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ON EXTENSION OF SUBMEASURES

IVAN DOBRAKOV

Let $\mathscr{R}$ be a ring of subsets of a non-empty set T. According to Definition 1 in [1] we say that a set function $\mu\colon \mathscr{R}\to[0, +\infty)$ is a submeasure if it is 1) monotone, 2) continuous: $A_n\in\mathscr{R}$, $n=1, 2, \ldots$, and $A_n\searrow\emptyset$ implies $\mu(A_n)\to 0$, and subadditively continuous: For every $A\in\mathscr{R}$ and $\varepsilon>0$ there is a $\delta>0$ such that $B\in\mathscr{R}$ and $\mu(B)<\delta$ implies $\mu(A)-\varepsilon\leqq\mu(A-B)\leqq\mu(A)\leqq\mu(A\cup B)\leqq\mu(A)+\varepsilon$. If the δ in condition 3) is uniform with respect to $A\in\mathscr{R}$, then we say that μ is a uniform submeasure. It is easy to verify, see page 68 in [2], that subadditive continuity is equivalent to the following property 3)*: If A, $A_n\in\mathscr{R}$, $n=1, 2, \ldots$ and $\mu(A\triangle A_n)\to 0$, then $\mu(A_n)\to\mu(A)$. Similarly, the uniform subadditive continuity is equivalent to the following one: 3u)*: for each $\varepsilon>0$ there is a $\delta>0$ such that A, $B\in\mathscr{R}$ and $\mu(A\triangle B)<\delta\Rightarrow|\mu(A)-\mu(B)|<\varepsilon$. If instead of 3) we have $\mu(A\cup B)\leqq\mu(A)+\mu(B)$ for every A, $B\in\mathscr{R}$, or $\mu(A\cup B)=\mu(A)+\mu(B)$ for every A, $B\in\mathscr{R}$, $A\cap B=\emptyset$, then we say that μ is a subadditive or an additive submeasure, respectively. Obviously subadditive, and particularly additive submeasures (i.e., countable additive measures) are uniform.

We say that a set function $\mu\colon \mathscr{R}\to[0, +\infty]$ is exhaustive if $\mu(A_n)\to 0$ for each infinite sequence $A_n\in\mathscr{R}$, $n=1, 2, \ldots$ of pairwise disjoint sets. In Theorem 18 in [1] we proved, see also [3] for another proof, that a uniform, subadditive or additive submeasure $\mu\colon \mathscr{R}\to[0, +\infty)$ has a unique extension of the same type to $\sigma(\mathscr{R})$ – the σ-ring generated by $\mathscr{R}$, if and only if it is exhaustive. Two additional, rather clumsy, conditions were needed to obtain the extension theorem for non-uniform submeasures. In this note, using a more transparent approach we show that these conditions may be replaced by the following: (ii) below, and $A_n\in\mathscr{R}$, $n=1, 2, \ldots$ and $\mu(A_n\triangle A_m)\to 0$ as n, $m\to\infty$ implies that $\mu(A_n)-\mu(A_m)\to 0$ as n, $m\to\infty$.

We start with a set function $\mu\colon \mathscr{R}\to[0, +\infty)$ having the following properties:

(i) μ is monotone and $\mu(\emptyset)=0$,

(ii) μ has the pseudometric generating property, briefly the (p.g.p.), see Theorem 1 in [2]: For each $\varepsilon>0$ there is a $\delta>0$ such that A, $B\in\mathscr{R}$, $\mu(A)$, $\mu(B)<\delta$ implies $\mu(A\cup B)<\varepsilon$, and

(iii) μ has the Fatou property, briefly the (F.p.): A, $A_n\in\mathscr{R}$, $n=1, 2, \ldots$ and $A_n\nearrow A$ implies $\mu(A_n)\nearrow\mu(A)$.

