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CONVEX AUTOMORPHISMS OF A LATTICE

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(Communicated by Tibor Katriňák)

ABSTRACT. Let $L$ be a lattice which can be decomposed into a direct product of finitely many directly indecomposable factors. There are described bijections $f$ of $L$ onto itself preserving convex sublattices in the sense that $A$ is a convex sublattice of $L$ if and only if so is $f(A)$.

V. I. Marzoevev investigated the lattice $C(L)$ of convex sublattices of a lattice $L$. Let $L = (L; \wedge, \vee)$ and $L' = (L'; \wedge, \vee)$ be lattices and $f : L \to L'$ a bijection. In [2] there is proved that $f$ has the property

$$A \in C(L) \iff f(A) = \{f(a) : a \in A\} \in C(L')$$

if and only if for any $a, b \in L$

$$f([a \wedge b, a \vee b]) = [f(a) \wedge f(b), f(a) \vee f(b)]$$

(*)

holds.

Marzoevev calls bijections having this property convex isomorphisms. The aim of Marzoevev’s paper [3] was to describe convex automorphisms of a lattice $L$ (i.e. convex isomorphisms of $L$ onto $L$) provided that

(a) any bounded chain in $L$ is finite, and

(b) $L$ can be decomposed into a direct product $L = L_1 \times \cdots \times L_n$ such that all $L_i$ are directly indecomposable.

However, the author uses non-explained (non-standard) terms in the formulation of results and there are only short sketches of proofs there. Hence it is not possible to find out what the author’s description says.

In the present paper we give a description of convex automorphisms of a lattice $L$ supposing only $(b)$. This is done in Theorem 10.

To prove the main theorem the following two lemmas are useful.

AMS Subject Classification (1991): Primary 06B99.

Key words: Lattice, Convex sublattice, Convex automorphism, Direct product decomposition.
1. **Lemma.** Let a lattice $L = (L; \wedge, \vee)$ be a direct product of lattices $L_1, \ldots, L_n$, $L = L_1 \times \cdots \times L_n$. A subset $A$ of $L$ forms a convex sublattice of $L$ if and only if $A$ is a direct product $A = A_1 \times \cdots \times A_n$ of convex sublattices $A_i$ ($i \in \{1, \ldots, n\}$) of $L_i$.

**Proof.** It is easy to see that if $A_i$ is a convex sublattice of $L_i$ ($i \in \{1, \ldots, n\}$), then $A_1 \times \cdots \times A_n$ is a convex sublattice of $L_1 \times \cdots \times L_n$.

To prove the converse let $A$ be a convex sublattice of $L_1 \times \cdots \times L_n$. Denote $A_i = \{x \in L_i :$ there exists $(y_1, \ldots, y_n) \in A$ with $y_i = x\}$. Evidently $A_i$ is a sublattice of $L_i$. Obviously $A \subseteq A_1 \times \cdots \times A_n$. To prove the opposite inclusion let $b = (b_1, \ldots, b_n) \in A_1 \times \cdots \times A_n$. For any $i \in \{1, \ldots, n\}$ there exists $i^{th}$ coordinate $i_c = (i_{c1}, \ldots, i_{cn}) \in A$ with $i_{ci} = b_i$. Set $u = ^{1c} \wedge \cdots \wedge ^{nc}$, $v = ^{1c} \vee \cdots \vee ^{nc}$. Evidently $u, v \in A$ and $u \leq b \leq v$, hence $b \in A$. It remains to show that every $A_i$ is a convex subset of $L_i$. Let $x, y \in A_i$, $z \in L_i$, $x \leq z \leq y$. Then there exist $\bar{x}, \bar{y} \in A$, $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_n)$, $\bar{y} = (\bar{y}_1, \ldots, \bar{y}_n)$ with $\bar{x}_i = x$, $\bar{y}_i = y$. Consider $\bar{z} = (\bar{x}_1 \wedge \bar{y}_1, \ldots, \bar{x}_{i-1} \wedge \bar{y}_{i-1}, z, \bar{x}_{i+1} \wedge \bar{y}_{i+1}, \ldots, \bar{x}_n \wedge \bar{y}_n)$. Evidently $\bar{x} \wedge \bar{y} \leq \bar{z} \leq \bar{y}$, hence $\bar{z} \in A$ and $z \in A_i$.

