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SEMI-CLOSED SETS AND THE ASSOCIATED TOPOLOGY

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Introduction

Let $(X, \mathfrak{T})$ be a topological space. Let $A \subset X$ then following N. Levine [4] A is said to be semi-open iff there exists a $\mathfrak{T}$-open set O such that $O \subset A \subset \bar{O}$ where $(\overline{\ })$ denotes the $\mathfrak{T}$-closure. A set $F \subset X$ is said to be semi-closed iff $X - F$ is semi-open. In [2] S. G. Crossley and S. K. Hildebrand have defined the semi-closure $scl\,A$ of a set $A \subset X$ as follows:

$scl\,A \cap \{F: F$ is a semi-closed set containing $A\}$. They have proved that A is semi-closed iff $A = scl\,A$. They have introduced in [2] another set $D_A \subset X$ for each set $A \subset X$ such that $scl\,A(A \cup D_A \cup B) = A \cup D_A \cup scl\,B$ for all subsets $B \subset X$, and that D_A is minimal in the sense that if any set $C \subset X$ satisfies this relation, we have $D_A \subset C$. By defining c: $\mathbf{P}(X) \to \mathbf{P}(X)$ by the rule: $cA = A \cup D_A$ for all $A \in \mathbf{P}(X)$ where $\mathbf{P}(X)$ denotes the family of all subsets of X. It follows that the operator c is a Kuratowski-closure operator in X. The topology induced by the Kuratowski closure operator c on X, is denoted by $\mathfrak{F}(\mathfrak{T})$. In [2] it has been shown that $\mathfrak{F}(\mathfrak{T})$ is finer than $\mathfrak{T}$ on X. The purpose of this paper is to study properties of $\mathfrak{F}(\mathfrak{T})$. In § 1 of this paper, we have established a characterization of open sets in $(X, \mathfrak{F}(\mathfrak{T}))$ and deduced that $(X, \mathfrak{F}(\mathfrak{T}))$ does not associate any more real-valued continuous function than $(X, \mathfrak{T})$ does, where we take the space $\mathcal{R}$ of reals with usual topology $\mathfrak{U}$. If $C_{\mathcal{R}}(\mathfrak{U})$ is the class of continuous functions ever $\mathcal{R}$ (into self). We have sought informations in § 2 for f: $(R, \mathfrak{F}(\mathfrak{U})) \to$ itself to be continuous whenever $f \in C_{\mathcal{R}}(\mathfrak{U})$. We have also studied conditions for continuity of functions over topological spaces whenever the spaces are equipped with finer topologies of the kind as stated above:

§ 1. Theorem 1.1. *For a topological space $(X, \mathfrak{T})$ a subset $G \subset X$ belongs to $\mathfrak{F}(\mathfrak{T})$ iff for each $x \in G$ there is an $\mathfrak{T}$-open neighbourhood N_0 of x such that $(\overline{G^\circ}) \supset N_0$, where $(\ )^\circ$ denotes the $\mathfrak{T}$-interior.*

Proof: The proof is based on the following characterisation of the set D_A for any set $A \subset X$:

$D_A = \{p \in X \colon p \notin A$ and for every neighbourhood N of p, $(\overline{N \cap A}) \neq \Phi\}$

This characterization has been obtained by C. Banerjee in her Ph. D. Thesis [1]. Its analogue in the classical setting of the space $(R, \mathfrak{U})$ can be found in [3]. For necessity part, let $G \in \mathfrak{F}(\mathfrak{T})$. So $X - G$ is $\mathfrak{F}(\mathfrak{T})$-closed. Put $F = X - G$. Clearly $cF = F$ and $D_F = \Phi$. Take a point $x \in G$. Clearly $x \notin F$ and since $D_F = \Phi$, it follows that there exists a $\mathfrak{T}$-opereniegbbourhood N_0 of x, such that $(\overline{N_0 \cap F})^\circ = \Phi$. So for $\mathfrak{T}$-opern subset $V \subset N_0$, there is a point $\xi \in V$ such that $\xi \notin \overline{N_0 \cap F}$. This shows that there exists a $\mathfrak{T}$-openneighbourhood N_ξ of ξ such that

$$N_\xi \cap (N_0 \cap F) = \Phi. \tag{1.1}$$

Put $W = V \cap N_\xi$. Clearly W is a $\mathfrak{T}$-open neighbourhood of ξ and

$$\xi \in W \subset V \subset N\,. \tag{1.2}$$

From (1.1) it follows that $W \cap (N_0 \cap F) = \Phi$. This implies that $W \cap F = \Phi$ i. e. $W \subset G$.