Put $\mathscr{R}_\sigma(\mathscr{R}_\delta)=\{A$; there are $A_n\in\mathscr{R}$, $n=1,2,\ldots$ such that $A_n\nearrow(\searrow)A\}$, and $\mathscr{R}^*=\{A\colon A\subset B$ for some $B\in\mathscr{R}_\sigma\}$. Clearly μ has a unique extension $\mu\colon\mathscr{R}_\sigma\to[0,+\infty]$ defined by the equality $\mu(A)=\lim_{n\to\infty}\mu(A_n)$, where $A_n\in\mathscr{R}$, $n=1,2,\ldots$ and $A_n\nearrow A$, and μ on $\mathscr{R}_\sigma$ shares the properties of μ on $\mathscr{R}$.

For $A\in\mathscr{R}^*$ define $\mu^*(A)=\inf\{\mu(B)\colon B\in\mathscr{R}_\sigma,\ B\supset A\}$. Then:

a) $\mu^*/\mathscr{R}_\sigma=\mu$,

b) μ^* is monotone, and

c) there is a sequence of positive numbers δ_k, $k=1,2,\ldots$ such that $\delta_k\searrow 0$, $0<\delta_k\leqq 2^{-k}$, and $A_k\in\mathscr{R}^*$, $\mu^*(A_k)<\delta_k$, $k=1,2,\ldots$ implies

$$\mu^*\left(\bigcup_{i=k+1}^{\infty}A_i\right)\leqq\delta_k.$$

Obviously $\mathscr{N}^*=\{N\colon N\in\mathscr{R}^*$ and $\mu^*(N)=0\}$ is a hereditary σ-ring.

We shall also need another extension of μ, namely we put

$\hat{\mathscr{R}}_\sigma=\{A\colon$ there are $A_n\in\mathscr{R}$, $n=1,2,\ldots$ such that $A_n\nearrow A$ and $\mu(A-A_n)\to 0\}$,

$\hat{\mathscr{R}}_\delta=\{A\colon$ there are $A_n\in\mathscr{R}$, $n=1,2,\ldots$ such that $A_n\searrow A$ and $\mu(A_n-A)\to 0\}$,

$\hat{\mathscr{R}}=\{A\colon A\subset B$ for some $B\in\hat{\mathscr{R}}_\sigma\}$, and for $A\in\hat{\mathscr{R}}$ we define $\hat{\mu}(A)=\inf\{\mu(B),\ B\in\hat{\mathscr{R}}_\sigma,\ B\supset A\}$.

Then it is easy to see that $\hat{\mathscr{R}}$ is a hereditary ring, the restriction of $\hat{\mu}$ to $\hat{\mathscr{R}}_\sigma$ equals μ, and

c) there is a sequence of positive numbers δ_k, $k=1,2,\ldots$ such that $\delta_k\searrow 0$, $0<\delta_k\leqq 2^{-k}$, and $A_k\in\hat{\mathscr{R}}$, $\hat{\mu}(A_k)<\delta_k$, $k=1,2,\ldots$ implies that $\bigcup_{i=k+1}^{\infty}A_i\in\hat{\mathscr{R}}$ and $\hat{\mu}\left(\bigcup_{i=k+1}^{\infty}A_i\right)\leqq\delta_k$.

Clearly $\hat{\mathscr{N}}=\{N\colon N\in\hat{\mathscr{R}},\ \hat{\mu}(N)=0\}$ is a hereditary σ-ring, and since $\hat{\mu}(A)\geqq\mu^*(A)$ for each $A\in\hat{\mathscr{R}}$, $\hat{\mathscr{N}}\subset\mathscr{N}^*$.

For $\mathscr{Q}\subset\mathscr{R}^*$ we define its closure $\bar{\mathscr{Q}}$ by the equality $\bar{\mathscr{Q}}=\{A\colon A\in\mathscr{R}^*$, and there are $A_n\in\mathscr{Q}$, $n=1,2,\ldots$ such that

$$\mu^*(A_n\triangle A)\to 0.$$

Similarly for $\mathscr{Q}\subset\hat{\mathscr{R}}$ we define its closure $\bar{\mathscr{Q}}$ using $\hat{\mathscr{R}}$ and $\hat{\mu}$.

