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ON WEAK ISOMETRIES IN MULTILATTICE GROUPS

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ABSTRACT. Let f be a weak isometry in a distributive multilattice group G. In this paper it is proved that f is a bijection and $f(U(L(x, y)) \cap L(U(x, y))) = U(L(f(x), f(y))) \cap L(U(f(x), f(y)))$ for each $x, y \in G$. This gives the positive answer to a question proposed in a recent paper by J. Jakubík concerning weak isometries in lattice ordered groups.

In [2] J. Jakubík proved that each weak isometry in a representable lattice ordered group is a bijection and put the question whether each weak isometry f in a lattice ordered group G satisfies the condition

$$f([x \wedge y, x \vee y]) = [f(x) \wedge f(y), f(x) \vee f(y)] \quad \text{for each} \quad x, y \in G.$$

In this paper it is proved that each weak isometry in a distributive multilattice group is a bijection. This generalizes the above mentioned result of J. Jakubík on lattice ordered groups.

Further, it is shown that for each weak isometry f in a distributive multilattice group G the relation

$$f(U(L(x, y)) \cap L(U(x, y))) = U(L(f(x), f(y))) \cap L(U(f(x), f(y)))$$

is valid for each $x, y \in G$.

From this follows that the answer to the question of J. Jakubík is positive.

First we recall some notions and notations used in the paper.

Let C be a partially ordered group (po-group). The group operation will be written additively. We denote $C^+ = \{x \in C, x \geq 0\}$. If $A \subseteq C$, then we denote by $U(A)$ and $L(A)$ the set of all upper bounds and the set of all lower bounds of the set A in C, respectively. If $A = \{a_1, \ldots, a_n\}$, we shall write $U(a_1, \ldots, a_n)$ for $U(A)$ and $L(a_1, \ldots, a_n)$ for $L(A)$. For each $a \in C$, $|a| = U(a, -a)$. If a and b are elements of C, then we denote by $a \vee_m b$ the set of all minimal elements of the set $U(a, b)$ and analogously $a \wedge_m b$ is defined to be the set of all maximal elements of the set $L(a, b)$. If for $a, b \in C$ there exists the least upper bound

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(greatest lower bound) of the set $\{a, b\}$ in C, then it will be denoted by $a \vee b$ ($a \wedge b$).

A mapping $f\colon C \to C$ is called a weak isometry in C if $|x-y| = |f(x) - f(y)|$ for each $x, y \in C$. A weak isometry f is called a weak 0-isometry if $f(0) = 0$.

The partially ordered set P is said to be a multilattice (Benado [1]) if it fulfils the following conditions for each pair $a, b \in P$:

(m_1) If $x \in U(a, b)$, then there is $x_1 \in a \vee {}_m b$ such that $x_1 \leq x$.

(m_2) If $y \in L(a, b)$, then there is $y_1 \in a \wedge {}_m b$ such that $y_1 \geq y$.

A multilattice P is called distributive if, whenever a, b, c are elements P such that

$$(a \vee {}_m b) \cap (a \vee {}_m c) \neq \emptyset \quad \text{and} \quad (a \wedge {}_m b) \cap (a \wedge {}_m c) \neq \emptyset,$$

then $b = c$.

Let G be a partially ordered group such that

(i) G is directed,

(ii) the partially ordered set $(G, \leq)$ is a multilattice. Then G is called a multilattice group.

A quadruple (a, b, u, v) of elements of a multilattice group G is said to be regular if $u \in a \wedge {}_m b$, $v \in a \vee {}_m b$ and $v - a = b - u$.

1. Theorem. *Let G be a distributive multilattice group, f a weak 0-isometry in G. Let $x \in G^+$. Then there exist $x_1, x_2 \in G^+$ such that $x = x_1 + x_2, f(x) = x_1 - x_2$, $x_1 + x_2 = x_2 + x_1, f(x_1) = x_1, f(x_2) = -x_2$. Moreover, $x_1 \vee x_2 = x, x_1 \wedge x_2 = 0$, $x_1 = f(x) \vee 0$, $x_2 = -f(x) \vee 0$.*

Proof. Since $x \geq 0$, from the relation $U(x) = |x| = |f(x)| = U(f(x), -f(x))$ we get $x = -f(x) \vee f(x)$. By 1 (i) [4], $(-f(x), f(x), -f(x) - x + f(x), x)$ is a regular quadruple in G. Clearly $-f(x) - x + f(x) \leq 0$. Let $y_2 \in -f(x) \wedge {}_m 0$, $y_2 \geq -f(x) - x + f(x)$. According to Theorem 5 [4], there exist elements $y_1 \in [-f(x) - x + f(x), f(x)]$, $x_1 \in [f(x), x]$, $x_2 \in [-f(x), x]$ such that $(-f(x), 0, y_2, x_2)$, $(0, f(x), y_1, x_1)$, $(y_2, y_1, -f(x) - x + f(x), 0)$, $(x_2, x_1, 0, x)$ are regular quadruples in G. clearly $x_1 \vee x_2 = x$, $x_1 \wedge x_2 = 0$. Thus $x = x_1 + x_2$ where $x_1 \in U(0, f(x))$, $x_2 \in U(0, -f(x))$. Let $z \in U(0, f(x))$, $t \in U(0, -f(x))$. Then $z + x_2$, $x_1 + t \in U(f(x), -f(x)) = |f(x)| = |x| = U(x_1 + x_2)$. From this we have $z \geq x_1$, $t \geq x_2$. Therefore $x_1 = f(x) \vee 0$, $x_2 = -f(x) \vee 0$. Then it is easy to verify that $x_2 = x_1 - f(x) = -f(x) + x_1$. From this we obtain $f(x) = x_1 - x_2$, $x_1 + x_2 = x_2 + x_1$.