2. **Lemma.** (cf. [1]) Let $L = (L; \wedge, \vee)$ and $L' = (L'; \wedge, \vee)$ be lattices and $f : L \to L'$ a bijection. The following two conditions are equivalent.

(i) $A \in C(L) \iff f(A) \in C(L')$.

(ii) There exist lattices $C = (C; \wedge, \vee)$, $D = (D; \wedge, \vee)$ and bijections $g : L \to C \times D$, $h : L' \to C \times D$ such that $g$ is an isomorphism of $L$ onto $C \times D$, $h$ is an isomorphism of $L'$ onto $C \times D^d$ and $g = f \circ h$ ($D^d$ is the dual of $D$).

3. **Theorem.** Let $L$ be a lattice which can be decomposed into a direct product of lattices $L_i$, $L = \prod(L_i \mid i \in I)$ ($I$ is any nonempty set). Further let $\pi$ be a bijection of $I$ onto $I$, $f_i$ for each $i \in I$ an isomorphism or a dual isomorphism of $L_i$ onto $L_{\pi(i)}$. Then the mapping $f : L \to L$ defined by

$$f(x)_{\pi(i)} = f_i(x_i) \quad \text{for all} \quad x \in L, \quad i \in I$$

is a convex automorphism of $L$. (For $y \in L$ and $j \in I$ by $y_j$ the jth component of $y$ is meant.)
Proof. Evidently $f$ is a bijection. By Marmazeev’s theorem it is sufficient to prove that for any $a, b \in L$ the condition (*) holds.

Let $a, b \in L$, $f(a) = u$, $f(b) = v$. By the definition of $f$ there is $u_{\pi(i)} = f_i(a_i)$, $v_{\pi(i)} = f_i(b_i)$ for each $i \in I$.

a) To prove the inclusion $f([a \land b, a \lor b]) \subseteq [f(a) \land f(b), f(a) \lor f(b)]$ let $s \in f([a \land b, a \lor b])$. Then there exists $c \in [a \land b, a \lor b]$ with $s = f(c)$. As $a \land b \leq c \leq a \lor b$, we have $a_i \land b_i \leq c_i \leq a_i \lor b_i$ for each $i \in I$, from which there follows $f_i(a_i) \land f_i(b_i) \leq f_i(c_i) \leq f_i(a_i) \lor f_i(b_i)$ for each $i \in I$ (it does not matter, if $f_i$ is an isomorphism or a dual isomorphism). Hence $u_{\pi(i)} \land v_{\pi(i)} \leq s_{\pi(i)} \leq u_{\pi(i)} \lor v_{\pi(i)}$ for each $i \in I$, which means $s \in [f(a) \land f(b), f(a) \lor f(b)]$.

b) Now we are going to prove the inclusion $[f(a) \land f(b), f(a) \lor f(b)] \subseteq f([a \land b, a \lor b])$. Let $s \in [f(a) \land f(b), f(a) \lor f(b)]$. Then $f(a)_{\pi(i)} \land f(b)_{\pi(i)} \leq s_{\pi(i)} \leq f(a)_{\pi(i)} \lor f(b)_{\pi(i)}$ for all $i \in I$.

Since $f$ is a bijection, there exists $c \in L$ such that $s = f(c)$ and the last inequalities can be rewritten as $f_i(a_i) \land f_i(b_i) \leq f_i(c_i) \leq f_i(a_i) \lor f_i(b_i)$. Applying $f_i^{-1}$, which is also an isomorphism or a dual isomorphism, we get $a_i \land b_i \leq c_i \leq a_i \lor b_i$ for each $i \in I$. Consequently $c \in [a \land b, a \lor b]$ and $s = f(c) \in f([a \land b, a \lor b])$.

