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INTERNAL, EXTERNAL BOUNDARIES AND CONTINUOUS MAPPINGS

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ABSTRACT. The notions of internal and external boundaries are introduced. Four weak forms of continuity are defined: internal, external continuities and internal, external boundary continuities. They compose two related decompositions of continuity: a mapping is continuous if and only if it is internally and externally continuous; a mapping is continuous if and only if it is internally and externally boundary continuous.

1. Introduction

In [6], the present author introduced the notions of $\mathcal{B}$*-set* and $\mathcal{B}$*-continuous* mapping. Together with the known notions of *pre-open set* and *pre-continuous mapping* [3] we obtained interesting decompositions: a set is open if and only if it is pre-open and is a $\mathcal{B}$-set; a mapping is continuous if and only if it is both pre-continuous and $\mathcal{B}$-continuous. In this paper, we first introduce the notions of *internal* and *external boundaries* of a set in a topological space, which describe the intrinsic properties of the pre-open set and the $\mathcal{B}$-set, then we prove an improvement of the above mentioned decompositions, in which the $\mathcal{B}$-set and $\mathcal{B}$-continuity can be replaced by weaker notions. In the last section, we prove a new *decomposition theorem* of continuity involving internal and external boundaries, which is closely related with the previous one.

2. Internal and external boundaries

We start from a very simple observation. Let $S = \{(x,y) \mid 0 < x^2 + y^2 \le 1\}$ be the punctured unit disc in the Euclidean plane. Let $I = \{(0,0)\}$, $E = \{(x,y) \mid x^2 + y^2 = 1\}$. It is easily seen that the boundary of S, $\operatorname{Fr} S = I \cup E$. Apparently there is some difference between set I and set E. To characterize the difference, we notice that $I = \operatorname{Int}\operatorname{Cl} S \setminus \operatorname{Int} S$ and $E = \operatorname{Cl} S \setminus \operatorname{Int}\operatorname{Cl} S$, where $\operatorname{Int} S$ and $\operatorname{Cl} S$ denote the interior and the closure of the set S, respectively. The following definitions are suggested.

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DEFINITION 1. *Let S be a set in a topological space. The internal boundary set of S, $I - \operatorname{Fr} S$ is defined by*

$$I - \operatorname{Fr} S = \operatorname{Int} \operatorname{Cl} S \setminus \operatorname{Int} S.$$

DEFINITION 2. *Let S be a set in a topological space. The external boundary set of S, $E - \operatorname{Fr} S$ is defined by*

$$E - \operatorname{Fr} S = \operatorname{Cl} S \setminus \operatorname{Int} \operatorname{Cl} S.$$

The propositions below are immediate.

PROPOSITION 1. *$E - \operatorname{Fr} S$ is a closed set.*

PROPOSITION 2. $(I - \operatorname{Fr} S) \cap (E - \operatorname{Fr} S) = \emptyset$.

Since $\operatorname{Fr} S = \operatorname{Cl} S \setminus \operatorname{Int} S$, we have:

PROPOSITION 3. $\operatorname{Fr} S = (I - \operatorname{Fr} S) \cup (E - \operatorname{Fr} S)$.

The definitions of internal and external boundaries are quite reasonable. See the propositions below.

PROPOSITION 4. ([2]). *A convex set A in a topological vector space has no internal boundary.*

An open set U is a regular open set [4] if $\operatorname{Int} \operatorname{Cl} U = U$.

PROPOSITION 5. *An open set U in a topological space is a regular open set if and only if $I - \operatorname{Fr} U = \emptyset$.*

A set A is said to be a *t-set* [6] if $\operatorname{Int} \operatorname{Cl} A = \operatorname{Int} A$.

PROPOSITION 6. *A set S in a topological space is a t-set if and only if $I - \operatorname{Fr} S = \emptyset$.*

A set S is said to be a *pre-open set* [3] if $S \subset \operatorname{Int} \operatorname{Cl} S$.

PROPOSITION 7. *A set S in a topological space is a pre-open set if and only if*

$$S \cap (E - \operatorname{Fr} S) = \emptyset.$$

Proof.

