $\boldsymbol{V}_i$ commutes with $\boldsymbol{V}_j$, $i, j = 1, \ldots, p$. This case is solved in the following theorem.

Theorem 2. *Let* $\boldsymbol{V}_i\boldsymbol{V}_j = \boldsymbol{V}_j\boldsymbol{V}_i$ *and*

(13) $$\mathrm{vec}\,\boldsymbol{V}_i \in \mathfrak{M}(\boldsymbol{H}),$$

where $\boldsymbol{H} = \sum_{i=1}^{p} \boldsymbol{W}_i \otimes \boldsymbol{W}_i$

(14) $$\boldsymbol{W}_i = \sum_{j=i}^{p} r_{ij}\boldsymbol{V}_j,$$

$$c_{ij} = \sum_{k=1}^{q} r_{ki} r_{kj}, \quad q = \min(i, j), \quad i,j = 1, \ldots, p.$$

a) *The* BAQUE *for the parametric function* $\gamma = \boldsymbol{f}'\boldsymbol{\theta}$ *in the model* (2) *exists iff*

(15) $$\boldsymbol{f} \in \mathfrak{M}(\boldsymbol{K}),$$

where

(16) $$\boldsymbol{K} = (\operatorname{tr}(\boldsymbol{M}_i \boldsymbol{V}_j)), \quad i,j = 1, \ldots, p,$$

(17) $$\boldsymbol{M}_i = \frac{1}{2} \boldsymbol{V}_i \left\{ \left(\sum_{i=1}^{p} \boldsymbol{W}_i^2 \right)^+ + \boldsymbol{L} \left[\boldsymbol{I} - \sum_{i=1}^{p} \boldsymbol{W}_i^2 \left(\sum_{i=1}^{p} \boldsymbol{W}_i^2 \right)^+ \right] \right\} +$$
$$+ \frac{1}{2} \left\{ \left(\sum_{i=1}^{p} \boldsymbol{W}_i^2 \right)^+ + \left[\boldsymbol{I} - \left(\sum_{i=1}^{p} \boldsymbol{W}_i^2 \right)^+ \sum_{i=1}^{p} \boldsymbol{W}_i^2 \right] \boldsymbol{L} \right\} \boldsymbol{V}_i,$$

$\boldsymbol{L}$ *is an arbitrary symmetric matrix which commutes with matrices* $\boldsymbol{V}_1, \ldots, \boldsymbol{V}_p$.

b) *The* BAQUE *is given by*

(18) $$\hat{\gamma} = \sum_{i=1}^{p} \lambda_i \boldsymbol{y}' \boldsymbol{M}_i \boldsymbol{y},$$

where $\boldsymbol{\lambda}$ *satisfies the unbiasedness condition*

(19) $$\boldsymbol{K\lambda} = \boldsymbol{f}.$$

c) *If the matrix* $\sum_{i=1}^{p} \boldsymbol{W}_i^2$ *is regular, then the BAQUE is uniquely given by* (18), (19), *where*

$$\boldsymbol{M}_i = \boldsymbol{V}_i \left(\sum_{i=1}^{p} \boldsymbol{W}_i^2 \right)^{-1}$$

and is therefore admissible.

d) *The sufficient conditions for the admissibility of the* BAQUE *is*

(20) $$\mathfrak{M}(\boldsymbol{V}_j) \subset \mathfrak{M}\left(\sum_{i=1}^{p} \boldsymbol{W}_i^2 \right), \quad j = 1, \ldots, p.$$

Proof. Using the notation from Remark 2, the equation (3) has the form

(3b) $$\sum_{i=1}^{p} \boldsymbol{W}_i \boldsymbol{A} \boldsymbol{W}_i = \sum_{j=1}^{p} \lambda_j \boldsymbol{V}_j.$$

Since (3b) is identical to

$$\text{(3c)} \qquad \sum_{i=1}^{p} (\boldsymbol{W}_i \otimes \boldsymbol{W}_i) \operatorname{vec} \boldsymbol{A} = \sum_{j=1}^{p} \lambda_j \operatorname{vec} \boldsymbol{V}_j,$$

the equation (3b) is consistent for all $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_p)'$ iff (13) holds.