Theorem 1. *Let $\mathscr{Q}\subset\mathscr{R}^*$ be a ring, and let $E_n\in\mathscr{Q}$, $n=1,2,\ldots$ be such that $\mu^*(E_n\triangle E_m)\to 0$ as $n,m\to\infty$. Then there is a subsequence $\{E_{n_k}\}_1^\infty\subset\{E_n\}_1^\infty$ such that:*

1) $F_k = \bigcup_{i=k}^{\infty} E_{n_i} \in \hat{\mathfrak{Q}}_\sigma$, $G_k = \bigcap_{i=k}^{\infty} E_{n_i} \in \hat{\mathfrak{Q}}_\delta$, *and* $\mu^*(F-G)=0$, *where* $F = \bigcap_{k=1}^{\infty} \bigcup_{i=k}^{\infty} E_{n_i} =$ *lim sup* $_k$ E_{n_k} *and* $G = \bigcup_{k=1}^{\infty} \bigcap_{i=k}^{\infty} E_{n_i} = \liminf_k E_{n_k}$ ($\hat{\mathfrak{Q}}_\sigma$ *and* $\hat{\mathfrak{Q}}_\delta$ *are defined using* μ^*), *and*

2) $\mu^*(E_n \triangle F) \to 0$ *as* $n \to \infty$.

Analogous results hold in $\hat{\mathfrak{R}}$ *with* $\hat{\mu}$.

Proof. Take a sequence $\{\delta_k\}_{k=1}^{\infty}$ according to the property c) of μ^* above, and then a subsequence $\{E_{n_k}\} \subset \{E_n\}$ such that $\mu^*(E_{n_{k+1}} \triangle E_{n_k}) < \delta_k$ for each $k = 1, 2, \ldots$. Then

$$\mu^*\left(\bigcup_{i=k}^{\infty}(E_{n_{i+1}} \triangle E_{n_i})\right) \leqq \delta_{k-1} \quad \text{for} \quad k = 2, 3, \ldots,$$

hence

$$F_k = \bigcup_{i=k}^{\infty} E_{n_i} = E_{n_k} \cup \bigcup_{i=k}^{\infty}(E_{n_{i+1}} \triangle E_{n_i}) \in \hat{\mathfrak{Q}}_\sigma,$$

and

$$G_k = \bigcap_{i=k}^{\infty} E_{n_i} = E_{n_k} - \bigcup_{i=k}^{\infty}(E_{n_{i+1}} \triangle E_{n_i}) \in \hat{\mathfrak{Q}}_\delta.$$

Further, since

$$F_k - G_k = \bigcup_{i=k}^{\infty}(E_{n_{i+1}} \triangle E_{n_i}),$$

$$0 \leqq \mu^*(F-G) \leqq \mu^*(F_k - G_k) \leqq \delta_{k-1} \to 0.$$

Hence $\mu^*(F-G)=0$.

2) now follows immediately from the inclusions

$$E_n \triangle F = E_n \triangle E_{n_k} \triangle E_{n_k} \triangle F_k \triangle F_k \triangle F \subset (E_n \triangle E_{n_k}) \cup$$

$$(E_{n_k} \triangle F_k) \cup (F_k \triangle F) \subset (E_n \triangle E_{n_k}) \cup \bigcup_{i=k}^{\infty}(E_{n_{i+1}} \triangle E_{n_i}).$$

Analogous arguments yield the corresponding assertions for $\hat{\mathfrak{R}}$ and $\hat{\mu}$.

Corollary 1. *Any* σ*-ring* $\mathfrak{Q} \subset \mathfrak{R}^*(\hat{\mathfrak{R}})$ *is complete with respect to* ϱ, $\varrho(E, F) = \mu^*(E \triangle F)$ $(=\hat{\mu}(E \triangle F))$.

Corollary 2. $\mathfrak{R}^*(\hat{\mathfrak{R}})$ *is complete with respect to* ϱ.

