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DIGRAPHS MAXIMAL WITH RESPECT TO CONNECTIVITY

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The digraphs considered in this paper are finite without loops and multiple arcs.

The strong (unilateral, weak) connectivity $\varkappa^3=\varkappa^3(G)$ ($\varkappa^2=\varkappa^2(G)$, $\varkappa^1=\varkappa^1(G)$) of a digraph G is the minimum number of points whose removal results in a not strong (unilateral, weak) or trivial digraph.

We shall describe constructively and determine the number of digraphs maximal with respect to the strong or unilateral or weak connectivity, respectively.

Maximal graphs with a given vertex (edge) connectivity have been studied in [1].

Let digraphs G_1 and G_2 have disjoint sets V_1 and V_2 of points and disjoint arc sets E_1 and E_2, respectively.

Their union is the digraph $G=G_1\cup G_2$, which has the point set $V=V_1\cup V_2$ and the arc set $E=E_1\cup E_2$.

Their join G_1+G_2 consists of $G_1\cup G_2$ and all arcs joining V_1 with V_2.

Their directional join $G_1\oplus G_2$ consists of $G_1\cup G_2$ and all arcs going from V_1 to V_2.

It is clear that the directional join is not a commutative operation.

Definition 1. *Let G be a not complete digraph and n a nonnegative integer. Then G is called $\varkappa_n^i$-maximal if $\varkappa^i(G)=n$ and $\varkappa^i(G+x) > \varkappa^i(G)$ holds for every* arc $x\in E(\bar{G})$ *(for $i=1, 2, 3$).*

All notions not defined here will be used in the sense of [2]. The symbol K_n denotes here the complete diagraph on n points.

Theorem 1. *Let G be a diagraph and n be a natural number. Then G is $\varkappa_n^i$-maximal if and only if $G\simeq K_n+D$, where D is a $\varkappa_0^i$-maximal digraph (for $i=1$,* 2, 3).

Proof. Let $G\simeq K_n+D$ and be $\varkappa_0^i$-maximal. Let us denote $V(G)=A\cup B$, where $A=V(k_n)$, $B=V(D)$.

Since $\varkappa^i(G-A)=\varkappa^i(D)=0$, we have $\varkappa^i(G)\leqslant n$. If $C\subset V(G)$, $|C|\leqslant n$, $C\neq A$, then by $G-C=K_m+D'$ it follows that the digraph $G-C$ is strong. Thus $\varkappa^i(G)=n$.

Let $x \in E(\bar{G})$. Then $\varkappa^i((G+x)-A)=\varkappa^i(D+x)>0$, because the digraph D is $\varkappa_0^i$-maximal. Now $\varkappa^i((G+x)-C) \geqslant \varkappa^i(G-C)>0$ implies $\varkappa^i(G+x)>n$, i.e. the digraph G is $\varkappa_n^i$-maximal.

Let G be $\varkappa_n^i$-maximal. Then there exists a set A of points of G such that $|A|=n$ and $\varkappa^i(G-A)=0$. We denote by D the digraph $G-A$. From the $\varkappa_n^i$-maximality of G it follows that $G \simeq K_n+D$. To finish our proof we must show that the digraph D is $\varkappa_0^i$-maximal. We shall prove it indirectly.

Let $x \in E(\bar{G})$ and $\varkappa^i(D+x)=0$. Then $\varkappa^i(G+x)=n$ and this is a conradiction because G is $\varkappa_n^i$-maximal. Q.E.D.

Theorem 2. *Let G be a digraph. Then G is*

a) *$\varkappa_0^1$-maximal if and only if $G \simeq K_a \cup K_b$,*

b) *$\varkappa_0^2$-maximal if and only if either $G \simeq K_a \cup K_b$, or $G \simeq K_c \oplus (K_a \cup K_b)$ or $G \simeq (K_a \cup K_b) \oplus K_c$ or $G \simeq K_d \oplus ((K_a \cup K_b) \oplus K_c)$,*

c) *$\varkappa_0^3$-maximal if and only if $G \simeq K_a \oplus K_b$.*

Proof. One can easily verify that the sufficient condition in all three assertions holds.