Hence $\xi \in G^\circ$, and from (1.2) we $V \cap G^\circ \neq \Phi$. This shows that G° is everywhere dense in N_0 i. e. $(\overline{G^\circ}) \supset N_0$ For sufficiency part let $G \subset X$ and suppose that condition of Theorem 1.1 holds for G. We show that G is $\mathfrak{F}(\mathfrak{T})$-open by showing $F = X - G$ to be $\mathfrak{F}(\mathfrak{T})$-closed. It suffices to show that $D_F = \Phi$. If possible, let $\xi \in D_F$. Then $\xi \notin F$ and for every neighbourhood N of ξ we have $(\overline{N \cap F})^\circ \neq \Phi$. Clearly $\xi \in G$ and let N_0 be a $\mathfrak{T}$-open neighbourhood of ξ such that G° is $\mathfrak{T}$-every where dense in N_0. Hence by the above argument, we have $(\overline{N_0 \cap F})^\circ \neq \Phi$. Let V be a nonempty $\mathfrak{T}$-open set such that

$$V \subset (\overline{N_0 \cap F}) \tag{1.3}$$

Clearly $V \cap N_0 \neq \Phi$. Putting V' $V \cap N_0$, V' is a non-empty $\mathfrak{T}$-open set such that $V' \cap G^\circ = \Phi$, for otherwise let $\beta \in V' \cap G^\circ$. So there exists a $\mathfrak{T}$-open set W such that $\beta \in W \subset G$. This implies that $W \cap F = \Phi$ i. e. $\beta \notin \bar{F}$, which is a contradiction of (1.3), since $V' \subset V$. Hence $V' \cap G^\circ = \Phi$.

Now this contradicts the fact that G° is $\mathfrak{T}$-everywhere dense in N_0. Thus we have shown $D_F = \Phi$ and the proof is now complete.

Corollary 1.1. *Let $G \in \mathfrak{F}(\mathfrak{T})$ then for each $x \in G$, there is a $\mathfrak{T}$-open neighbourhood N_0 of x such that G is $\mathfrak{T}$-everywhere dense in N_0.*

We now recall the following definition as in [5].

Definition 1.1. *If $(X, \mathfrak{T})$ and $(Y, \mathfrak{T}')$ are topological spaces, then a function $f\colon (X, \mathfrak{T}) \to (Y, \mathfrak{T}')$ is said to be quasi-continuous(q-continuous) at $\xi \in X$ if for every $\mathfrak{T}$-open set U containing ξ and every $\mathfrak{T}'$-open set V containing $f(\xi)$, we have $(f^{-1}(V) \cap U)^\circ \neq \Phi$.*

f is said to be q-continuous whenever f is q-continuous everywhere in X.

Theorem 1.2. *Every real-valued continuous function in* $(X, \mathfrak{F}(\mathfrak{T}))$ *is continuous in* $(X, \mathfrak{T})$.

To prove this theorem we need a lemma.

Lemma 1.1. *Every real-valued continuous function in* $(X, \mathfrak{F}(\mathfrak{T}))$ *is* q*-continuous in* $(X, \mathfrak{T})$.

Proof: Let f be a real-valued continuous function on $(X, \mathfrak{F}(\mathfrak{T}))$. Let $\xi \in X$ and let N_ξ be an $\mathfrak{T}$-open neighbourhood of ξ and let $\varepsilon > 0$. Then we have $N_\xi \cap f^{-1}(f(\xi)-\varepsilon,\ f(\xi)+\varepsilon) = A$ as $\mathfrak{F}(\mathfrak{T})$-open set containing ξ (since N_ξ is also $\mathfrak{F}(\mathfrak{T})$-open). By Theorem 1.1 there exists a $\mathfrak{T}$-open neighbourhood N_0 of ξ such that $A^\circ \cap N_0 \neq \Phi$. i. e., there is an non-empty $\mathfrak{T}$-open set B such that $B \subset A^\circ \cap N_0$.