By [3, § 1c.3, Theorem II], there exists an orthogonal matrix $\boldsymbol{Q}$ and diagonal matrices $\boldsymbol{D}_i$, $i = 1, \ldots, p$ so that

$$\boldsymbol{Q}'\boldsymbol{V}_i\boldsymbol{Q} = \boldsymbol{D}_i, \quad \text{i.e.} \quad \boldsymbol{V}_i = \boldsymbol{Q}\boldsymbol{D}_i\boldsymbol{Q}', \quad i = 1, \ldots, p.$$

Therefore

$$\boldsymbol{Q}'\boldsymbol{W}_i\boldsymbol{Q} = \boldsymbol{\Lambda}_i, \quad \text{i.e.} \quad \boldsymbol{\Lambda}_i = \boldsymbol{Q}\boldsymbol{W}_i\boldsymbol{Q}', \quad i = 1, \ldots, p,$$

where $\boldsymbol{\Lambda}_i = \sum_{j=i}^{p} r_{ij}\boldsymbol{D}_j$, $i = 1, \ldots, p$ are diagonal matrices. Substituting in (3b), we get

$$\sum_{i=1}^{p} \boldsymbol{Q}\boldsymbol{\Lambda}_i\boldsymbol{Q}'\boldsymbol{A}\boldsymbol{Q}\boldsymbol{\Lambda}_i\boldsymbol{Q}' = \sum_{j=1}^{p} \lambda_j\boldsymbol{Q}\boldsymbol{D}_j\boldsymbol{Q}'.$$

Hence

$$\sum_{i=1}^{p} \boldsymbol{\Lambda}_i\boldsymbol{Q}'\boldsymbol{A}\boldsymbol{Q}\boldsymbol{\Lambda}_i = \sum_{j=1}^{p} \lambda_j\boldsymbol{D}_j.$$

Putting $\boldsymbol{Z} = \boldsymbol{Q}'\boldsymbol{A}\boldsymbol{Q}$, we obtain

$$\text{(3d)} \qquad \sum_{i=1}^{p} \boldsymbol{\Lambda}_i\boldsymbol{Z}\boldsymbol{\Lambda}_i = \sum_{j=1}^{p} \lambda_j\boldsymbol{D}_j.$$

This equation has, under the condition (13), the following symmetric solution

$$\boldsymbol{A} = \boldsymbol{Q}\boldsymbol{Z}\boldsymbol{Q}' = \frac{1}{2}\sum_{j=1}^{p} \lambda_j\boldsymbol{V}_j\left\{\boldsymbol{Q}\left(\sum_{i=1}^{p} \boldsymbol{Q}'\boldsymbol{W}_i^2\boldsymbol{Q}\right)^{+}\boldsymbol{Q}' + \right.$$

$$+ \boldsymbol{Q}\boldsymbol{D}\boldsymbol{Q}'\left[\boldsymbol{I} - \sum_{i=1}^{p} \boldsymbol{W}_i^2\boldsymbol{Q}\left(\sum_{i=1}^{p} \boldsymbol{Q}'\boldsymbol{W}_i^2\boldsymbol{Q}\right)^{+}\boldsymbol{Q}'\right]\right\} + \frac{1}{2}\left\{\boldsymbol{Q}\left(\sum_{i=1}^{p} \boldsymbol{Q}'\boldsymbol{W}_i^2\boldsymbol{Q}\right)^{+}\boldsymbol{Q}' + \right.$$

$$\left. + \left[\boldsymbol{I} - \sum_{i=1}^{p} \boldsymbol{W}_i^2\boldsymbol{Q}\left(\sum_{i=1}^{p} \boldsymbol{Q}'\boldsymbol{W}_i^2\boldsymbol{Q}\right)^{+}\boldsymbol{Q}'\right]\boldsymbol{Q}\boldsymbol{D}\boldsymbol{Q}'\right\}\sum_{j=1}^{p} \lambda_j\boldsymbol{V}_j =$$