Let G be $\varkappa_0^i$-maximal. If S is a strong component of the diagraph G, then the $\varkappa_0^i$-maximality of G implies $S \simeq K_a$.

Let $S_1, S_2, \ldots, S_n$ be the strong components of G.

a) If $i=1$, then the digraph G is disconnected and from the $\varkappa_0^1$-maximality of G it follows that G consists of exactly two strong components. Thus $G \simeq K_a \cup K_b$.

b) If $i=2$, then the digraph G is not unilateral, hence there exist two points $u \in S_r$, $v \in S_t$ such that u cannot be reached from v and v cannot be reached from u.

Let us put

$$\mathcal{M}=\{S_j, j \neq r, j \neq t, S_j \text{ can be reached from } S_t \text{ or from } S_r\}$$

$$\mathcal{N}=\{S_j, j \neq r, j \neq t, S_j \notin \mathcal{M}\}.$$

The $\varkappa_0^2$-maximality of G implies that $\mathcal{M}$ (and analogously $\mathcal{N}$) is empty or it contains exactly one strong component.

We have to consider only four cases.

1. Let $\mathcal{M}=\emptyset$, $\mathcal{N}=\emptyset$. Then $G \simeq K_a \cup K_b$.
2. Let $\mathcal{M}=\emptyset$, $\mathcal{N} \neq \emptyset$. Then $G \simeq K_c \oplus (K_a \cup K_b)$.
3. Let $\mathcal{M} \neq \emptyset$, $\mathcal{N}=\emptyset$. Then $G \simeq (K_a \cup K_b) \oplus K_c$.
4. Let $\mathcal{M} \neq \emptyset$, $\mathcal{N} \neq \emptyset$. Then $G \simeq K_d \oplus ((K_a \cup K_b) \oplus K_c)$.

We can prove 1—4 by the $\varkappa_0^2$-maximality of G.

c) If $i=3$, then the diagraph G is unilateral. The strong components of a unilateral diagraph G can be denoted in such a way that S_c can be reached from S_d if and only if $c \geqslant d$ (see [2], p. 200). Let $n \geqq 3$, then the digraph $G+x$ (where

$x \in E(\bar{G})$, $x = uv$, $v \notin S_1$) is not strong. This is a contradiction, because G is $\varkappa_0^3$- maximal. Hence $n = 2$ and from the $\varkappa_0^3$-maximality of G it follows that $G \simeq K_a \oplus K_b$. Q.E.D.

By using Theorem 1 and 2 we prove the following inequalities.

Theorem 3. *Let G be a digraph with p points and q* arcs. *Let* $\varkappa^i(G) = n$. *Then we have:*

a) $q(G) \leq (p-1)(p-2) + 2n$ (*for* $i = 1, 2$),

b) $q(G) \leq (p-1)^2 + n$ (*for* $i = 3$).

Proof. Let $\varkappa^i(G) = n$. Then there exists an $\varkappa_n^i$-maximal diagraph H such that G is a factor of H.

By Theorem 1 we have $H \simeq K_n + D$, where D is $\varkappa_0^i$-maximal. Then

$$q(G) \leq q(H) = n(n-1) + 2n(p-n) + q(D). \tag{1}$$

a) Let D be $\varkappa_0^1$-maximal. According to Theorem 2 $D \simeq K_a \cup K_b$. Then $q(D) = a(a-1) + b(b-1)$, where $a + b = p - n$.

The maximum of the function $q(D)$ (for $1 \leq a \leq p - n - 1$) is reached in the digraph $K_1 \cup K_{p-n-1}$. We have

$$q(D) \leq (p-n-1)(p-n-2). \tag{2}$$

If $\varkappa^1(G) = n$, then from (1) and (2) it follows that

$$q(G) \leq (p-1)(p-2) + 2n.$$

b) Let D be $\varkappa_0^2$-maximal. According to Theorem 2 either $D \simeq K_a \cup K_b$ or $D \simeq (K_a \cup K_b) \oplus K_c$ or $D \simeq K_a \oplus (K_b \cup K_c)$ or $D \simeq K_a \oplus ((K_b \cup K_c) \oplus K_d)$.

