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ON EXTENSION OF SUBMEASURES

IVAN DOBRAKOV

Let \( \mathcal{R} \) be a ring of subsets of a non-empty set \( T \). According to Definition 1 in [1] we say that a set function \( \mu : \mathcal{R} \to [0, +\infty) \) is a submeasure if it is 1) monotone, 2) continuous: \( A_n \in \mathcal{R}, \ n = 1, 2, \ldots, \) and \( A_n \setminus \emptyset \) implies \( \mu(A_n) \to 0 \), and subadditively continuous: For every \( A \in \mathcal{R} \) and \( \varepsilon > 0 \) there is a \( \delta > 0 \) such that \( B \in \mathcal{R} \) and \( \mu(B) < \delta \) implies \( \mu(A) - \varepsilon \leq \mu(A - B) \leq \mu(A) \leq \mu(A \cup B) \leq \mu(A) + \varepsilon \). If the \( \delta \) in condition 3) is uniform with respect to \( A \in \mathcal{R} \), then we say that \( \mu \) is a uniform submeasure. It is easy to verify, see page 68 in [2], that subadditive continuity is equivalent to the following property 3)*: If \( A, A_n \in \mathcal{R}, \ n = 1, 2, \ldots \) and \( \mu(A \Delta A_n) \to 0 \), then \( \mu(A_n) \to \mu(A) \). Similarly, the uniform subadditive continuity is equivalent to the following one: 3u)*: for each \( \varepsilon > 0 \) there is a \( \delta > 0 \) such that \( A, B \in \mathcal{R} \) and \( \mu(A \Delta B) < \delta \Rightarrow |\mu(A) - \mu(B)| < \varepsilon \). If instead of 3) we have \( \mu(A \cup B) \leq \mu(A) + \mu(B) \) for every \( A, B \in \mathcal{R} \), or \( \mu(A \cup B) = \mu(A) + \mu(B) \) for every \( A, B \in \mathcal{R} \), \( A \cap B = \emptyset \), then we say that \( \mu \) is a subadditive or an additive submeasure, respectively. Obviously subadditive, and particularly additive submeasures (i.e., countable additive measures) are uniform.

We say that a set function \( \mu : \mathcal{R} \to [0, +\infty) \) is exhaustive if \( \mu(A_n) \to 0 \) for each infinite sequence \( A_n \in \mathcal{R}, \ n = 1, 2, \ldots \) of pairwise disjoint sets. In Theorem 18 in [1] we proved, see also [3] for another proof, that a uniform, subadditive or additive submeasure \( \mu : \mathcal{R} \to [0, +\infty) \) has a unique extension of the same type to \( \sigma(\mathcal{R}) \) the \( \sigma \)-ring generated by \( \mathcal{R} \), if and only if it is exhaustive. Two additional, rather clumsy conditions were needed to obtain the extension theorem for non-uniform submeasures. In this note, using a more transparent approach we show that these conditions may be replaced by the following: (ii) below, and \( A_n \in \mathcal{R}, \ n = 1, 2, \ldots \) and \( \mu(A_n \Delta A_m) \to 0 \) as \( n, m \to \infty \) implies that \( \mu(A_n) - \mu(A_m) \to 0 \) as \( n, m \to \infty \).

We start with a set function \( \mu : \mathcal{R} \to [0, +\infty) \) having the following properties:

