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ON LONGEST CIRCUITS IN CERTAIN NON-REGULAR PLANAR GRAPHS

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ABSTRACT. An edge h of a graph G is of type $(a,b;m,n)$ if its vertices are of degrees a and b, and the two faces incident with h are an m-gon and an n-gon. It is shown that the infinite class T of 3-polytopial graphs whose edges are of types $(3,5;4,4)$ and $(5,4;4,6)$ has shortness coefficient equal to 24/31 and all graphs dual to those from T are Hamiltonian.

1. Introduction

There are many papers studying simple circuits in various classes of planar 3-connected graphs (or, equivalently, 3-polytopial graphs), see, e.g., Ewald and others [1], Grünbaum [2], Grünbaum and Walther [3], Harant and Walther [4], Owens [10], Jackson [5] and others. In [3], Grünbaum and Walther introduced several numbers that measure, in a certain sense, the size of the longest simple circuits in graphs belonging to a given class of graphs. We recall one of them.

For any graph G let $v(G)$ denote the number of vertices and $h(G)$ the maximum length of simple circuits in G. The shortness coefficient $\varrho(\mathcal{G})$ of an infinite class $\mathcal{G}$ of graphs is defined by

$$\varrho(\mathcal{G}) = \liminf_{G\in\mathcal{G}} \frac{h(G)}{v(G)}\,.$$

We recall that G is Hamiltonian if $v(G) = h(G)$. The class of graphs $\mathcal{G}$ is Hamiltonian provided that all its members are Hamiltonian, and $\mathcal{G}$ is strongly non-Hamiltonian if it contains no Hamiltonian graph.

Now we consider a planar graph G. An edge h of G is of type $(a,b;m,n)$ if its vertices are of degrees a and b, and the two faces incident with h are an m-gon and an n-gon. The present paper deals with 3-polytopial graphs having

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edges of exactly two types. Let $S(a,b,c;m,n,k)$ denote the class of 3-polytopial graphs with edges of types $(a,b;m,n)$ and $(b,c;n,k)$ (see [6]). There are several papers which deal with the longest circuits in regular graphs from classes $S(a,a,a;m,n,k)$, $a \in \{3,4,5\}$ or with graphs dual to those (see [7] and [8]). In [9] or [10], it has been shown that the shortness coefficient is less than one for many classes of simple 3-polytopial graphs with edges of only two types. In the present paper, we investigate the maximum length of simple circuits in non-regular graphs from some classes of graphs with two types of edges.

Let R denote the class of 3-polytopial graphs with edges of types $(4,4;3,5)$ and $(4,6;5,4)$ and let T denote the class of graphs dual to those from R. It is easy to see that $R = S(4,4,6;3,5,4)$ and $T = S(3,5,4;4,4,6)$. In [6], it was shown that both of these classes contain infinitely many graphs.

The main results of this paper are summed up in the following theorem.

THEOREM.

(1) *The class T is strongly non-Hamiltonian.*

(2) *Let G be a graph from T. Then*

$$h(G) = \frac{24\big(v(G)-1\big)}{31} .$$

(3) $\varrho(T) = 24/31$.

(4) *The class R is Hamiltonian.*

2. Constructions and proof of theorem

We begin to describe our constructions. Certain graphs which occur repeatedly as subgraphs will be denoted by capital letters. As the first example, Fig. 1 shows a subgraph A. The "dangling" edges are not part of the subgraph, but show how it is to be joined into a graph. Let W be a subgraph obtained from A by replacing all its interior 10-gons with copies of the subgraph Z, as shown in Fig. 2. Two vertices with numerical labels 1 and 2 show how a particular 10-gon is to be replaced with the copy of Z in the subgraph W. Ten vertices of the inside face of A (or W respectively) are labelled by integer labels from $\{3,4,\dots,12\}$.