So $B \subset A$, and consequently $(N_\xi \cap f^{-1}(f(\xi)-\varepsilon,\ f(\xi)+\varepsilon))^\circ \neq \Phi$. This shows that f is q-continuous at ξ. Since ξ is any point in X, f is q-continuous.

Proof of the theorem 1.2. Suppose theorem 1.2 is false. We seek a contradiction. Let there exist a point $\xi \in X$ where f is $\mathfrak{F}(\mathfrak{T})$-continuous but $\mathfrak{T}$-discontinuous. So, there exists a $\delta > 0$ such that for every $\mathfrak{T}$-open set V containing ξ, there exists a point $\eta \in V$ such that $f(\eta) \notin (f(\xi)-\delta,\ f(\xi)+\delta)$. Choose $\delta' > 0$ such that

$$[f(\xi)-\delta',\ f(\xi)+\delta'] \subset (f(\xi)-\delta,\ f(\xi)+\delta) \tag{1.4}$$

It follows that $f^{-1}(f(\xi)-\delta',\ f(\xi)+\delta')$ is an $\mathfrak{F}(\mathfrak{T})$-open set containing ξ. So by Corollary 1.1 there is a $\mathfrak{T}$-open set G containing ξ such that $f^{-1}(f(\xi)-\delta',\ f(\xi)+\delta')$ is $\mathfrak{T}$-every where dense in G. Clearly $f^{-1}(f(\xi)-\delta',\ f(\xi)+\delta')$ is $\mathfrak{T}$-everywhere dense in $G \cap V$, which is a $\mathfrak{T}$-open set containing ξ. So, there is a point $\eta' \in G \cap V$ such that by (1.4) $f(\eta') \notin [f(\xi)-\delta',\ f(\xi)+\delta']$. Choose $\delta'' > 0$ such that $(f(\eta')-\delta'',\ f(\eta')+\delta'') \cap (f(\xi)-\delta',\ f(\xi)+\delta') = \Phi$. Now by Lemma 1.1, f is q-continuous. So f is q-continuous at η'. Therefore, there exists a non-empty $\mathfrak{T}$-open set O such that $O \subset G \cap V$ and $f(O) \subset (f(\eta')-\delta'',\ f(\eta')+\delta'')$. Clearly then $f(O) \cap (f(\xi)-\delta',\ f(\xi)+\delta') = \Phi$. This contradicts the fact that $f^{-1}(f(\xi)-\delta',\ f(\xi)+\delta')$ is $\mathfrak{T}$-everywhere dense in $G \cap V$. This is the desired contradiction, and we have proved the theorem.

Theorem 1.3. *Every* $\mathfrak{T}$*-nowhere dense set* A *in* $(X, \mathfrak{T})$ *is closed in* $(X, \mathfrak{F}(\mathfrak{T}))$.

Proof: Let A be a $\mathfrak{T}$-nowhere dense set in $(X, \mathfrak{T})$. Then $(\bar{A})^\circ = \Phi$. It suffices to show that $D_A = \Phi$. If possible, let $\xi \in D_A$, then for every $\mathfrak{T}$-open neighbourhood N of ξ, we have $(\overline{N \cap A})^\circ \neq \Phi$. i. e. $(\bar{A})^\circ \neq \Phi$, a contradiction. Hence $D_A = \Phi$.

Corollary 1.2. *Every* $\mathfrak{T}$*-1st category set in* $(X, \mathfrak{T})$ *is an* F_σ*-set of 1st category relative to* $\mathfrak{F}(\mathfrak{T})$.

We close this section with the following remark.

Remark 1.1. The closed interval $[0, 1]$ is not compact in $(R, \mathfrak{F}(\mathfrak{U}))$.