(i) \( \mu \) is monotone and \( \mu(\emptyset) = 0 \),
(ii) \( \mu \) has the pseudometric generating property, briefly the (p.g.p.), see Theorem 1 in [2]: For each \( \varepsilon > 0 \) there is a \( \delta > 0 \) such that \( A, B \in \mathcal{R} \), \( \mu(A) \), \( \mu(B) < \delta \) implies \( \mu(A \cup B) < \varepsilon \), and
(iii) \( \mu \) has the Fatou property, briefly the (F.p.): \( A, A_n \in \mathcal{R}, \ n = 1, 2, \ldots \) and \( A_n \nearrow A \) implies \( \mu(A_n) \nearrow \mu(A) \).
Put $\mathcal{R}_\sigma(\mathcal{R}_\alpha) = \{ A : \text{there are } A_n \in \mathcal{R}, n = 1, 2, \ldots \text{ such that } A_n \nrightarrow A \}$, and $\mathcal{R}^* = \{ A : A \subseteq B \text{ for some } B \in \mathcal{R}_\sigma \}$. Clearly $\mu$ has a unique extension $\mu : \mathcal{R}_\sigma \to [0, +\infty]$ defined by the equality $\mu(A) = \lim_{n\to\infty} \mu(A_n)$, where $A_n \in \mathcal{R}, n = 1, 2, \ldots$ and $A_n \nrightarrow A$, and $\mu$ on $\mathcal{R}_\sigma$ shares the properties of $\mu$ on $\mathcal{R}$.

For $A \in \mathcal{R}^*$ define $\mu^*(A) = \inf \{ \mu(B) : B \in \mathcal{R}_\sigma, B \supseteq A \}$. Then:

a) $\mu^*/\mathcal{R}_\sigma = \mu$,

b) $\mu^*$ is monotone, and

c) there is a sequence of positive numbers $\delta_k, k = 1, 2, \ldots$ such that $\delta_k \downarrow 0$, $0 < \delta_k \leq 2^{-k}$, and $A_k \in \mathcal{R}^*, \mu^*(A_k) < \delta_k, k = 1, 2, \ldots$ implies

$$\mu^* \left( \bigcup_{i=k+1}^{\infty} A_i \right) \leq \delta_k.$$  

Obviously $\mathcal{N}^* = \{ N : N \in \mathcal{R}^* \text{ and } \mu^*(N) = 0 \}$ is a hereditary $\sigma$-ring.

We shall also need another extension of $\mu$, namely we put

$$\hat{\mathcal{R}}_\sigma = \{ A : \text{there are } A_n \in \mathcal{R}, n = 1, 2, \ldots \text{ such that } A_n \nrightarrow A \text{ and } \mu(A - A_n) \to 0 \},$$

$$\hat{\mathcal{R}}_\alpha = \{ A : \text{there are } A_n \in \mathcal{R}, n = 1, 2, \ldots \text{ such that } A_n \nrightarrow A \text{ and } \mu(A_n - A) \to 0 \},$$

$$\hat{\mathcal{R}} = \{ A : A \subseteq B \text{ for some } B \in \hat{\mathcal{R}}_\sigma \},$$

and for $A \in \hat{\mathcal{R}}$ we define $\hat{\mu}(A) = \inf \{ \mu(B), B \in \hat{\mathcal{R}}_\sigma, B \supseteq A \}$.

Then it is easy to see that $\hat{\mathcal{R}}$ is a hereditary ring, the restriction of $\hat{\mu}$ to $\hat{\mathcal{R}}_\sigma$ equals $\mu$, and

c) there is a sequence of positive numbers $\delta_k, k = 1, 2, \ldots$ such that $\delta_k \downarrow 0$, $0 < \delta_k \leq 2^{-k}$, and $A_k \in \hat{\mathcal{R}}$, $\hat{\mu}(A_k) < \delta_k, k = 1, 2, \ldots$ implies that $\bigcup_{i=k+1}^{\infty} A_i \in \hat{\mathcal{R}}$ and

$$\hat{\mu} \left( \bigcup_{i=k+1}^{\infty} A_i \right) \leq \delta_k.$$  

Clearly $\hat{\mathcal{N}} = \{ N : N \in \hat{\mathcal{R}}, \hat{\mu}(N) = 0 \}$ is a hereditary $\sigma$-ring, and since $\hat{\mu}(A) \geq \mu^*(A)$ for each $A \in \hat{\mathcal{R}}, \hat{\mathcal{N}} \subseteq \mathcal{N}^*$.