Corollary 3. *The closure* $\bar{\mathfrak{Q}}$ $(\bar{\mathfrak{Q}})$ *of a ring* $\mathfrak{Q} \subset \mathfrak{R}^*$ $(\hat{\mathfrak{R}})$ *is a ring which is complete in* ϱ, *and* $\bar{\mathfrak{Q}} \subset \sigma(\mathfrak{Q}) \cup \mathcal{N}^*$ $(\bar{\mathfrak{Q}} \subset \sigma(\mathfrak{Q}) \cup \hat{\mathcal{N}})$.

Theorem 2. *Let $\mathcal{Q} \subset \mathcal{R}^*$ be a ring and let $\mu^*: \mathcal{Q} \to [0, +\infty]$ be exhaustive. Then $\bar{\mathcal{Q}} = \sigma(\mathcal{Q}) \cup \mathcal{N}^*$, and $\mu^*(A_n \triangle A) \to 0$ whenever $A_n \in \bar{\mathcal{Q}}$, $n = 1, 2, \dots$ and $A_n \to A$ (i.e. if $\liminf_n A_n = \limsup_n = A$), particularly μ^* is exhaustive on $\bar{\mathcal{Q}}$. Analogous results hold in $\hat{\mathcal{R}}$ with $\hat{\mu}$. Particularly, if $\mu: \mathcal{R} \to [0, +\infty)$ is exhaustive, then $\hat{\mathcal{R}}_\sigma = \mathcal{R}_\sigma$, hence $\hat{\mathcal{R}} = \mathcal{R}^*$, $\hat{\mu} = \mu^*$ on $\mathcal{R}^*$, $\mathcal{N}^* = \hat{\mathcal{N}} = \mathcal{N}$, $\bar{\mathcal{R}} = \bar{\hat{\mathcal{R}}} = \sigma(\mathcal{R}) \cup \mathcal{N}$, and $\mu^*: \sigma(\mathcal{R}) \cup \mathcal{N} \to [0, +\infty]$ is continuous.*

Proof. First we show that $\mu^*: \bar{\mathcal{Q}} \to [0, +\infty]$ is exhaustive. Suppose the contrary. Take a sequence $\{\delta_k\}_1^\infty$ according to the property c) of μ^*. Then there is a positive integer k and a sequence of pairwise disjoint sets $A_n \in \bar{\mathcal{Q}} - \mathcal{Q}$, $n = 1, 2, \dots$ such that $\mu^*(A_n) > \delta_k$ for each $n = 1, 2, \dots$. For each $n = 1, 2, \dots$ take $B_n \in \mathcal{Q}$ so that $\mu^*(A_n \triangle B_n) < \delta_{k+3+n}$. Since for $n \neq m$ $B_n \cap B_m \subset (A_n \triangle B_n) \cup (A_m \triangle B_m)$, $\mu^*(B_n \cap B_m) < \delta_{k+2+n\wedge m}$. Put $C_1 = B_1$ and $C_n = B_n - (B_1 \cup \dots \cup B_{n-1})$ for $n > 1$. Then C_n, $n = 1, 2, \dots$ are pairwise disjoint elements of $\mathcal{Q}$, hence by exhaustivity of μ^* on $\mathcal{Q}$ there is an n_0 such that $\mu^*(C_n) < \delta_{k+3}$ for each $n \geqq n_0$. Since $B_n - C_n = (B_1 \cap B_n) \cup \dots \cup (B_{n-1} \cap B_n)$, $\mu^*(B_n - C_n) < \delta_{k+2}$ for each $n = 1, 2, \dots$. Thus $\mu^*(B_n) \leqq \mu^*((B_n - C_n) \cup C_n) < \delta_{k+1}$ for each $n \geqq n_0$. Hence for $n \geqq n_0$ we have the contradiction $\mu^*(A_n) \leqq \mu^*((A_n \triangle B_n) < \delta_k$.