Proof. Suppose the contrary. Then every $\mathfrak{F}(\mathfrak{U})$-closed subset in $[0, 1]$ is $\mathfrak{F}(\mathfrak{U})$-compact in $[0, 1]$. Let us consider the subset $B = \left\{1, \frac{1}{2}, \frac{1}{3} \dots\right\}$. By theorem 1.3

it follows that B is $\mathfrak{S}(\mathfrak{U})$-closed in $[0, 1]$. But B is not $\mathfrak{S}(\mathfrak{U})$-compact in $[0, 1]$. In fact let for each i $A_i = \left\{\frac{1}{i}, \frac{1}{i+1}, \ldots\right\}$. Then arguing as above A_i's are $\mathfrak{S}(\mathfrak{U})$-closed subsets of B with finite intersection property, but $\bigcap_{i=1} A_i = \Phi$. Thus B fails to be $\mathfrak{S}(\mathfrak{U})$-compact and hence $[0, 1]$ is not $\mathfrak{S}(\mathfrak{U})$-compact.

§**2. Theorem 2.1.** *Let f: $(R, \mathfrak{U}) \to (R, \mathfrak{U})$ be a continuous function such that f does not remain constant over any non-degenerate subinterval of $\mathscr{R}$ then f: $(R, \mathfrak{S}(\mathfrak{U})) \to (R, \mathfrak{S}(\mathfrak{U}))$ is continuous.*

Proof: If possible, let f be $(\mathfrak{S}(\mathfrak{U}) - \mathfrak{S}(\mathfrak{U}))$ discontinuous at a point $\xi \in R$. Then there exists an $\mathfrak{S}(\mathfrak{U})$-open set G containing $f(\xi)$ such that for every $\mathfrak{S}(\mathfrak{U})$-open set V containing ξ, we have $f(V) \not\subset G$. Since $f(\xi)$ is an $\mathfrak{S}(\mathfrak{U})$-interior point of G, it follows that there exists an $\mathfrak{U}$-open neighbourhood N_0 of $f(\xi)$ such that G° is $\mathfrak{U}$-everywhere dense in N_0. Since f: $(R, \mathfrak{U}) \to (R, \mathfrak{U})$ is continuous, it follows that $f^{-1}(N_0)$ is an $\mathfrak{U}$-open set containing the point ξ. Clearly $(f^{-1}(N_0)$ is an $\mathfrak{U}$-open set containing the point ξ. Clearly $(f^{-1}(G))^\circ$ is not $\mathfrak{U}$-everywhere dense In $f^{-1}(N_0)$, otherwise it follows from Theorem 1.1 that ξ is a $\mathfrak{S}(\mathfrak{U})$-interior point of $f^{-1}(G)$ — a contradiction. Since $(f^{-1}(G))$ is not $\mathfrak{U}$-everywhere dense in $f^{-1}(N_0)$, there is an open interval (non-degenerate) $I \subset f^{-1}(N_0)$ such that $I \cap (f^{-1}(G)) = \Phi$. So $R \mid f^{-1}(G)$ is $\mathfrak{U}$-everywhere dense in I. Since f does not remain constant over any subinterval of R, it follows that there are two points $\alpha, \beta \in I$ such that $\alpha < \beta$ and $f(\alpha) \neq f(\beta)$. Without loss of gererality, we take $f(\alpha) < f(\beta)$. Since $f(\alpha), f(\beta) \in N_0$ and N_0 is open and since G° is $\mathfrak{U}$-everywhere dense in N_0, we find a $v \in G^\circ$ such that $v \in (f(\alpha), f(\beta))$. Since f: $(R, \mathfrak{U}) \to (R, \mathfrak{U})$ is continuous, it satisfies the Dabroux Property. So find $x' \in (\alpha, \beta)$ such that $f(x') = v$. Choose $\varepsilon > 0$ such that $(v - \varepsilon, v + \varepsilon) \subset G$. As f is $(\mathfrak{U} - \mathfrak{U})$ continuous at x', we can find $\delta > 0$ such that $f(x' - \delta, x' + \delta) \subset (v - \varepsilon, v + \varepsilon) \subset G$ without loss of generality, take δ such that $(\xi' - \delta, x' + \delta) \subset (\alpha, \beta)$. Clearly $(x' - \delta, x' + \delta) \cap R \mid f^{-1}(G) = \Phi$. This contradicts the fact that $R \mid f^{-1}(G)$ is $\mathfrak{U}$-dense in I. Hence f: $(R, \mathfrak{S}(\mathfrak{U})) \to (R, \mathfrak{S}(\mathfrak{U}))$ is shown to be continuous.