For $\mathcal{Q} \subseteq \mathcal{R}^*$ we define its closure $\mathcal{Q}$ by the equality $\mathcal{Q} = \{ A : A \in \mathcal{R}^*, \text{ and there are } A_n \in \mathcal{Q}, n = 1, 2, \ldots \text{ such that } \mu^*(A_n \Delta A) \to 0 \}$.

Similarly for $\mathcal{Q} \subseteq \hat{\mathcal{R}}$ we define its closure $\mathcal{Q}$ using $\hat{\mathcal{R}}$ and $\hat{\mu}$.

**Theorem 1.** Let $\mathcal{Q} \subseteq \mathcal{R}^*$ be a ring, and let $E_n \in \mathcal{Q}, n = 1, 2, \ldots$ be such that $\mu^*(E_n \Delta E_m) \to 0$ as $n, m \to \infty$. Then there is a subsequence $\{ E_{n_k} \}$ in $\{ E_n \}$ such that:

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1) \( F_k = \bigcup_{i = k}^{\infty} E_i \in \mathcal{J}_{\sigma}, \ G_k = \bigcap_{i = k}^{\infty} E_i \in \mathcal{J}_{\delta}, \) and \( \mu^*(F - G) = 0, \) where \( F = \bigcap_{k = 1}^{\infty} \bigcup_{i = k}^{\infty} E_i = \limsup_{n \to \infty} E_n \) and \( G = \bigcup_{k = 1}^{\infty} \bigcap_{i = k}^{\infty} E_i = \liminf_{n \to \infty} E_n \) (\( \mathcal{J}_{\sigma} \) and \( \mathcal{J}_{\delta} \) are defined using \( \mu^* \)), and

2) \( \mu^*(E_n \Delta F) \to 0 \) as \( n \to \infty. \)

Analogous results hold in \( \mathcal{R} \) with \( \hat{\mu}. \)

**Proof.** Take a sequence \( \{\delta_k\}_{k = 1}^{\infty} \) according to the property c) of \( \mu^* \) above, and then a subsequence \( \{E_{n_k}\} \subset \{E_n\} \) such that \( \mu^*(E_{n_{k+1}} \Delta E_{n_k}) < \delta_k \) for each \( k = 1, 2, \ldots. \) Then

\[
\mu^* \left( \bigcup_{i = k}^{\infty} (E_{n_{i+1}} \Delta E_{n_i}) \right) \leq \delta_{k-1} \quad \text{for} \quad k = 2, 3, \ldots,
\]

hence

\[
F_k = \bigcup_{i = k}^{\infty} E_i = E_{n_k} \cup \bigcup_{i = k}^{\infty} (E_{n_{i+1}} \Delta E_{n_i}) \in \mathcal{J}_{\sigma},
\]

and

\[
G_k = \bigcap_{i = k}^{\infty} E_i = E_{n_k} - \bigcup_{i = k}^{\infty} (E_{n_{i+1}} \Delta E_{n_i}) \in \mathcal{J}_{\delta}.
\]

Further, since

\[
F_k - G_k = \bigcup_{i = k}^{\infty} (E_{n_{i+1}} \Delta E_{n_i}),
\]

\[
0 \leq \mu^*(F - G) \leq \mu^*(F_k - G_k) \leq \delta_{k-1} \to 0.
\]

Hence \( \mu^*(F - G) = 0. \)

2) now follows immediately from the inclusions

\[
E_n \Delta F = E_n \Delta E_{n_k} \Delta E_{n_k} \Delta F_k \Delta F_k \Delta F \subset (E_n \Delta E_{n_k}) \cup (E_{n_k} \Delta E_{n_k}) \cup \bigcup_{i = k}^{\infty} (E_{n_{i+1}} \Delta E_{n_i}).
\]

Analogous arguments yield the corresponding assertions for \( \mathcal{R} \) and \( \hat{\mu}. \)

**Corollary 1.** Any \( \sigma \)-ring \( \mathcal{R} \subset \mathcal{R}^*(\mathcal{R}) \) is complete with respect to \( \varphi, \) \( \varphi(E, F) = \mu^*(E \Delta F) \) \( (= \hat{\mu}(E \Delta F)). \)