$$= \frac{1}{2}\sum_{j=1}^{p} \lambda_j\boldsymbol{V}_j\left\{\left(\sum_{i=1}^{p} \boldsymbol{W}_i^2\right)^{+} + \boldsymbol{L}\left[\boldsymbol{I} - \sum_{i=1}^{p} \boldsymbol{W}_i^2\left(\sum_{i=1}^{p} \boldsymbol{W}_i^2\right)^{+}\right]\right\} +$$

$$+ \frac{1}{2}\left\{\left(\sum_{i=1}^{p} \boldsymbol{W}_i^2\right)^{+} + \left[\boldsymbol{I} - \left(\sum_{i=1}^{p} \boldsymbol{W}_i^2\right)^{+}\sum_{i=1}^{p} \boldsymbol{W}_i^2\right]\boldsymbol{L}'\right\}\sum_{j=1}^{p} \lambda_j\boldsymbol{V}_j,$$

where $\boldsymbol{L} = \boldsymbol{QDQ}'$, $\boldsymbol{D}$ is an arbitrary diagonal matrix. $\boldsymbol{DD}_i = \boldsymbol{D}_i\boldsymbol{D}$ holds and therefore $\boldsymbol{Q}'\boldsymbol{LQQ}'\boldsymbol{V}_i\boldsymbol{Q} = \boldsymbol{Q}'\boldsymbol{V}_i\boldsymbol{QQ}'\boldsymbol{LQ}$, i.e. $\boldsymbol{LV}_i = \boldsymbol{V}_i\boldsymbol{L}$, $i = 1, \ldots, p$. Substituting in (4), we get equations for $\lambda_1, \ldots, \lambda_p$

$$\sum_{j=1}^{p} \lambda_j \operatorname{tr} \boldsymbol{M}_j \boldsymbol{V}_k = f_k, \quad k = 1, \ldots, p,$$

where $\boldsymbol{M}_j$ is given by (17). These equations have a solution iff the condition (15) holds. Therefore the BAQUE of the parametric function $\boldsymbol{f}'\boldsymbol{\theta}$ has the form (18), where $\lambda = (\lambda_1, \ldots, \lambda_p)'$ satisfies the conditions (19).

From the form of $\boldsymbol{A}$ and the risk function $\mathrm{R}(\hat{\gamma}, \boldsymbol{\theta}) = \operatorname{tr}(\boldsymbol{AV}(\boldsymbol{\theta})\boldsymbol{AV}(\boldsymbol{\theta}))$, we see that the risk function is invariant with respect to the choice of the matrix $\boldsymbol{L}$, and therefore all the BAQUE are admissible if the condition (20) holds.

Remark 3. If the condition (20) holds, then

$$\boldsymbol{M}_i = \boldsymbol{V}_i\left(\sum_{i=1}^{p} \boldsymbol{W}_i^2\right)^{+}. \tag{21}$$

Proof. Since $\boldsymbol{V}_i\left(\sum_{i=1}^{p} \boldsymbol{W}_i^2\right) = \left(\sum_{i=1}^{p} \boldsymbol{W}_i^2\right)\boldsymbol{V}_i$ we get

$$\left(\sum_{i=1}^{p} \boldsymbol{W}_i^2\right)^{+} \boldsymbol{V}_i\left(\sum_{i=1}^{p} \boldsymbol{W}_i^2\right)\left(\sum_{i=1}^{p} \boldsymbol{W}_i^2\right)^{+} = \left(\sum_{i=1}^{p} \boldsymbol{W}_i^2\right)^{+}\left(\sum_{i=1}^{p} \boldsymbol{W}_i^2\right)\boldsymbol{V}_i\left(\sum_{i=1}^{p} \boldsymbol{W}_i^2\right)^{+}$$

and using the conditions (20), we have $\left(\sum_{i=1}^{p} \boldsymbol{W}_i^2\right)^{+} \boldsymbol{V}_i = \boldsymbol{V}_i\left(\sum_{i=1}^{p} \boldsymbol{W}_i^2\right)^{+}$ from which (21) follows.