Necessity. $S \subset \operatorname{Int}\operatorname{Cl} S$ implies $S \cap (E - \operatorname{Fr} S) = S \cap (\operatorname{Cl} S \setminus \operatorname{Int}\operatorname{Cl} S) \subset S \cap (\operatorname{Cl} S \setminus S) = \emptyset$.

Sufficiency. $S \cap (E - \operatorname{Fr} S) = \emptyset$ implies $S \cap (\operatorname{Cl} S \setminus \operatorname{Int}\operatorname{Cl} S) = \emptyset$. Since $S \subset \operatorname{Cl} S$, we have $S \subset \operatorname{Int}\operatorname{Cl} S$.

Proposition 7 suggests the following definitions.

DEFINITION 3. *A set S in a topological space is said to be external boundary free if $S \cap (E - \operatorname{Fr} S) = \emptyset$.*

DEFINITION 4. *A set S in a topological space is said to be internal boundary free if $S \cap (I - \operatorname{Fr} S) = \emptyset$.*

Definition 3 does not give a new concept, because a set is external boundary free if and only if it is pre-open by Proposition 7. But we will see in Proposition 8 that an internal boundary free set is a new concept including t-set and $\mathcal{B}$-set, where a $\mathcal{B}$-set B is defined by $B = U \cap S$ with an open set U and a t-set S.

We need a lemma to prove Proposition 8.

LEMMA 1. ([1, p. 53, Prob. 48]). *Let S be a set in a topological space, U be an open set. Then* $\operatorname{Int}\operatorname{Cl}(U \cap S) = (\operatorname{Int}\operatorname{Cl} U) \cap (\operatorname{Int}\operatorname{Cl} S)$.

PROPOSITION 8. *A $\mathcal{B}$-set in a topological space is internal boundary free.*

Proof. Let $B = U \cap S$, where U is open and $\operatorname{Int}\operatorname{Cl} S = \operatorname{Int} S$.

$$\begin{aligned}
&B \cap (I - \operatorname{Fr} B)\\
=&B \cap \big(\operatorname{Int}\operatorname{Cl}(U \cap S) \setminus \operatorname{Int}(U \cap S)\big)\\
=&B \cap \big((\operatorname{Int}\operatorname{Cl} U \cap \operatorname{Int}\operatorname{Cl} S) \setminus (\operatorname{Int} U \cap \operatorname{Int} S)\big)\\
=&B \cap \big((\operatorname{Int}\operatorname{Cl} U \cap \operatorname{Int} S) \setminus (U \cap \operatorname{Int} S)\big)\\
=&(U \cap S) \cap \big((\operatorname{Int}\operatorname{Cl} U \setminus U) \cap \operatorname{Int} S\big)\\
=&\big(U \cap (\operatorname{Int}\operatorname{Cl} U \setminus U)\big) \cap (S \cap \operatorname{Int} S)\\
=&\emptyset \cap \operatorname{Int} S = \emptyset .
\end{aligned}$$

The converse is not correct, see the example below.

Example 1. Let $X = \{a, b\}$ with topology $\{\emptyset, \{a\}, X\}$. Then $I - \operatorname{Fr}\{a\} = \operatorname{Int}\operatorname{Cl}\{a\} \setminus \operatorname{Int}\{a\} = \operatorname{Int} X \setminus \{a\} = \{b\}$. Hence $\{a\}$ is an internal boundary free set. It is easily seen that $\{a\}$ is not a $\mathcal{B}$-set since $\operatorname{Int}\operatorname{Cl}\{a\} = X \neq \{a\} = \operatorname{Int}\{a\}$.

The following proposition shows the importance of internal and external boundary free sets.

PROPOSITION 9. *A set S in a topological space is open if and only if it is both internal and external boundary free.*

Proof.