**Corollary 2.** \( \mathcal{R}^*(\mathcal{R}) \) is complete with respect to \( \varphi. \)

**Corollary 3.** The closure \( \mathcal{J} (\mathcal{J}) \) of a ring \( \mathcal{J} \subset \mathcal{R}^*(\mathcal{R}) \) is a ring which is complete in \( \varphi, \) and \( \mathcal{J} \subset \sigma(\mathcal{J}) \cup \mathcal{N}^* \) \( (\mathcal{J} \subset \sigma(\mathcal{J}) \cup \mathcal{N}). \)
Theorem 2. Let \( \mathcal{D} \subset \mathcal{R}^* \) be a ring and let \( \mu^*: \mathcal{J} \to [0, +\infty] \) be exhaustive. Then 
\[ \mu^*(A_n \triangle A) = 0 \quad \text{whenever} \quad A_n \in \mathcal{J}, \quad n = 1, 2, \ldots \] and \( A_n \to A \) (i.e. 
if \( \liminf_n A_n = \limsup_n A_n = A \)), particularly \( \mu^* \) is exhaustive on \( \mathcal{J} \). Analogous results hold in \( \mathcal{R} \) with \( \hat{\mu} \). Particularly, if \( \mu: \mathcal{R} \to [0, +\infty] \) is exhaustive, then \( \hat{\mathcal{J}} = \hat{\mathcal{R}}, \quad \hat{\mu} = \mu^* \) on \( \mathcal{R}^* \), \( N^* = \hat{N} = \mathcal{N} \), \( \hat{\mathcal{R}} = \sigma(\mathcal{R}) \cup \mathcal{N} \), and \( \mu^*: \sigma(\mathcal{R}) \cup \mathcal{N} \to [0, +\infty] \) is continuous.

Proof. First we show that \( \mu^*: \mathcal{J} \to [0, +\infty] \) is exhaustive. Suppose the contrary. Take a sequence \( \{ \delta_n \}_{n=1}^\infty \) according to the property c) of \( \mu^* \). Then there is a positive integer \( k \) and a sequence of pairwise disjoint sets \( A_n \in \mathcal{J} \), \( n = 1, 2, \ldots \) such that 
\[ \mu^*(A_n) > \delta_k \quad \text{for each} \quad n = 1, 2, \ldots \]
For each \( k = 1, 2, \ldots \), take \( B_n \) so that 
\[ \mu^*(A_n \triangle B_n) < \delta_{k+3} \cdot n \cdot n_k \]
Since for \( n \neq m \) \( B_n \cap B_m \subset (A_n \triangle B_n) \cup (A_m \triangle B_m) \), 
\[ \mu^*(B_n \cap B_m) < \delta_{k+2+n \cdot n_k} \]
Put \( C_1 = B_1 \) and \( C_n = B_n - (B_1 \cup \ldots \cup B_{n-1}) \) for \( n > 1 \). Then 
\( C_n, n = 1, 2, \ldots \) are pairwise disjoint elements of \( \mathcal{J} \), hence by exhaustivity of \( \mu^* \) on \( \mathcal{J} \) there is an \( n_0 \) such that 
\[ \mu^*(C_{n_0}) < \delta_{k+3} \quad \text{for each} \quad n \geq n_0. \]
Since \( B_n - C_n = (B_1 \cap B_n) \cup \ldots \cup (B_{n-1} \cap B_n) \), 
\[ \mu^*(B_n - C_n) < \delta_{k+2} \quad \text{for each} \quad n = 1, 2, \ldots \]
Thus \( \mu^*(B_n) \leq \mu^*(B_n - C_n) + \mu^*(C_n) < \delta_{k+4} \quad \text{for each} \quad n \geq n_0. \)
Hence for \( n \geq n_0 \) we have the contradiction 
\[ \mu^*(A_n) = \mu^*(A_n \triangle B_n) < \delta_k. \]

The inclusion \( \mathcal{J} \subset \sigma(\mathcal{J}) \cup \mathcal{N}^* \) follows from Corollary 3 above. Since clearly \( \mathcal{J} \) is a ring containing \( \mathcal{J} \) and \( \mathcal{N}^* \), to show that \( \sigma(\mathcal{J}) \cup \mathcal{N}^* \subset \mathcal{J} \) it is enough to prove that \( \mathcal{J} \) contains the union of any sequence of pairwise disjoint sets from \( \mathcal{J} \).

