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LATIN PARALLELEPIPEDS
NOT COMPLETING TO A CUBE

MARTIN KOCHOL

ABSTRACT. In this paper we construct a latin \((n \times n \times (n - d))\)-parallelepiped that
cannot be extended to a latin cube of order \(n\), for every \(d \geq 3\) and \(n \geq 6d\) or
\(n = 3d, 4d, 5d\). For \(d = 2\), it is similar to the construction already known.

1. Introduction

A latin square of the elements \(z_1, \ldots, z_n\) is an \(n \times n\) array such that the entries
are members of \(\{z_1, \ldots, z_n\}\) and no member occurs in any row or column more
than once. Moreover, if some cells may be empty we have an incomplete latin
square of the elements \(z_1, \ldots, z_n\).

Let \(A_1 = [a_{i,j,1}], A_2 = [a_{i,j,2}], \ldots, A_k = [a_{i,j,k}]\) be latin squares of the elements
\(z_1, \ldots, z_n\). The ordered \(k\)-tuple \(A = (A_1, A_2, \ldots, A_k)\) is called a latin \((n \times n \times k)\)-
parallelepiped of elements \(z_1, \ldots, z_n\) if the elements \(a_{i,j,1}, \ldots, a_{i,j,k}\) are mutually
distinct, for every \(1 \leq i, j \leq n\). In the case \(k = n\), \(A\) is called a latin cube of the elements
\(z_1, \ldots, z_n\).

Usually \(z_i = i, 1 \leq i \leq n\). In this case we speak in abbreviation about latin
squares or cubes of order \(n\) and about \((n \times n \times k)\)-parallelepipeds (and do not
use the words “of elements 1, 2, \ldots, \(n^\)”).

A latin cube \(A'\) of order \(n\) is an extension of a latin \((n \times n \times k)\)-parallelepiped
\(A = (A_1, \ldots, A_k)\) if there exist latin squares \(A_{k+1}, \ldots, A_n\) such that \(A' = (A_1, \ldots,
A_k, A_{k+1}, \ldots, A_n)\).

The following problem (see [4]) was mentioned during the Sixth Hungarian
Colloquium on Cmbinatorics, Eger 1981. Given a latin \((n \times n \times k)\)-paral­
leelepiped \(A\), does there exist a latin cube of order \(n\), which is an extension of \(A\)?
An analogous problem for latin rectangles was answered in the affirmative by
Hall in [3]. On the contrary there are known constructions of the latin
\((n \times n \times (n - 2))\)-parallelepipeds that cannot be extended to a latin cube of
order \(n\): these constructions are done for \(n = 2^k, k \geq 3\), in [4], for \(n = 6\) and

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In this part we prove the following theorem.

**Theorem:** Let $d \geq 3$, $n = 3d$, $4d$, $5d$ or $n \geq 6d$. Then there exists a latin $(n \times n \times (n - d))$-parallelepiped that cannot be extended to a latin cube of order $n$.

**Proof:** Let $d \geq 3$. Take a latin cube $B = (B_1, \ldots, B_d)$ of order $d$ such that $b_{i,j,k}$, the entry in the $i$-th row and the $j$-th column of $B_k$ satisfies $b_{i,j,k} \equiv i + j + k - 2 \pmod{d}$, and $b_{i,j,k} = d$ if $i + j + k - 2 \equiv 0 \pmod{d}$.

Replace, in the latin cube $B$, each number $t \in \{1, \ldots, d\}$ by an arbitrary latin $(3 \times 3 \times 2)$-parallelepiped $C^{(t)}$ of the elements $t, d + t, 2d + t$. We get a latin $(3d \times 3d \times 2d)$-parallelepiped. The same idea will be used in the following construction.

Let $d \geq 3$. Let $\phi$ be a map of $\{\langle i, j \rangle; 1 \leq i, j \leq d\}$ onto the five element set $\{p, r, s, t, u\}$ satisfying:

\[
\begin{align*}
\phi \langle 1, 1 \rangle &= p, \\
\phi \langle i, 1 \rangle &= r, \text{ for } 2 \leq i \leq d, \\
\phi \langle 1, j \rangle &= s, \text{ for } 2 \leq j \leq d, \\
\phi \langle 2, j \rangle &= t, \text{ for } 2 \leq j \leq d, \\
\phi \langle i, j \rangle &= u, \text{ for } 3 \leq i \leq d, 2 \leq j \leq d.
\end{align*}
\]

We will use five distinct latin $(3 \times 3 \times 2)$-parallelepipeds $C^{(u, y)}$ (where $y \in \{p, r, s, t, u\}$) if $t = 1, 2$. Let us construct.