Necessity. If S is open, then $S = \text{Int}(\text{Int}\, S) \subset \text{Int}(\text{Cl}\, S)$. Hence $S \cap (E - \text{Fr}\, S) = S \cap (\text{Cl}\, S \setminus \text{Int}\,\text{Cl}\, S) = \emptyset$, while $S \cap (I - \text{Fr}\, S) = S \cap (\text{Int}\,\text{Cl}\, S \setminus \text{Int}\, S) = S \cap (\text{Int}\,\text{Cl}\, S \setminus S) = \emptyset$.

Sufficiency. Assume $S \cap (E - \text{Fr}\, S) = S \cap (\text{Cl}\, S \setminus \text{Int}\,\text{Cl}\, S) = \emptyset$. Then $S \subset \text{Cl}\, S$ implies $S \subset \text{Int}\,\text{Cl}\, S$. Assume $S \cap (I - \text{Fr}\, S) = \emptyset$. Then $S \subset \text{Int}\,\text{Cl}\, S$ implies $S \subset \text{Int}\, S$. Hence S is an open set.

Now the following definitions are natural.

DEFINITION 5. *A mapping $f\colon X \to Y$ is said to be internally continuous if $f^{-1}(V)$ is an internal boundary free set in X for each open set V in Y.*

DEFINITION 6. *A mapping $f\colon X \to Y$ is said to be externally continuous if $f^{-1}(V)$ is an external boundary free set in X for each open set V in Y.*

A mapping $f\colon X \to Y$ is said to be *pre-continuous* if $f^{-1}(V)$ is a pre-open set for each open set V in Y.

By Proposition 7, a mapping f is externally continuous if and only if it is pre-continuous. Therefore external continuity is not a new concept. But internal continuity is new.

A mapping f is said to be *$\mathcal{B}$-continuous* if $f^{-1}(V)$ is a $\mathcal{B}$-set for each open set V in Y.

By Proposition 8, a $\mathcal{B}$-continuous mapping is internally continuous. The converse is not true, see example below.

Example 2. Let $X = \{x, y\}$ with topology $\{\emptyset, \{x\}, X\}$, $Y = \{a, b\}$ with discrete topology. Define $f\colon X \to Y$ by $f(x) = a$, $f(y) = b$. Then f is internally continuous but not $\mathcal{B}$-continuous.

By Proposition 9, the following decomposition theorem is correct.

THEOREM 1. *A mapping $f\colon X \to Y$ is continuous if and only if it is both internally and externally continuous.*

The decomposition in [6] states that a mapping f is continuous if and only if it is pre-continuous and $\mathcal{B}$-continuous. Theorem 1 improves it by replacing $\mathcal{B}$-continuity with the weaker internal continuity.

3. Decomposition of continuity involving internal and external boundaries

Internal and external boundaries can be used to characterize continuity directly. It is not always necessary to use the notions of internal and external boundary free sets.

Let $f\colon X \to Y$ be a continuous mapping, $S \subset Y$ be an arbitrary set. Then $f^{-1}(\operatorname{Fr} S) = f^{-1}(\operatorname{Cl} S \backslash \operatorname{Int} S) = f^{-1}(\operatorname{Cl} S) \backslash f^{-1}(\operatorname{Int} S)$. Since $f^{-1}(\operatorname{Cl} S)$ is closed and $f^{-1}(S) \subset f^{-1}(\operatorname{Cl} S)$, we have $\operatorname{Cl} f^{-1}(S) \subset f^{-1}(\operatorname{Cl} S)$. Since $f^{-1}(\operatorname{Int} S)$ is open and $f^{-1}(S) \supset f^{-1}(\operatorname{Int} S)$, we have $\operatorname{Int} f^{-1}(S) \supset f^{-1}(\operatorname{Int} S)$. Therefore $\operatorname{Fr} f^{-1}(S) = \operatorname{Cl} f^{-1}(S) \setminus \operatorname{Int} f^{-1}(S) \subset f^{-1}(\operatorname{Cl} S) \setminus f^{-1}(\operatorname{Int} S) = f^{-1}(\operatorname{Fr} S)$. This simple fact suggests the following definitions.