Let \( A_n \in \mathcal{J}, \quad n = 1, 2, \ldots \) be pairwise disjoint sets. Since \( \mu^*: \mathcal{J} \to [0, +\infty] \) is exhaustive, for each \( k = 2, 3, \ldots \) there is an \( n_k > n_k - 1 \) such that 
\[ \mu^*(\bigcup_{i=1}^{n_k} A_i) < \delta_k \quad \text{for each} \quad p = 1, 2, \ldots \]
Thus \( \mu^*(\bigcup_{i=n_k}^{n_k+p-1} A_i) < \delta_k \quad \text{for each} \quad j = 1, 2, \ldots \), hence 
\[ \mu^*(\bigcup_{i=1}^{n_k+p-1} A_i - \bigcup_{i=1}^{n_k-1} A_i) = \mu^*(\bigcup_{i=n_k}^{n_k+p} A_i) = \left( \bigcup_{i=n_k}^{n_k+p} A_i \right) \leq \delta_k. \]
for each \( k = 2, 3, \ldots \). Hence \( \bigcup_{n=1}^\infty A_n \in \mathcal{J}, \) which we wanted to show. Thus \( \mathcal{J} = \sigma(\mathcal{J}) \cup \mathcal{N}^* \).

Since \( A_n \to A \) means \( \limsup_n (A_n \triangle A) = 0 \), and since \( \mathcal{J} \) is a \( \sigma \)-ring, for the second assertion of the theorem it is enough to show that \( \mu^* \) is continuous on \( \mathcal{J} \). Let 
\( A_n \in \mathcal{J}, \quad n = 1, 2, \ldots \), and let \( A_n \setminus \emptyset \). Then \( B_n = A_n - A_{n+1}, n = 1, 2, \ldots \) are pairwise disjoint and \( A_n = \bigcup_{i=1}^\infty B_i \). Now in the same way as in the paragraph above we obtain that \( \mu^*(A_n) \to 0 \).

Analogous arguments yield the results for \( \mathcal{R} \) and \( \hat{\mu} \). The rest of the theorem is evident.
Let \( \mu : \mathcal{R} \to [0, +\infty) \) be a subadditive or a uniform submeasure. Then it is easy to see that \( \mu^*: \mathcal{R}^* \to [0, +\infty] \) is subadditive, or is uniformly subadditively continuous, respectively. Hence, as a corollary, we immediately have the extension theorem for such submeasures, see also Theorem 18 in [1].

**Corollary.** An additive, subadditive or uniform submeasure \( \mu : \mathcal{R} \to [0, +\infty) \) has a unique extension \( \mu : \sigma(\mathcal{R}) \to [0, +\infty) \) of the same type if and only if it is exhaustive.

The uniqueness of the extension follows immediately from Corollary 3 of Theorem 15 in [1]. If \( \mu : \mathcal{R} \to [0, +\infty) \) is additive, then the additivity of \( \mu^*: \sigma(\mathcal{R}) \to [0, +\infty) \) may be proved in the following way: Let \( A, B \in \sigma(\mathcal{R}) \), and let \( A \cap B = \emptyset \). Take \( A_n, B_n \in \mathcal{R} \), \( n = 1, 2, \ldots \) so that \( \mu(A_n \Delta A) \to 0 \) and \( \mu(B_n \Delta B) \to 0 \). Then \( \mu(A_n) \to \mu(A) \) and \( \mu(B_n) \to \mu(B) \), hence by additivity of \( \mu \) on \( \mathcal{R} \) we have:

\[
\mu(A \cup B) = \lim_{n \to \infty} \mu(A_n \cup B_n) = \lim_{n \to \infty} \mu(A_n - B_n) + \lim_{n \to \infty} \mu(B_n - A_n) = 2\mu(A \cup B) - \mu(A) - \mu(B),
\]

hence \( \mu(A \cup B) = \mu(A) + \mu(B) \).