In the sections 4–8 we will suppose that $f$ is a convex automorphism of a lattice $L = L_1 \times \cdots \times L_n$, where $L_i$ are directly indecomposable lattices. Without loss of generality we can also assume that every $L_i$ has more than one element. The denotation $\overline{n}$ will be used for the set $\{1, \ldots, n\}$.

4. Lemma. Let $a = (a_1, \ldots, a_n)$ be any fixed element of $L$, $f(a) = u = (u_1, \ldots, u_n)$. For every $i \in \overline{n}$ there exists $k \in \overline{n}$ such that

$$f(\{a_1\} \times \cdots \times L_i \times \cdots \times \{a_n\}) = \{u_1\} \times \cdots \times U_k \times \cdots \times \{u_n\}$$

for a convex sublattice $U_k$ or $L_k$. Moreover the mapping $f_i : L_i \to U_k$ derived from

$$f : \{a_1\} \times \cdots \times L_i \times \cdots \times \{a_n\} \to \{u_1\} \times \cdots \times U_k \times \cdots \times \{u_n\}$$

in a natural way is an isomorphism or a dual isomorphism.
Proof. Let $L'_i = \{a_1^i \times \cdots \times L_i \times \cdots \times \{a_n\}\}$. As $L'_i$ is a convex sublattice of $L$, $f(L'_i)$ is a convex sublattice of $L$, too. By 1, $f(L'_i) = U_1 \times \cdots \times U_n$, where $U_j$ is a convex sublattice of $L_j$ for each $j \in \bar{n}$. Evidently the restriction of $f$ to $L'_i$, $f'_i = f \upharpoonright L'_i$, is a convex isomorphism of $L'_i$ onto $f(L'_i)$, hence there exist lattices $C$, $D$ and isomorphisms $g: L'_i \rightarrow C \times D$, $h: f(L'_i) \rightarrow C \times D^d$ with $g = f'_i \circ h$, by 2. Since $L_i$ is directly indecomposable, so is $L'_i$ and consequently either $C$ or $D$ is a one-element set.

a) Suppose $C$ has only one element. Let $\overline{g}$ and $\overline{h}$ be mappings $L'_i \rightarrow D$ and $f(L'_i) \rightarrow D^d$, respectively, assigned to $g$ and $h$. Evidently $\overline{g}$, $\overline{h}$ are isomorphisms and $\overline{g} = f'_i \circ \overline{h}$. Since $\overline{g}$ is also an isomorphism of $L'_i^d$ onto $D^d$ and $\overline{h}^{-1}$ is an isomorphism of $D^d$ onto $f(L'_i)$, their composition $\overline{g} \circ \overline{h}^{-1} = f'_i$ is an isomorphism of $L'_i^d$ onto $f(L'_i)$ and a dual isomorphism of $L'_i$ onto $f(L'_i)$. Using the fact that $L'_i$ is directly indecomposable and hence so is $f(L'_i)$, we obtain that there is an index $k \in \bar{n}$ such that each $U_j$ for $j \neq k$ is a one-element lattice. Because $a = (a_1, \ldots, a_n) \in L'_i$, we have $f(a) = u = (u_1, \ldots, u_n) \in f(L'_i)$, from which it follows that $f(\{a_1\} \times \cdots \times L_i \times \cdots \times \{a_n\}) = \{u_1\} \times \cdots \times U_k \times \cdots \times \{u_n\}$. Moreover since $f'_i$ is a dual isomorphism of $L'_i$ onto $f(L'_i)$, $f_i$ is a dual isomorphism of $L_i$ onto $U_k$.

b) Assume $D$ is a one-element lattice. Analogously as in a) it can be proved that $f(L'_i) = \{u_1\} \times \cdots \times U_k \times \cdots \times \{u_n\}$ for a $k \in \bar{n}$ the mapping $f_i: L_i \rightarrow U_k$ is an isomorphism.