**Construction A:**

Take partial latin squares $D^{(u, y)}_x$ of the elements $t, d + t, 2d + t$ (for $x \in \{2, 3\}, t \in \{1, 2\}, y \in \{p, r, s, t, u\}$) as it is illustrated in Fig. 1. We can check that there exist latin cubes $E^{(u, y)}_x = (E^{(u, y)}_1, E^{(u, y)}_2, E^{(u, y)}_3)$ of the elements $t, d + t, 2d + t$ for $t \in \{1, 2, \ldots, d\}, y \in \{p, r, s, t, u\}$ satisfying (1) and (2):

1. If $t = 1, 2$, then $E^{(u, y)}_x$ is an extension of $D^{(u, y)}_x$, where $x \in \{2, 3\}, y \in \{p, r, s, t, u\}$.
2. If $t = 3, \ldots, d$, then the entry in the first row and the first column of $E^{(u, y)}_3$ is equal to $t$. Furthermore, all $E^{(u, y)}_3$ are the same for all $y \in \{p, r, s, t, u\}$.

Then let us define $C^{(u, y)} = (E^{(u, y)}_1, E^{(u, y)}_2, E^{(u, y)}_3)$, the $(3 \times 3 \times 2)$-parallelepiped of the elements $t, d + t, 2d + t$ for any $t \in \{1, \ldots, d\}, y \in \{p, r, s, t, u\}$.
Construction B:

We have the latin cube $B = (B_1, \ldots, B_d)$, $B_k = [b_{i,j,k}]$, $1 \leq k \leq d$. Replace each $t = b_{i,j,k} = i + j + k - 2$ (mod $d$) by $C^{(t, \varphi(i,j))}$. We get a new latin $(3d \times 3d \times 2d)$-parallelepiped $F = (F_1, \ldots, F_{2d})$. The latin square $F_k$, $k = 1, \ldots, d$, arises from $B_k$ if we replace $t = b_{i,j,k}$ by $E_1^{(t, \varphi(i,j))}$. Similarly the latin square $F_{k-1}$, $k = 1, \ldots, d$, arises from $B_k$ if we replace $t = b_{i,j,k}$ by $E_2^{(t, \varphi(i,j))}$. 

Construction C:

Now we construct a new latin $(3d \times 3d \times 2d)$-parallelepiped $G$ from $F$. Take the members $1, 2$ from $F_2, F_4, \ldots, F_{2d}$ as shown in Fig. 2. for $d = 4$. More precisely, take the numbers 1, 2 which are in the intersections of the 1st, 5th, 7th, $\ldots$, $3(d-1) + 2nd$ rows and the 2nd, $\ldots$, $3(l+1) + 2nd$, $3l + 1st$, $3(d-1) + 2nd$ columns of $F_{2k}$, where $l = d - k + 1$ if $k \neq 1$. In every $F_2, F_4, \ldots, F_{2d}$ we interchange this 1 and 2. We get new latin squares $G_2, G_4, \ldots, G_{2d}$.

Let $F_k = [f_{i,j,k}]$, $1 \leq k \leq 2d$. If $1 = f_{i,j,k}$ is interchanged in $F_k$ by 2, then (3) or (4) holds:

(3) There exists $l \in \{2, 4, \ldots, 2d\}$ such that $f_{i,j,l} = 2$ is interchanged in $F_l$ by 1.

(4) No member $f_{i,j,l}$ is equal to 2 for any $l \in \{1, \ldots, 2d\}$ (this follows from the condition (1) for $y \in \{s, t\}$).

Similarly, if $2 = f_{i,j,k}$ is interchanged in $F_k$ by 1, then (5) or (6) holds:

(5) There exists $l \in \{2, 4, \ldots, 2d\}$ such that $f_{i,j,l} = 1$ is interchanged in $F_l$ by 2.

(6) No member $f_{i,j,l}$ is equal to 1 for any $l \in \{1, \ldots, 2d\}$.

Thus $G = (G_1, G_2, G_3, \ldots, G_{2d})$ is a latin $(3d \times 3d \times 2d)$-parallelepiped provided $G_{2k+1} = F_{2k+1}$ for $k = 0, \ldots, d - 1$.

Now we prove that $G$ cannot be extended to a latin cube of order 3d. Let $G_k = [g_{i,j,k}]$, $1 \leq k \leq 2d$. Let us denote by $M_{i,j}(G)$ the subset of the members 1, 2, $\ldots$, $3d$ which do not occur in the set $\{g_{i,j,1}, g_{i,j,2}, \ldots, g_{i,j,2d}\}$, $1 \leq i, j \leq 3d$. $M_{i,j}(F)$ can be defined similarly.

From (1), (2) and the construction of $C^{(r,s)}$ it follows that:

$M_{3k+1,1}(F) = \{1, 2, \ldots, d\}(0 \leq k \leq d - 1),$

$M_{3k+1,3l+1}(F) = \{2, 3, \ldots, d, d + 1\}(0 \leq k \leq d - 1, 1 \leq l \leq d - 1).$

From the Construction C we can see that:

$M_{3k+1,1}(G) = M_{3k+1,1}(F) = \{1, 2, \ldots, d\}(0 \leq k \leq d - 1),$

$M_{1,3l+1}(G) = \{1, 3, 4, \ldots, d, d + 1\}(1 \leq l \leq d - 1),$

$M_{3k+1,3l+1}(G) = M_{3k+1,3l+1}(F) = \{2, 3, \ldots, d, d + 1\}(1 \leq k, l \leq d - 1).$

Denote $I = \{\langle 3k + 1, 3l + 1 \rangle; 0 \leq k, l \leq d - 1\}.$

Let $H = [h_{i,j}]$ be a latin square of order $3d$ such that $h_{i,1} = 1$ and $h_{i,j} \in M_{i,j}(G)$ for all $1 \leq i, j \leq 3d$.