Concerning subadditively continuous extensions we have

**Theorem 3.** The following conditions are equivalent:

a) \( \hat{\mu} : \mathcal{R} \to [0, +\infty) \) is subadditively continuous,

b) If \( A_n \in \mathcal{R} \), \( n = 1, 2, \ldots \) and \( \mu(A_n \Delta A_m) \to 0 \) as \( n, m \to \infty \), then for each \( \varepsilon > 0 \) there is a \( \delta > 0 \) such that \( B \in \mathcal{R} \) and \( \mu(B) < \delta \) implies \( \mu(A_n) - \varepsilon \leq \mu(A_n - B) \leq \mu(A_n) \leq \mu(A_n \cup B) \leq \mu(A_n) + \varepsilon \) for each \( n = 1, 2, \ldots \) and

c) If \( A_n \in \mathcal{R} \), \( n = 1, 2, \ldots \) and \( \mu(A_n \Delta A_m) \to 0 \) as \( n, m \to \infty \), then \( \mu(A_n) - \mu(A_m) \to 0 \) as \( n, m \to \infty \).

**Proof.** a) \( \Rightarrow \) b). Let \( A_n \in \mathcal{R} \), \( n = 1, 2, \ldots \) be such that \( \mu(A_n \Delta A_m) \to 0 \) as \( n, m \to \infty \). By Corollary 2 of Theorem 1 there is an \( A \in \mathcal{R} \) such that \( \hat{\mu}(A_n \Delta A) \to 0 \). Let \( \varepsilon > 0 \). By the subadditivity of \( \hat{\mu} \) on \( \mathcal{R} \) there is a \( \delta_0 > 0 \) such that \( B \in \mathcal{R} \) and \( \hat{\mu}(B) < \delta_0 \) implies \( \hat{\mu}(A) - 2^{-1} \cdot \varepsilon \leq \hat{\mu}(A - B) \leq \hat{\mu}(A) \leq \hat{\mu}(A \cup B) \leq \hat{\mu}(A) + 2^{-1} \cdot \varepsilon \). Further, by the (p.g.p.) of \( \hat{\mu} \) there is a \( \delta_0 < \delta_0 \) such that \( B, B_1 \in \mathcal{R} \) and \( \hat{\mu}(B), \hat{\mu}(B_1) < \delta_0 \) implies \( \hat{\mu}(B \cup B_1) < \delta_0 \). Take \( n_0 \) so that \( \hat{\mu}(A \Delta A_n) < \delta_0 \) for \( n \geq n_0 \). Then for \( n \geq n_0 \) and for \( B \in \mathcal{R} \) with \( \hat{\mu}(B) < \delta_0 \) we have the inequalities \( \hat{\mu}(A) - 2^{-1} \cdot \varepsilon \leq \hat{\mu}(A - (B \cup (A - A_n))) \leq \hat{\mu}(A_n - B) \leq \hat{\mu}(A_n) \leq \hat{\mu}(A_n \cup B) \leq \hat{\mu}(A \cup (A_n - A) \cup B) \leq \hat{\mu}(A) + 2^{-1} \cdot \varepsilon \). Hence for such \( n \) and \( B \) we have the inequalities \( \hat{\mu}(A_n) - \varepsilon \leq \hat{\mu}(A_n - B) \leq \hat{\mu}(A_n) \leq \hat{\mu}(A_n \cup B) \leq \hat{\mu}(A_n) + \varepsilon \). Finally, by the subadditivity of \( \hat{\mu} \) we take \( \delta_1, \ldots, \delta_{n_0} \) corresponding to \( \varepsilon \) and \( A_1, \ldots, A_{n_0} \) respectively, and we put \( \delta = \min \{ \delta_0, \delta_1, \ldots, \delta_{n_0} \} \).