5. Lemma. Under the assumptions and denotations as in the previous lemma there is $U_k = L_k$.

Proof. $f^{-1}$ is also a convex automorphism of $L$ and $f^{-1}(u) = a$, so using 4 for $f^{-1}$ and elements $u$ and $f^{-1}(u) = a$ we get

$$f^{-1}(\{u_1\} \times \cdots \times L_k \times \cdots \times \{u_n\}) = \{a_1\} \times \cdots \times B_j \times \cdots \times \{a_n\}$$

for a $j \in \bar{n}$. Applying $f^{-1}$ to

$$\{u_1\} \times \cdots \times U_k \times \cdots \times \{u_n\} \subseteq \{u_1\} \times \cdots \times L_k \times \cdots \times \{u_n\}$$

we obtain

$$\{a_1\} \times \cdots \times L_i \times \cdots \times \{a_n\} \subseteq \{a_1\} \times \cdots \times B_j \times \cdots \times \{a_n\}.$$

Since $L_i$ contains more than one element, it must be $i = j$, $L_i = B_j$. Hence $f(L'_i) = f(\{a_1\} \times \cdots \times L_i \times \cdots \times \{a_n\}) = f(\{a_1\} \times \cdots \times B_j \times \cdots \times \{a_n\}) = \{u_1\} \times \cdots \times L_k \times \cdots \times \{u_n\}$, so that $U_k = L_k$.
6. COROLLARY. Let \( a = (a_1, \ldots, a_n) \) be any fixed element of \( L \), \( f(a) = u = (u_1, \ldots, u_n) \). Then there exists a permutation \( \pi \) of the set \( \overline{n} \) such that for each \( i \in \overline{n} \)

\[
\begin{align*}
    f \left( \{a_1\} \times \cdots \times L_i \times \cdots \times \{a_n\} \right) &= \{u_1\} \times \cdots \times L_{\pi(i)} \times \cdots \times \{u_n\}
\end{align*}
\]

and moreover the mapping \( f_i : L_i \rightarrow L_{\pi(i)} \) derived from

\[
\begin{align*}
    f : \{a_1\} \times \cdots \times L_i \times \cdots \times \{a_n\} &\rightarrow \{u_1\} \times \cdots \times L_{\pi(i)} \times \cdots \times \{u_n\}
\end{align*}
\]

in a natural way is an isomorphism or a dual isomorphism.

Proof. Using denotation as in 4, set \( \pi(i) = k \). To verify that \( \pi \) is a permutation, it is sufficient to show that \( \pi(i_1) = \pi(i_2) \) implies \( i_1 = i_2 \). Now if \( \pi(i_1) = \pi(i_2) = k \), then

\[
\begin{align*}
    f \left( \{a_1\} \times \cdots \times L_{i_1} \times \cdots \times \{a_n\} \right) &= \{u_1\} \times \cdots \times L_k \times \cdots \times \{u_n\} \\
    &= f \left( \{a_1\} \times \cdots \times L_{i_2} \times \cdots \times \{a_n\} \right).
\end{align*}
\]

Recall that \( f \) is a bijection. Hence it must be

\[
\{a_1\} \times \cdots \times L_{i_1} \times \cdots \times \{a_n\} = \{a_1\} \times \cdots \times L_{i_2} \times \cdots \times \{a_n\}.
\]

By assumption \( L_{i_1}, L_{i_2} \) are not one-element sets, therefore \( i_1 = i_2 \). In view of 4 and 5 the further assertions are evident.

We want to make clear if the permutation \( \pi \) and the mappings \( f_i \) depend on the choice of \( a \in L \) or not.

7. LEMMA. Let \( a = (a_1, \ldots, a_n) \) be any fixed element of \( L \), \( f(a) = u = (u_1, \ldots, u_n) \). Then for any \( i, j \in \overline{n}, i < j \)

\[
\begin{align*}
    f \left( \{a_1\} \times \cdots \times L_i \times \cdots \times L_j \times \cdots \times \{a_n\} \right) &= \{u_1\} \times \cdots \times L_k \times \cdots \times L_l \times \cdots \times \{u_n\}
\end{align*}
\]

holds, where \( \{k, l\} = \{\pi(i), \pi(j)\} \).