DEFINITION 7. *A mapping $f\colon X \to Y$ is said to be internally boundary continuous if $I - \operatorname{Fr} f^{-1}(S) \subset f^{-1}(\operatorname{Fr} S)$ for any set $S \subset Y$.*

DEFINITION 8. *A mapping $f\colon X \to Y$ is said to be externally boundary continuous if $E - \operatorname{Fr} f^{-1}(S) \subset f^{-1}(\operatorname{Fr} S)$ for any set $S \subset Y$.*

In the sequel we will discuss the relations among the four weak forms of continuity.

Example 3. Internal continuity $\not\Rightarrow$ external continuity, internal boundary continuity or external boundary continuity.

Let $X = \{x, y\}$ with topology $\{\emptyset, \{x\}, X\}$, $Y = \{a, b\}$ with topology $\{\emptyset, \{b\}, Y\}$. Define $f\colon X \to Y$ by $f(x) = a$, $f(y) = b$. Then f is internally continuous but not externally continuous, not internally or externally boundary continuous.

Example 4. External continuity $\not\Rightarrow$ internal continuity, internal boundary continuity or external boundary continuity.

Let $X = \{x, y, z\}$ with topology $\{\emptyset, \{x\}, X\}$, $Y = \{a, b, c\}$ with topology $\{\emptyset, \{a\}, \{a, b\}, Y\}$. Define $f\colon X \to Y$ by $f(x) = a$, $f(y) = b$, $f(z) = c$. Then f is externally continuous but not internally continuous, not internally boundary continuous or externally boundary continuous.

Example 5. Internal boundary continuity $\not\Rightarrow$ external continuity or external boundary continuity.

Let $X = \{x, y, z\}$ with topology $\{\emptyset, \{x\}, \{z\}, \{x, z\}, X\}$, $Y = (a, b, c)$ with topology $\{\emptyset, \{a\}, \{a, b\}, Y\}$. Define $f\colon X \to Y$ by $f(x) = a$, $f(y) = b$, $f(z) = c$. Then f is internally boundary continuous but not externally continuous or externally boundary continuous.

Example 6. External boundary continuity $\not\Rightarrow$ internal continuity or internal boundary continuity.

Let $X = \{x, y\}$ with topology $\{\emptyset, X\}$, $Y = \{a, b\}$ with discrete topology. Define $f: X \to Y$ by $f(x) = a$, $f(y) = b$. Then f is externally boundary continuous but not internally continuous or internally boundary continuous.

The following two propositions give the relations between internal continuity and internal boundary continuity, and between external continuity and external boundary continuity.

PROPOSITION 10. *An internally boundary continuous mapping is internally continuous.*

Proof. Let $U \subset Y$ be an open set. Then $I - \operatorname{Fr} f^{-1}(U) \subset f^{-1}(\operatorname{Fr} U)$. Hence

$$f^{-1}(U) \cap I - \operatorname{Fr} f^{-1}(U) \subset f^{-1}(U) \cap f^{-1}(\operatorname{Fr} U)$$
$$= f^{-1}(U) \cap \big(f^{-1}(\operatorname{Cl} U) \setminus f^{-1}(\operatorname{Int} U)\big) = f^{-1}(U) \cap \big(f^{-1}(\operatorname{Cl} U) \setminus f^{-1}(U)\big) = \emptyset .$$

PROPOSITION 11. *An externally boundary continuous mapping is externally continuous.*

Proof. For an open set U in Y we have

$$f^{-1}(U) \cap E - \operatorname{Fr} f^{-1}(U) = f^{-1}(U) \cap \big(f^{-1}(\operatorname{Cl} U) \setminus f^{-1}(U)\big) = \emptyset.$$

By Theorem 1, the following decomposition is immediate.

THEOREM 2. *A mapping $f: X \to Y$ is continuous if and only if it is internally boundary continuous and externally boundary continuous.*

By Proposition 3, we have a characterization of continuity.

COROLLARY 1. *A mapping $f: X \to Y$ is continuous if and only if*

$$\operatorname{Fr} f^{-1}(S) \subset f^{-1}(\operatorname{Fr} S)$$

for each set $S \subset Y$.

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