Clearly b) \( \Rightarrow \) c).
c) $\Rightarrow$ a). For $A \in \mathcal{R}$ put $\mu(A) = \lim_{n \to \infty} \mu(A_n)$, where $A_n \in \mathcal{R}$, $n = 1, 2, ...$ and $\mu(A_n \Delta A) \to 0$. By c) $\mu$ is clearly unambiguously defined. First we show that $\mu: \mathcal{R} \to [0, +\infty)$ is subadditively continuous, and then that $\mu(A) = \hat{\mu}(A)$ for each $A \in \mathcal{R}$.

Suppose $\hat{\mu}: \mathcal{R} \to [0, +\infty)$ is not subadditively continuous. Then there is an $\varepsilon > 0$ and $A, A_n \in \mathcal{R}$, $n = 1, 2, ...$ such that $\mu(A_n \Delta A) \to 0$ and $|\mu(A_n) - \mu(A)| > \varepsilon$ for each $n = 1, 2, ...$. Take $A_{0,k}, A_{n,k} \in \mathcal{R}$, $k = 1, 2, ...$ so that $\hat{\mu}(A_{0,k} \Delta A_n) \to 0$ as $k \to \infty$, for each $n = 1, 2, ...$. Then $\mu(A) = \lim_{k \to \infty} \mu(A_{0,k}),$

$$\mu(A_n) = \lim_{k \to \infty} \mu(A_{n,k}) \text{ for each } n = 1, 2, ... \text{ and } \lim_{n \to \infty} \mu(A \Delta A_n) = \lim_{k \to \infty} \lim_{n \to \infty} \mu(A_{0,k} \Delta A_{n,k}) = 0.$$

Take a sequence $\{\delta_i\}_{i=1}^\infty$ according to the property c) of $\mu^*$. By the last equality for each $i = 1, 2, ...$ there is an $n_i$ such that $\lim_{k \to \infty} \mu(A_{0,k} \Delta A_{n_i,k}) < \delta_i$. But then for each $i$ there is a $k_i$ such that $\mu(A_{0,k_i} \Delta A_{n_i,k_i}) < \delta_i$ and $|\mu(A_{n_i,k_i}) - \mu(A_{n_i})| < i^{-1}$.

By the properties of the sequence $\{\delta_i\}_{i=1}^\infty$ the first inequality implies that the sequence $\{A_{0,k_i}, A_{n_i,k_i}, ..., A_{0,k_i}, A_{n_i,k_i} \}$ is $\delta$-Cauchy, where $\delta(E, F) = \mu(E \Delta F)$, hence by c) and the second inequality we have the contradiction

$$\mu(A) = \lim_{i \to \infty} \mu(A_{n_i,k_i}) = \lim_{i \to \infty} \mu(A_{n_i}).$$

There remains to be shown that $\mu(E) = \hat{\mu}(E)$ for each $E \in \mathcal{R}$. Let $E \in \mathcal{R}$. Take a sequence $E_n \in \mathcal{R}$, $n = 1, 2, ...$ so that $\mu(E \Delta E_n) \to 0$, and let have the notations of Theorem 1. Then $\hat{\mu}(E) = \inf \{\mu(B): E \subset B, B \in \mathcal{R}_o\} \leq \inf \mu(F_k) = \lim_{k \to \infty} \mu(F_k) = \lim_{k \to \infty} \mu(E_{n_k}) = \mu(E)$, since $\mu(F_k \Delta E_{n_k}) \to 0$ as $k \to \infty$.