Proof. Applying \( f \) to

\[
\{a_1\} \times \cdots \times L_i \times \cdots \times \{a_n\} \subseteq \{a_1\} \times \cdots \times L_i \times \cdots \times L_j \times \cdots \times \{a_n\}
\]

and to

\[
\{a_1\} \times \cdots \times L_j \times \cdots \times \{a_n\} \subseteq \{a_1\} \times \cdots \times L_i \times \cdots \times L_j \times \cdots \times \{a_n\},
\]
we get

\[ \{u_1\} \times \cdots \times L_{\pi(i)} \times \cdots \times \{u_n\} \subseteq f(\{a_1\} \times \cdots \times L_i \times \cdots \times L_j \times \cdots \times \{a_n\}), \]

(1)

\[ \{u_1\} \times \cdots \times L_{\pi(j)} \times \cdots \times \{u_n\} \subseteq f(\{a_1\} \times \cdots \times L_i \times \cdots \times L_j \times \cdots \times \{a_n\}), \]

(2)

by 6. Since \( \{a_1\} \times \cdots \times L_i \times \cdots \times L_j \times \cdots \times \{a_n\} \) is a convex sublattice of \( L \), so is \( f(\{a_1\} \times \cdots \times L_i \times \cdots \times L_j \times \cdots \times \{a_n\}) \). Therefore by 1 \( f(\{a_1\} \times \cdots \times L_i \times \cdots \times L_j \times \cdots \times \{a_n\}) = U_1 \times \cdots \times U_n \), with \( U_t \) being a convex sublattice of \( L_t \) for each \( t \in \bar{n} \).

Now \( U_1 \times \cdots \times U_n \supseteq \{u_1\} \times \cdots \times L_{\pi(i)} \times \cdots \times \{u_n\} \) by (1) and \( U_1 \times \cdots \times U_n \supseteq \{u_1\} \times \cdots \times L_{\pi(j)} \times \cdots \times \{u_n\} \) by (2). So \( L_{\pi(i)} = U_{\pi(i)} \) and \( L_{\pi(j)} = U_{\pi(j)} \) and we have

\[ f(\{a_1\} \times \cdots \times L_i \times \cdots \times L_j \times \cdots \times \{a_n\}) = U_1 \times \cdots \times L_k \times \cdots \times L_l \times \cdots \times U_n, \]

where \( \{k, l\} = \{\pi(i), \pi(j)\} \).

Consider the mapping \( f^{-1} \), which is also a convex automorphism of \( L \), and elements \( u \) and \( f^{-1}(u) = a \). Evidently there holds an analogous statement as in 6, the belonging permutation is \( \pi^{-1} \). Using analogous considerations as above we get

\[ f^{-1}(\{u_1\} \times \cdots \times L_k \times \cdots \times L_l \times \cdots \times \{u_n\}) = V_1 \times \cdots \times L_i \times \cdots \times L_j \times \cdots \times V_n, \]

where \( V_t \) is a convex sublattice of \( L_t \), for each \( t \in \bar{n} - \{i, j\} \).

Applying \( f \) to

\[ \{a_1\} \times \cdots \times L_i \times \cdots \times L_j \times \cdots \times \{a_n\} \subseteq V_1 \times \cdots \times L_i \times \cdots \times L_j \times \cdots \times V_n \]

(\( a_t \in V_t \) for each \( t \in \bar{n} - \{i, j\} \) because \( a = f^{-1}(u) \in f^{-1}(\{u_1\} \times \cdots \times L_k \times \cdots \times L_l \times \cdots \times \{u_n\}) = V_1 \times \cdots \times L_i \times \cdots \times L_j \times \cdots \times V_n \)), we get

\[ U_1 \times \cdots \times L_k \times \cdots \times L_l \times \cdots \times U_n \subseteq \{u_1\} \times \cdots \times L_k \times \cdots \times L_l \times \cdots \times \{u_n\}, \]

so that \( U_1 = \{u_1\}, \ldots, U_n = \{u_n\} \). Hence

\[ f(\{a_1\} \times \cdots \times L_i \times \cdots \times L_j \times \cdots \times \{a_n\}) = \{u_1\} \times \cdots \times L_k \times \cdots \times L_l \times \cdots \times \{u_n\} \]

and the proof is finished.
8. COROLLARY. The permutation \( \pi \) and the mappings \( f_i : L_i \to L_{\pi(i)} \) mentioned in 6 do not depend on the choice of the element \( a \in L \).