Theorem 2.2. *Let $(X, \mathfrak{T})$ and $(Y, \mathfrak{T}')$ are topological spaces. If f: $(X, \mathfrak{S}(\mathfrak{T})) \to (Y, \mathfrak{S}(\mathfrak{T}'))$ is continuous and if $(Y, \mathfrak{S}(\mathfrak{T}'))$ is regular then f: $(X, \mathfrak{T}) \to (Y, \mathfrak{S}(\mathfrak{T}'))$ is continuous.*

Proof: If possible, let f: $(X, \mathfrak{T}) \to (Y, \mathfrak{S}(\mathfrak{T}'))$ be not continuous at a point $x_0 \in X$. Then there is a $\mathfrak{S}(\mathfrak{T}')$-neighbourhood G of $f(x_0)$ in Y such that for any $\mathfrak{T}$-open neighbourhood U of x_0 in X, there is a point $\alpha \in U$ such that $f(\alpha) \notin G$. Since $(Y, \mathfrak{S}(\mathfrak{T}'))$ is regular, we can find a $\mathfrak{S}(\mathfrak{T}')$-open set V such that

$$f(x_0) \in V \subset \mathfrak{S}(\mathfrak{T}') - cl V \subset G. \tag{2.1}$$

Clearly, $f^{-1}(V)$ is a $\mathfrak{S}(\mathfrak{T})$-open set containing x_0. So by Theorem 1.1, there is a $\mathfrak{T}$-open neighbourhood O of x_0 such that $(f^{-1}(V))^{\circ}$ is $\mathfrak{T}$-everywhere dense in O. Now there exists a point $\beta \in O$ such that $f(\beta) \notin G$. From (2.1) $f(\beta) \notin \mathfrak{F}(\mathfrak{T}') - clV$. There exists a $\mathfrak{F}(\mathfrak{T}')$-open neighbourhood V' of $f(\beta)$ in Y such that $V' \cap V = \Phi$. As $f\colon (X, \mathfrak{F}(\mathfrak{T})) \to (Y, \mathfrak{F}(\mathfrak{T}'))$ is continuous, it follows that $f^{-1}(V')$ is an $\mathfrak{F}(\mathfrak{T})$-open set containing β. Put

$$W = O \cap f^{-1}(V'). \tag{2.2}$$

Clearly W is a non-empty $\mathfrak{F}(\mathfrak{T})$-open set containing β. So by Theorem 1.1 there is a $\mathfrak{T}$-open neighbourhood N_0 of β such that W° is $\mathfrak{T}$-every where dense in N_0. Put $O' = N_0 \cap W^{\circ}$. Clearly O' is $\mathfrak{T}$-open set $\subset O$. Clearly $O' \cap f^{-1}(V) = \Phi$. Otherwise $V' \cap V \neq \Phi$. Thus $O' \cap (f^{-1}(V))^{\circ} = \Phi$. This is impossible since $(f^{-1}(V))^{\circ}$ is $\mathfrak{T}$-everywhere dense in O. This is the desired contradiction and we have proved the theorem.

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ПОЛУЗАМКНУТЫЕ МНОЖЕСТВА И СООТВЕТСТВУЮЩАЯ ТОПОЛОГИЯ

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Резюме

В топологическом пространстве класс полузамкнутых множеств определяет более тонкую топологию. В этой работе изучаются квази-непрерывные функции из точки зрения этой топологии. Оказывается, что более тонкая топология этого пространства никак не связана с действительными непрерывными функциями над этим пространством.