Since $x_1 \geq 0$, from the relation $|x_1| = |f(x_1)|$ we get $f(x_1) \leq x_1, f(x_1) \geq -x_1$. Then we have $f(x_1) + x_2 \geq -x_1 + x_2 = x_2 - x_1 = -f(x)$. Because of $x_2 \geq 0$, the relation $|x_2| = |x_1 + x_2 - x_1| = |x - x_1| = |f(x) - f(x_1)| = |x_1 - x_2 - f(x_1)|$ implies that $x_2 \geq x_1 - x_2 - f(x_1)$. Thus $f(x_1) + x_2 \geq x_1 - x_2 = f(x)$. Therefore $f(x_1) + x_2 \in U(-f(x), f(x))$. Then $f(x_1) + x_2 \geq x$. From this we have $f(x_1) \geq x_1$. Thus $f(x_1) = x_1$.

Since $x_2 \geq 0$, from the relation $|x_2| = |f(x_2)|$ we obtain $x_2 \geq f(x_2)$, $x_2 \geq -f(x_2)$. Hence $-f(x_2) + x_1 \geq -x_2 + x_1 = x_1 - x_2 = f(x)$. Further, from the relation $|x_1| = |x - x_2| = |f(x) - f(x_2)|$ we get $x_1 \geq f(x_2) - f(x)$. Then we have $-f(x_2) + x_1 \geq -f(x)$. Because of $x = -f(x) \vee f(x)$, we infer that $-f(x_2) + x_1 \geq x$. Therefore $f(x_2) = -x_2$.

Let g be a weak 0-isometry in a po-group H, $A_1 = \{x \in H^+, g(x) = x\}$, $B_1 = \{x \in H^+, g(x) = -x\}$, $A = A_1 - A_1$, $B = B_1 - B_1$. In [3] it was proved that A is a group [Lemma 1.8], B is an abelian group [Lemma 1.9] and $f(a + b) = a - b$ for each $a \in A$, $b \in B$ [Theorem 1.13].

Under these denotations, we now establish the following two theorems.

2. Theorem. *Let $a \in A_1$, $b \in B_1$. Then $a + b = b + a = a \vee b$.*

Proof. Let $x = a + b$, where $a \in A_1$, $b \in B_1$. By 1.13 [3], $g(a + b) = a - b$. Thus $a \in U(0, g(x))$, $b \in U(0, -g(x))$. Let $z \in U(0, g(x))$, $t \in U(0, -g(x))$. Hence $z + b$, $a + t \in U(g(x), -g(x)) = |g(x)| = |x| = U(a + b)$. From this we have $z \geq a$, $t \geq b$. Therefore $a = g(x) \vee 0 = (a - b) \vee 0$, $b = -g(x) \vee 0 = (b - a) \vee 0$. Then it is easy to verify that $a + b = b + a = a \vee b$.

3. Theorem. *Let $a \in A$, $b \in B$. Then $a + b = b + a$.*

Proof. It is a consequence of 2.

4. Theorem. *Each weak isometry in a distributive multilattice group is a bijection.*

Proof. Since each weak isometry in a po-group is an injection [3, Lemma 1.2] and a superposition of a weak 0-isometry and a right translation [3, Lemma 1.1], it suffices to prove that a weak 0-isometry f in a distributive multilattice group G is a surjection. Let $x \in G$. Since G is a directed group, $x = y - z$ where $y, z \in G^+$. By Theorem 1, there exist $y_1, y_2, z_1, z_2 \in G^+$ such that $f(y_1) = y_1$, $f(y_2) = -y_2$, $f(z_1) = z_1$, $f(z_2) = -z_2$, $y = y_1 + y_2$, $z = z_1 + z_2$. From 3, 1.9 and 1.13 [3] it follows that $f(y_1 - z_1 + (-y_2 + z_2)) = y_1 - z_1 - z_2 + y_2 = y_1 - z_1 + y_2 - z_2 = y_1 + y_2 - z_1 - z_2 = y - z = x$. This ends the proof.

Theorem 4 generalizes the proposition (A) of J. Jakubík [2] on lattice ordered groups.

5. Theorem. *Let f be a weak isometry in a distributive multilattice group G. Then $f(U(L(x, y)) \cap L(U(x, y))) = U(L(f(x), f(y))) \cap L(U(f(x), f(y)))$ for each $x, y \in G$.*

Proof. In [4] the desired relation was proved for any weak isometry in a distributive multilattice group which is a bijection (Theorem 18). Hence the theorem is a consequence of 4 and Theorem 18 [4].

Theorem 5 gives the positive answer to a question proposed by J. Jakubík in [2].

REFERENCES

[1] BENADO, M.: Sur la théorie de la divisibilité. Acad. R. P. Romine. Bul. Sti. Sect. Mat. Fyz. 6, 1954, 263—270.

[2] JAKUBÍK, J.: Weak isometries of lattice ordered groups. Math. Slovaca 38, 1988, 133—138.

[3] JASEM, M.: Weak isometries and isometries in partially ordered groups. Algebra Universalis (submitted).

[4] JASEM, M.: Isometries in non-abelian multilattice groups. Czech. Math. J. (submitted).

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