For any graph G, let $v_i(G)$ or v_i denote the number of i-valent vertices of G and $s_i(G)$ or s_i denote the number of i-gons of G. By a path through a subgraph H we mean a path whose ends are not in the subgraph H. By a path of type P^H_{ij} we mean a path through a subgraph H that contains dangling edges of H which are incident with the vertices with labels i and j. It is easy to see that all "heavy" edges determine a path in W (see Fig. 1 and 2). We denote it by Q. Note that Q is of type P^W_{34} and contains all 5-valent vertices of W.

Let K be the subgraph shown in Fig. 3 and let Q' denote the path which is determined by all heavy edges of K. Let G_1 be the graph obtained from one copy of W and one copy of K by adding ten "new" edges as follows: For any $i, j \in \{3, 4, \dots, 12\}$, a new edge of G_1 will join the vertex i of W to the vertex j of K if and only if the following condition is satisfied: $i + j \equiv 7 \pmod{10}$. Now we label the vertices 3 and 4 of K by X and Y and the vertices 4 and 3 of W by X' and Y' in the graph G_1, respectively. Then we delete all numerical labels in G_1. It is easy to verify that G_1 is a graph from T, moreover $Q \cup Q' \cup XX' \cup YY'$ is a circuit in G_1 (denote it by C_1) which contains all 5-valent vertices of G_1. Note that every subgraph K' of G_1 which is isomorphic to K has the following property (denote it by $\mathcal{P}$):

$K' \cap C_1$ is a path of type $P_{xy}^{K'}$ in G_1 which contains all 5-valent vertices of K'.

Let T_1 be the class of graphs which contains only the graph G_1. For $n \geq 2$, we shall say that the graph G is in the class of graphs T_n if and only if it can be obtained from a graph G_{n-1} of T_{n-1} when one (suitably chosen) copy of K in G_{n-1} is replaced by a copy of W in G in such a way that vertices X and Y of the copy of K are replaced by vertices 3 and 4 of the copy of W.

Let G_n be a graph from T_n, $n \geq 1$. It is easy to see that all heavy edges determine a circuit (denote it by C_n) in G_n which contains all 5-valent vertices of G_n, moreover, every subgraph K' of G_n which is isomorphic to K has the property $\mathcal{P}$.

Now we consider the class of graphs $T' = \bigcup_{i \geq 1} T_i$.

In [11; Theorem (1)], it was shown in a dual form that $T' = T$. Let G be a graph from $T = T' = S(3,5,4;4,4,6)$. Note that G contains only edges of type $(3,5;4,4)$ or $(5,4;4,6)$, and so G is a bipartite graph and the following conditions are satisfied:

$$v(G) = v_3 + v_4 + v_5\,, \tag{1}$$

$$5v_5 = 3v_3 + 4v_4\,, \tag{2}$$

$$3s_6 = 2v_4\,. \tag{3}$$

From Euler's famous formula,

$$\sum_{i \geq 1} (4 - i)(s_i + v_i) = 8\,.$$

By using (1), (2) and (3), it follows that

$$v_5 = \frac{12}{31}\big(v(G) - 1\big)\,.$$

Since every circuit in the bipartite graph G which contains n vertices must contain $n/2$ vertices of degree 5, then the length of any longest circuit is less than or equal to $2v_5$. From the constructions shown before it follows that for every graph G from T

$$h(G) = \frac{24\big(v(G) - 1\big)}{31} .$$

Parts (1) and (3) of Theorem follow immediately. The proof of part (4) uses a similar construction to that shown before.

More precisely, let G_0^* denote the medial graph of the dodecahedron, shown in Fig. 4a. It is a planar 3-connected graph with all edges of type $(4,4;3,5)$, and it is easy to see that it is the unique graph with these properties. The edge A_1A_2 of G_0^* is directed by a couple of arrows. This "double direction" we will use in our construction. Let L be the configuration obtained from G_0^* by deleting all "dashed" edges in Fig. 4a.

Let G_1^* denote the graph obtained as follows: Embed in each face α of a dodecahedron graph a configuration L in such a way that the vertices $A_1, A_2, \dots, A_5$ coincide with the vertices of α, and the double direction of the edge A_1A_2 coincides with the double direction of a directed edge of α such that its small crossing arrow tends to the inside of α (see Fig. 5). Then delete all original edges (or the directed edges) of the dodecahedron graph. Note that G_1^* is from R.