Since $h_{1,1} = 1$, there exists exactly one $\langle i, j \rangle \in I$ such that $h_{i,j} = 1$.

Clearly $h_{1,3l+1} \neq 2$ for any $l = 0, \ldots, d - 1$. Thus there exist at most $d - 1$ members $\langle i, j \rangle$ of $I$ such that $h_{i,j} = 2$. 
Similarly there exist at most $d - 1$ members $\langle i, j \rangle$ of $I$ such that $h_{i, j} = d + 1$. There exist at most $d(d - 2)$ members $\langle i, j \rangle$ of $I$ such that $h_{i, j} = 3, \ldots, d$.

Thus there exist at most $d^2 - 1$ members $\langle i, j \rangle$ of $I$ such that $h_{i, j} \in \{1, 2, \ldots, d, d + 1\}$. But if $\langle i, j \rangle \in I$, then $h_{i, j} \in \{1, 2, \ldots, d, d + 1\}$ — a contradiction with the fact that $|I| = d^2$. Thus $G$ cannot be extended to a latin cube of order $3d$. Note that we do not know whether $G$ can be extended to a latin cube.
By H.-L. Fu [1], [2] every latin cube of order \( m \) can be embedded in a latin cube of order \( n \) for every \( n \geq 2m \). Using this we can easily to see that \( G \) can be embedded in the latin \((n \times n \times (n - d))\)-parallelepiped \( H \), where \( n \geq 6d \) and \( M_{i,j}(G) = M_{i,j}(H) \) for \( 1 \leq i, j \leq 3d \). Therefore \( H \) cannot be extended to a latin cube of order \( n \). Thus we have proved the theorem for \( n = 3d \) and \( n \geq 6d \).

We prove the theorem if \( n = 4d \). For this purpose let \( V^{(t,y)} \) be the partial latin squares of the elements \( t, d + t, 2d + t, 3d + t \) (for \( x \in \{3, 4\}, t \in \{1, 2\}, y \in \{p, r, s, t, u\} \)) satisfying (7) and (8):

(7) The 4th row and the 4th column of \( V^{(t,y)}_x \) are empty.

(8) Removing the 4th row and the 4th column of \( V^{(t,y)}_x \) we get \( D^{(t,y)}_x \).

Analogously to the Construction A there exist latin cubes \( Q^{(t,y)}_x = (Q^{(t,y)}_1, \ldots, Q^{(t,y)}_d) \) of the elements \( t, d + t, 2d + t, 3d + t \) for \( t \in \{1, 2, \ldots, d\}, y \in \{p, r, s, t, u\} \) satisfying (9) and (10):

(9) If \( t = 1, 2 \), then \( Q^{(t,y)}_x \) is an extension of \( V^{(t,y)}_x \), where \( x \in \{3, 4\}, y \in \{p, r, s, t, u\} \).
(10) If \( t = 3, \ldots, d \), then the entry in the first row and the first column of \( Q_4^{(t,y)} \) is equal to \( t \). Furthermore, all \( Q_4^{(t,y)} \) are the same for all \( y \in \{p, r, s, t, u\} \).

Let us define analogously \( W^{(t,y)} = (Q_1^{(t,y)}, Q_2^{(t,y)}, Q_3^{(t,y)}) \), the \((4 \times 4 \times 3)\)-parallelepiped of the elements \( t, d + t, 2d + t, 3d + t \), for all \( t \in \{1, \ldots, d\} \), \( y \in \{p, r, s, t, u\} \).

We can continue in the construction in the same way as for \( n = 3d \) (i.e. we can replace each member of the latin cube \( B \) by an appropriate \( W^{(t,y)} \) as in the Construction B and use a similar switching as in the Construction C) to get a latin \((4d \times 4d \times 3d)\)-parallelepiped which cannot be extended to a latin cube of order \( 4d \).
The case $n = 5d$ can be proved in the same way as the cases $n = 3d, 4d$, concluding the proof of the theorem.

Note that in [5] we have proved that there exists a latin $(n \times n \times (n - 2))$-parallelepiped that cannot be extended to a latin cube of order $n$ if and only if $n \geq 5$. That is why we conjecture that the above theorem hold if and only if $n \geq 2d + 1$, for every $d \geq 2$, i.e. each latin $(n \times n \times (n - d))$-parallelepiped can be extended to a latin cube of order $n$ whenever $n \leq 2d$, but there exists a latin $(n \times n \times (n - d))$-parallelepiped that cannot be extended to a latin cube whenever $n \geq 2d + 1$.

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