Proof. Obviously it is sufficient to prove that if elements \( a, a' \) differ in one component, then the belonging permutations \( \pi \) and \( \pi' \) and mappings \( f_i : L_t \to L_{\pi(t)} \) \((t \in \mathbb{N})\) are equal.

Let \( a = (a_1, \ldots, a_i, \ldots, a_n) \), \( a' = (a_1, \ldots, a'_i, \ldots, a_n) \), hence \( a, a' \) differ in the \( i \)th component. Further let \( f(a) = u = (u_1, \ldots, u_n) \). First we show that \( \pi = \pi' \). Since \( a' \in \{a_1\} \times \cdots \times L_i \times \cdots \times \{a_n\} \), there is

\[
f(a') \in f(\{a_1\} \times \cdots \times L_i \times \cdots \times \{a_n\}) = \{u_1\} \times \cdots \times L_{\pi(i)} \times \cdots \times \{u_n\},
\]

by 4, 5, 6. Therefore \( f(a') \) differs from \( u = f(a) \) only in the \( \pi(i) \)th component, \( f(a') = (u_1, \ldots, u'_i, \ldots, u_n) \). By 4, 5, 6 used for \( a' \) instead of \( a \) we obtain

\[
f(\{a_1\} \times \cdots \times L_i \times \cdots \times \{a_n\}) = \{u_1\} \times \cdots \times L_{\pi(i)} \times \cdots \times \{u_n\}
\]

and comparing (3) and (4) we have \( \pi(i) = \pi'(i) \). Now let \( j \in \mathbb{N}, j \neq i \), e.g. \( j > i \). Use 7 first for \( a \), then for \( a' \). We get that the set \( f(\{a_1\} \times \cdots \times L_i \times \cdots \times L_j \times \cdots \times \{a_n\}) \) is equal to \( \{u_1\} \times \cdots \times L_k \times \cdots \times L_i \times \cdots \times \{u_n\} \) \((\{k, l\} = \{\pi(i), \pi(j)\})\) and also to \( \{u_1\} \times \cdots \times L_k' \times \cdots \times L_j' \times \cdots \times \{u_n\} \) \((\{k', l'\} = \{\pi'(i), \pi'(j)\})\). Hence \( \{k, l\} = \{k', l'\} \) and as \( \pi(i) = \pi'(i) \), it must be also \( \pi(j) = \pi'(j) \). We have proved \( \pi = \pi' \).

If \( x_i \in L_i \), then \( f_i(x_i) \) is the \( \pi(i) \)th component of \( f(a_1, \ldots, x_i, \ldots, a_n) \) and \( f'_i(x_i) \) is the \( \pi'(i) \)th component of the same element \( f(a_1, \ldots, x_i, \ldots, a_n) \). But we already know that \( \pi(i) = \pi'(i) \), so that \( f_i(x_i) = f'_i(x_i) \). We have proved that \( f_i = f'_i \). Now let \( j \in \mathbb{N}, j \neq i \). Without loss of generality we can suppose that \( j > i \). Take any \( x_j \in L_j \). Using 4, 5, 6 for the element \( a \) we get that \( f(a_1, \ldots, a_i, \ldots, x_j, \ldots, a_n) \) has the same components as \( f(a) \), with the exception of the \( \pi(j) \)th component, which is \( f_j(x_j) \). If we use 4, 5, 6 for \( a' \), we obtain that \( f(a_1, \ldots, a_i', \ldots, x_j, \ldots, a_n) \) has such components as \( f(a') \) besides the \( \pi(j) \)th component, which is \( f'_j(x_j) \). Use once more 4 and 5, now for \( (a_1, \ldots, a_i, \ldots, x_j, \ldots, a_n) \). We get

\[
f(\{a_1\} \times \cdots \times L_i \times \cdots \times \{x_j\} \times \cdots \times \{a_n\}) = \{u_1\} \times \cdots \times \{f_j(x_j)\} \times \cdots \times \{u_n\}
\]