The heavy edges (see Fig. 4b and Fig. 5) determine a circuit in G_1^* which contains all vertices of G_1^*, except several "white" vertices of some copies of configuration L. It is easy to verify that this circuit can be "enlarged" into a Hamiltonian circuit which contains no dashed edges in G_1^*. Hence G_1^* is Hamiltonian.

Let M denote the configuration marked by a thick boundary line in Fig. 4a, and let N denote the configuration obtained from G_1^* by deleting one (any one) copy of the configuration M.

Let R_1 be the class of graphs which contains only the graph G_1^*. For $n \geq 2$, we shall say that the graph G^* is in the class of graphs R_n if and only if it can be obtained from a graph G_{n-1}^* of R_{n-1} when one (suitably chosen) copy of M in G_{n-1}^* is replaced with a copy of N in G^* in such a way that vertices X and Y of the copy of M coincide with the vertices Z and Y of the copy of N (see Fig. 4b).

It is easy to see that all graphs in R_n are Hamiltonian, and that R_n contains exactly all graphs dual to those from T_n. Since $R = \bigcup_{i \geq 1} R_i$, part (4) of the theorem follows.

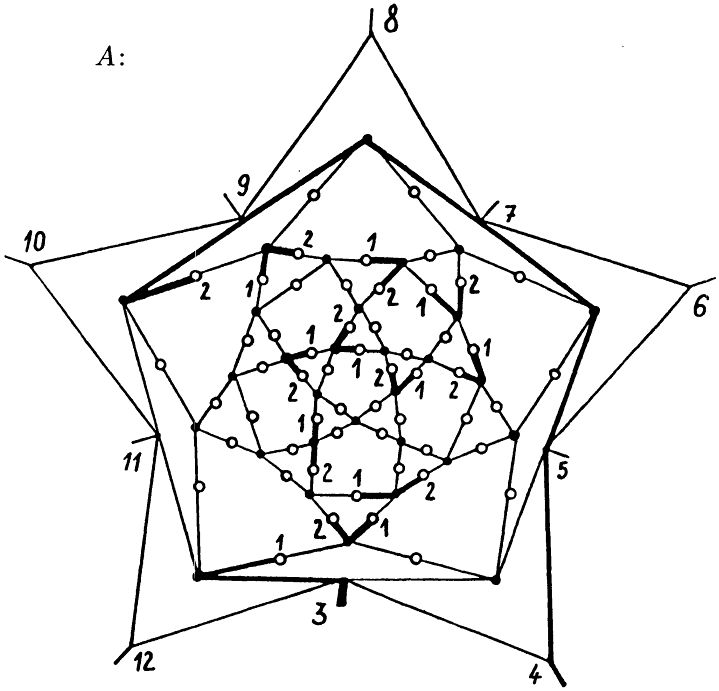

Figure 1.

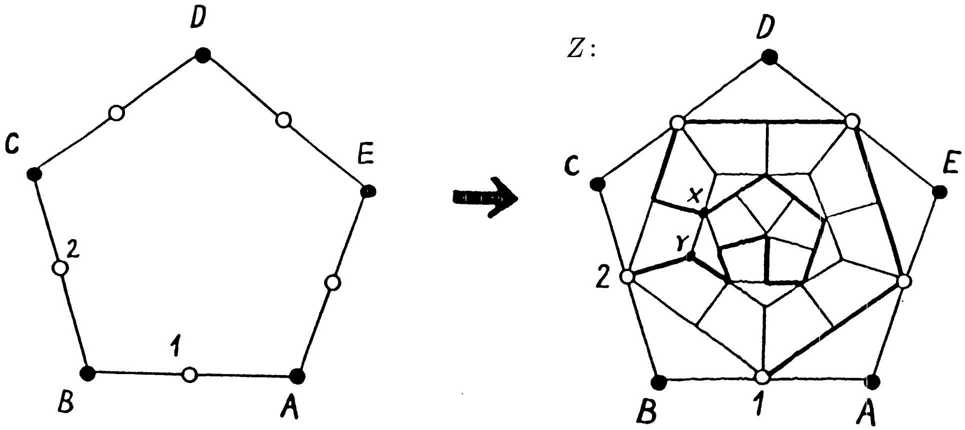

Figure 2.