On the other hand, for each $\varepsilon > 0$ there is a $B \in \mathcal{R}_o$ such that $B \supset F$ and $\hat{\mu}(F) + \varepsilon \geq \mu(B) \geq \mu(B \cap F_k) \geq \mu(F)$ for each $k$, hence $\hat{\mu}(F) \geq \mu(F) = \mu(E)$. There remains to be shown that $\hat{\mu}(F) = \hat{\mu}(E)$. Since $\mu: \mathcal{R} \to [0, +\infty)$ is subadditively continuous, and since $\hat{\mu} = \mu$ on $\mathcal{R}_o$, by the definition of $\hat{\mu}$, $\hat{\mu}: \mathcal{R} \to [0, +\infty)$ is subadditively continuous from the right, i.e., for each $A \in \mathcal{R}$ and $\varepsilon > 0$ there is a $\delta > 0$ such that $B \in \mathcal{R}$, $\hat{\mu}(B) < \delta$ implies $\hat{\mu}(A \cup B) \leq \hat{\mu}(B) + \varepsilon$. From this, since $\hat{\mu}(E \Delta F) = 0$ we immediately have the required equality $\hat{\mu}(F) = \hat{\mu}(E)$. The theorem is proved.

From Theorems 2 and 3, and Theorem 3-b) in [1] we immediately have (the uniqueness follows easily from Corollary 3 of Theorem 15 in [1]) our extension theorem for submeasures, compare with Theorem 18 in [1].

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Theorem 4. (Extension Theorem for Submeasures.) A submeasure \( \mu : \mathcal{R} \to [0, +\infty) \) has a unique extension to \( \sigma(\mathcal{R}) \)- the \( \sigma \)-ring generated by \( \mathcal{R} \), if and only if it is exhaustive on \( \mathcal{R} \), \( A_n \in \mathcal{R}, n = 1, 2, \ldots \) and \( \mu(A_n \Delta A_m) \to 0 \) as \( n, m \to \infty \) implies \( \mu(A_n) - \mu(A_m) \to 0 \) as \( n, m \to \infty \), and for each \( \varepsilon > 0 \) there is a \( \delta > 0 \) such that \( A, B \in \mathcal{R} \) and \( \mu(A), \mu(B) < \delta \) implies \( \mu(A \cup B) < \varepsilon \).

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О РАСШИРЕНИИ СУБМЕР

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Резюме

Пусть \( \mathcal{R} \) кольцо подмножеств непустого множества \( T \). Согласно с [1] функция множеств \( \mu : \mathcal{R} \to (0, +\infty) \) называется субмерой, если она монотонна, непрерывна \( (A_n \setminus \emptyset \Rightarrow \mu(A_n) \to 0) \), и полуаддитивно непрерывна \( (\forall A \in \mathcal{R} \text{ и } \forall \varepsilon > 0 \exists \delta > 0, \ B \in \mathcal{R}, \ \mu(B) < \delta \Rightarrow \mu(A) - \varepsilon \leq \mu(A - B) \leq \mu(A) \leq \mu(A \cup B) \leq \mu(A) + \varepsilon) \). Последнее условие можно заменить следующим: \( A, A_n \in \mathcal{R}, n = 1, 2, \ldots \) и \( \mu(A_n \Delta A) \to 0 \Rightarrow \mu(A_n) \to \mu(A) \). Необходимые и достаточные условия для расширения субмеры из кольца \( \mathcal{R} \) на порожденное им сигма кольцо были установлены Теоремой 18 в [1]. Условия II и III этой теоремы слишком громоздкие. В настоящей работе показывается, что их можно заменить более простыми условиями. А, именно, справедлива следующая

Теорема о расширении субмеры. Субмера \( \mu : \mathcal{R} \to (0, +\infty) \) однозначно расширяется до субмеры на сигма кольцо, порожденном \( \mathcal{R} \) тогда и только тогда, когда она не имеет ускользающей нагрузки на \( \mathcal{R} \), \( A_n \in \mathcal{R}, n = 1, 2, \ldots \) и \( \mu(A_n \Delta A_m) \to 0 \) для \( n, m \to \infty \) \( \Rightarrow \mu(A_n) - \mu(A_m) \to 0 \) для \( n, m \to \infty \), и для каждого \( \varepsilon > 0 \) существует \( \delta > 0 \) так, что \( A, B \in \mathcal{R} \) и \( \mu(A), \mu(B) < \delta \Rightarrow \mu(A \cup B) < \varepsilon \).