(the \( \pi(i) \)th factor is \( L_{\pi(i)} \), \( \{f_j(x_j)\} \) is the \( \pi(j) \)th factor). Evidently \( (a_1, \ldots, a_i, \ldots, x_j, \ldots, a_n) \in \{a_1\} \times \cdots \times L_i \times \cdots \times \{x_j\} \times \cdots \times \{a_n\} \), so \( f(a_1, \ldots, a_i', \ldots, x_j, \ldots, a_n) \in \{u_1\} \times \cdots \times \{f_j(x_j)\} \times \cdots \times \{u_n\} \). We know that the \( \pi(j) \)th component of \( f(a_1, \ldots, a_i', \ldots, x_j, \ldots, a_n) \) is \( f_j(x_j) \), therefore \( f'_j(x_j) \in \{f_j(x_j)\} \), which means \( f'_j(x_j) = f_j(x_j) \). We have proved also \( f_j = f'_j \) for each \( j \in \mathbb{N}, j \neq i \).
9. Theorem. Let $L$ be a lattice, that is a direct product of finitely many directly indecomposable lattices $L_i$, $L = L_1 \times \cdots \times L_n$. Further let $f$ be a convex automorphism of $L$. Then there exist a permutation $\pi$ of $\overline{n}$ and mappings $f_i: L_i \to L_{\pi(i)}$, all being isomorphisms or dual isomorphisms, such that for every $x \in L$

$$f(x)_{\pi(i)} = f_i(x_i)$$

holds.

Proof. First suppose that every $L_i$ has more than one element. Let $a = (a_1, \ldots, a_n)$ be any fixed element of $L$. Take the permutation $\pi$ and the mappings $f_i: L_i \to L_{\pi(i)}$ belonging to the element $a$ by 6. Now let $x = (x_1, \ldots, x_n)$ be any element of $L$. It is sufficient to show that for each $i \in \overline{n}$ there holds $f(x)_{\pi(i)} = f_i(x_i)$. In view of 8 the permutation $\pi$ and the mappings $f_i$ belong also to $x$, so that if $f(x) = y = (y_1, \ldots, y_n)$, then for any $i \in \overline{n}$ $y_{\pi(i)} = f_i(x_i)$, by 6.

Now assume that some of $L_i$ are one-element lattices. Without loss of generality we can suppose that one-element lattices are those for $i \in \overline{n} - \overline{k}$ ($k \in \overline{n}$, $k < n$). Evidently $f$ determines a convex automorphism $f'$ of $L' = L_1 \times \cdots \times L_k$. Hence there exist a permutation $\pi'$ of $\overline{k}$ and mappings $f_i: L_i \to L_{\pi'(i)}$ for $i \in \overline{k}$, all being isomorphisms or dual isomorphisms, such that for every $x' \in L'$ there holds $f'(x')_{\pi(i)} = f_i(x'_i)$. Now extend $\pi'$ to a permutation $\pi$ of $\overline{n}$ and define $f_i: L_i \to L_{\pi(i)}$ for $i \in \overline{n} - \overline{k}$ in a natural way (i.e. if $L_i = \{a_i\}$, $L_{\pi(i)} = \{a_{\pi(i)}\}$, then $f_i(a_i) = a_{\pi(i)}$). It is easy to see that such a $\pi$ and $f_i: L_i \to L_{\pi(i)}$ are as we need. The proof is complete.

The following theorem is obtained by combining the preceding one with 3.

10. Theorem. Let $L$ be a lattice which can be decomposed into a direct product $L = L_1 \times \cdots \times L_n$, where all $L_i$ are directly indecomposable. Convex automorphisms of $L$ are just the mappings obtained as follows: we take a permutation $\pi$ of the set $\overline{n}$ such that there exist bijections $f_i: L_i \to L_{\pi(i)}$, each of them being either an isomorphism or a dual isomorphism, and we set for any $x \in L$

$$f(x)_{\pi(i)} = f_i(x_i)$$

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CONVEX AUTOMORPHISMS OF A LATTICE


Received February 10, 1992
Revised July 9, 1992

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