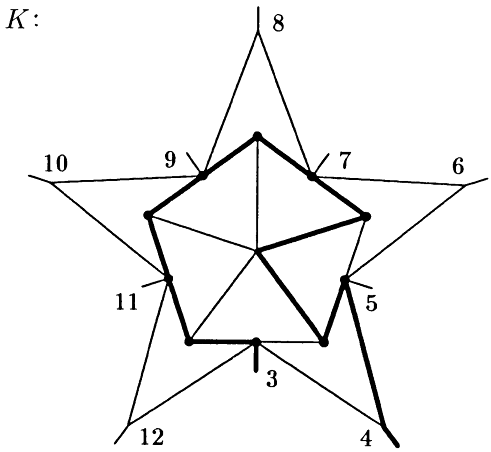

Figure 3.

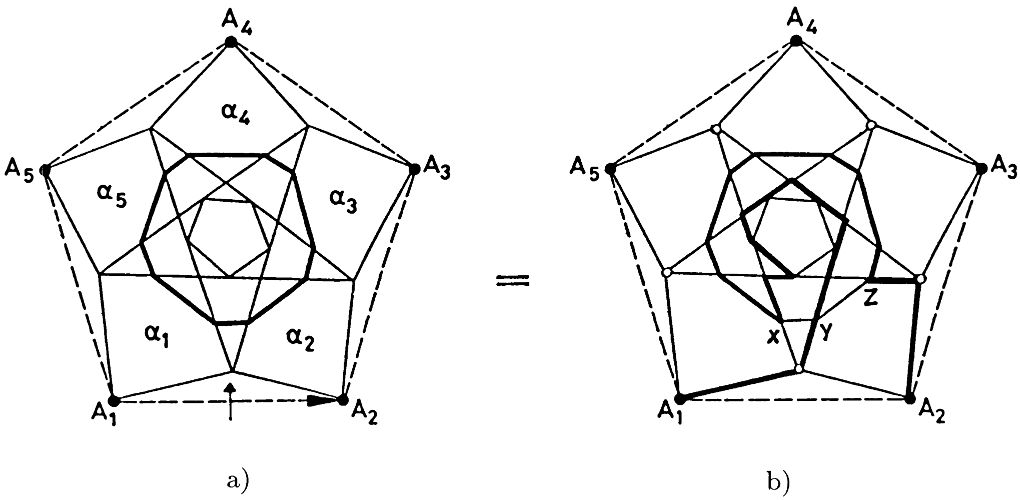

Figure 4.

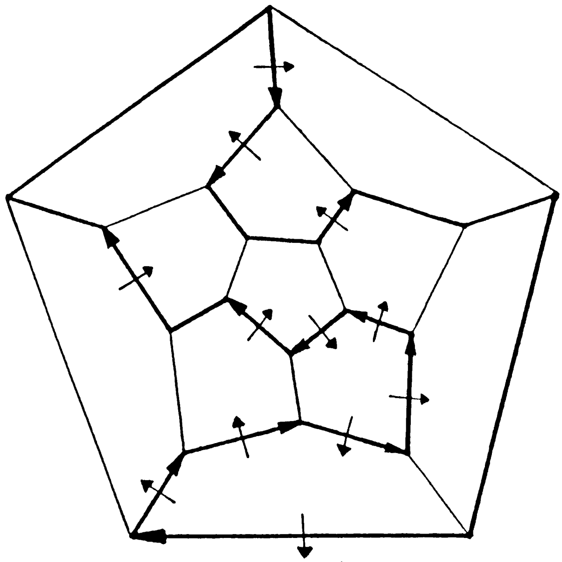

Figure 5.

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