Without loss of generality we can suppose that $b \leq c$. One can easily verify that

$$\begin{aligned} q(K_a \oplus ((K_b \cup K_c) \oplus K_d)) &\leq q((K_b \cup K_c) \oplus K_{a+d}) = \\ &= q(K_{a+d} \oplus (K_b \cup K_c)) \leq q(K_b \cup K_{a+c+d}). \end{aligned} \tag{3}$$

If $\varkappa^2(G) = n$, from (1), (2), (3) it follows that

$$q(G) \leq (p-1)(p-2) + 2n.$$

c) Let D be $\varkappa_0^3$-maximal. According to Theorem 2 $D \simeq K_a \oplus K_b$. Then $q(D) = a(a-1) + b(b-1) + ab$, where $a + b = p - n$. The maximum of the function $q(D)$ (for $1 \leq a \leq p - n - 1$) is reached in the digraph $K_1 \oplus K_{p-n-1}$. We have

$$q(D) \leq (p-n-1)^2. \tag{4}$$

If $\varkappa^3(G) = n$, then (1) and (4) imply

$$q(G) \leq (p-1)^2 + n \qquad \text{Q.E.D.}$$

By using Theorem 1 and 2 we determine the number of maximal digraphs with respect to the connectivity.

Theorem 4. *The number of nonisomorphic $\varkappa_n^i$-maximal digraphs with p points is*

a) $\left[\frac{m}{2}\right]$ *if* $i=1, m\geqq 2$,

b) $\frac{1}{24}\left(2m^3+3m^2-5m+6\left[\frac{m}{2}\right]\right)$, *if* $i=2, m\geqq 2$,

c) $m-1$ *if* $i=3, m\geqq 2$,

where $m=p-n$.

Proof. One can easily verify that parts a) and c) hold. Let $|V(G)|=p$. There are

$\left[\frac{m}{2}\right]$ nonisomorphic $\varkappa_n^2$-maximal digraphs of the form $K_a\cup K_b$,

$\sum_{i=2}^{m-1}\left[\frac{i}{2}\right]=\frac{1}{4}\left(m^2-m-2\left[\frac{m}{2}\right]\right)$ nonisomorphic $\varkappa_n^2$-maximal digraphs of the form $K_a\oplus(K_b\cup K_c)$ and of the form $(K_a\cup K_b)\oplus K_c$, too, and $\sum_{j=3}^{m-1}\frac{1}{4}\left(j^2-j-2\left[\frac{j}{2}\right]\right)$ $=\frac{1}{24}\left(2m^3-9m^2+7m+6\left[\frac{m}{2}\right]\right)$ nonisomorphic $\varkappa_n^2$-maximal digraphs of the form $K_a\oplus((K_b\cup K_c)\oplus K_d)$, where $m=p-n$. Thus there are $\frac{1}{24}\left(2m^3+3m^2-5m+6\left[\frac{m}{2}\right]\right)$ nonisomorphic $\varkappa_n^2$-maximal digraphs with $p=m+n$ points. Q.E.D.

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ОРГРАФЫ МАКСИМАЛЬНЫЕ ОТНОСИТЕЛЬНО СВЯЗНОСТИ

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Резюме

Сильной (односторонней, слабой) связностью $\varkappa^3=\varkappa^3(K)$ $\varkappa^2=\varkappa^2(K)$, $\varkappa^1=\varkappa^1(K)$ орграфа K называется наименьшее число вершин, удаление которых приводит к не сильному (одностороннему, слабому), или же тривиальному орграфу.

Конструктивно описано и определено число орграфов максималшных относительно сильнои или односторонней или слабой связности.