Remark 4. If the matrices $\boldsymbol{V}_1, \ldots, \boldsymbol{V}_p$ are positive semidefinite and $r_{1i} \neq 0$, $i = 1, \ldots, p$, then the condition (20) is fulfilled and all the BAQUE's are admissible. These conditions are also sufficient for the solvability of the equation (3d). If for some i ($i = 1, \ldots, p$) $r_{1i} = 0$, then $\mathrm{P}(\theta_i = 0) = 1$ and we can pass to the model with a less number of the variance components.

In the similar way as in Collorary 1, we can rewrite Theorem 2 for the model (1) as follows.

Corollary 2. *Let* $\boldsymbol{MU}_i\boldsymbol{MU}_j\boldsymbol{M} = \boldsymbol{MU}_j\boldsymbol{MU}_i\boldsymbol{M}$ *and* $\bar{\boldsymbol{M}} \operatorname{vec} \boldsymbol{U}_i \in \mathfrak{M}(\bar{\boldsymbol{M}}\boldsymbol{G}\bar{\boldsymbol{M}})$, $\boldsymbol{G} = \sum_{i=1}^{p} \boldsymbol{Z}_i \otimes \boldsymbol{Z}_i$, $\boldsymbol{Z}_i = \sum_{j=i}^{p} r_{ij}\boldsymbol{U}_j$, $c_{ij} = \sum_{k=1}^{q} r_{ki}r_{kj}$, $q = \min(i,j)$, $i,j = 1, \ldots, p$, $\boldsymbol{M} = \boldsymbol{P}'\boldsymbol{P}$, $\bar{\boldsymbol{M}} = \boldsymbol{M} \otimes M$.

a) *The* BAIQUE *of the parametric function* $\gamma = \boldsymbol{f}'\boldsymbol{\theta}$ *in the model* (1) *exists iff*

$$\boldsymbol{f} \in \mathfrak{M}(\boldsymbol{Q}),$$

where

$$\boldsymbol{Q} = (\operatorname{tr}(\boldsymbol{N}_i\boldsymbol{U}_j)), \quad i,j = 1, \ldots, p,$$

$$N_i = \frac{1}{2} MV_iM\left\{\left(\sum_{i=1}^{p} (MZ_iM)^2\right)^+ + Z\left[I - \sum_{i=1}^{p} (MZ_iM)^2\left(\sum_{i=1}^{p} (MZ_iM)^2\right)^+\right]\right\} +$$

$$+ \frac{1}{2}\left\{\left(\sum_{i=1}^{p} (MZ_iM)^2\right)^+ + \left[I - \left(\sum_{i=1}^{p} (MZ_iM)^2\right)^+ \sum_{i=1}^{p} (MZ_iM)^2\right] Z'\right\} MV_iM,$$

where $\mathbf{Z}$ *is an orbitrary symmetric matrix such that the matrix* $\mathbf{MZM}$ *commutes with the matrices* $\mathbf{MU}_1\mathbf{M}, \ldots, \mathbf{MU}_p\mathbf{M}$.

b) *The* BAIQUE *is given by*

$$\hat{\gamma} = \sum_{i=1}^{p} \lambda_i t' N_i t,$$

where λ *satisfies the unbiasedness condition*

$$Q\lambda = f.$$

c) *If the condition* $\mathfrak{M}\left[\sum_{i=1}^{p} (MZ_iM)^2\right] \supset \mathfrak{M}(M)$ *holds then the* BAIQUE *is uniquely given by the previous expressions, where*

$$N_i = MV_iM\left[\sum_{i=1}^{p} (MZ_iM)^2\right]^{-1}.$$

and therefore is admissible.

d) *The sufficient conditions for the admissibility of the* BAIQUE *is*

$$\mathfrak{M}(MU_jM) \subset \mathfrak{M}\left[\sum_{i=1}^{p} (MZ_iM)^2\right], \quad j = 1, \ldots, p.$$

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Katedra matematiky VŠDS
Marxa—Engelsa 25
010 26 Žilina