The inclusion $\bar{\mathcal{Q}} \subset \sigma(\mathcal{Q}) \cup \mathcal{N}^*$ follows from Corollary 3 above. Since clearly $\bar{\mathcal{Q}}$ is a ring containing $\mathcal{Q}$ and $\mathcal{N}^*$, to show that $\sigma(\mathcal{Q}) \cup \mathcal{N}^* \subset \bar{\mathcal{Q}}$ it is enough to prove that $\bar{\mathcal{Q}}$ contains the union of any sequence of pairwise disjoint sets from $\bar{\mathcal{Q}}$.

Let $A_n \in \bar{\mathcal{Q}}$, $n = 1, 2, \dots$ be pairwise disjoint sets. Since $\mu^*: \bar{\mathcal{Q}} \to [0, +\infty]$ is exhaustive, for each $k = 2, 3, \dots$ there is an $n_k > n_{k-1}$ such that $\mu^*\left(\bigcup_{i=n_k}^{n_k+p} A_i\right) < \delta_k$ for each $p = 1, 2, \dots$. Thus $\mu^*\left(\bigcup_{i=n_j}^{n_j+1} A_i\right) < \delta_j$ for each $j = 1, 2, \dots$, hence

$$\mu^*\left(\bigcup_{i=1}^{\infty} A_i - \bigcup_{i=1}^{n_k-1} A_i\right) = \mu^*\left(\bigcup_{i=n_k}^{\infty} A_i\right) = \left(\bigcup_{j=k}^{\infty} \bigcup_{i=n_j}^{n_{j+1}} A_i\right) \leqq \delta_{k-1}$$

for each $k = 2, 3, \dots$. Hence $\bigcup_{n=1}^{\infty} A_n \in \bar{\mathcal{Q}}$, which we wanted to show. Thus $\bar{\mathcal{Q}} = \sigma(\mathcal{Q}) \cup \mathcal{N}^*$.

Since $A_n \to A$ means $\limsup_n (A_n \triangle A) = \emptyset$, and since $\bar{\mathcal{Q}}$ is a σ-ring, for the second assertion of the theorem it is enough to show that μ^* is continuous on $\bar{\mathcal{Q}}$. Let $A_n \in \bar{\mathcal{Q}}$, $n = 1, 2, \dots$, and let $A_n \searrow \emptyset$. Then $B_n = A_n - A_{n+1}$, $n = 1, 2, \dots$ are pairwise disjoint and $A_n = \bigcup_{i=n}^{\infty} B_i$. Now in the same way as in the paragraph above we obtain that $\mu^*(A_n) \to 0$.

Analogous arguments yield the results for $\hat{\mathcal{R}}$ and $\hat{\mu}$. The rest of the theorem is evident.

Let $\mu\colon \mathcal{R}\to[0,+\infty)$ be a subadditive or a uniform submeasure. Then it is easy to see that $\mu^*\colon \mathcal{R}^*\to[0,+\infty]$ is subadditive, or is uniformly subadditively continuous, respectively. Hence, as a corollary, we immediately have the extension theorem for such submeasures, see also Theorem 18 in [1].

Corollary. *An additive, subadditive or uniform submeasure $\mu\colon \mathcal{R}\to[0,+\infty)$ has a unique extension $\mu\colon \sigma(\mathcal{R})\to[0,+\infty)$ of the same type if and only if it is exhaustive.*

The uniqueness of the extension follows immediately from Corollary 3 of Theorem 15 in [1]. If $\mu\colon \mathcal{R}\to[0,+\infty)$ is additive, then the additivity of $\mu^*\colon \sigma(\mathcal{R})\to[0,+\infty)$ may be proved in the following way: Let $A, B\in\sigma(\mathcal{R})$, and let $A\cap B=\emptyset$. Take $A_n, B_n\in\mathcal{R}$, $n=1,2,\dots$ so that $\mu^*(A_n\triangle A)\to 0$ and $\mu^*(B_n\triangle B)\to 0$. Then $\mu^*(A_n)\to\mu^*(A)$ and $\mu^*(B_n)\to\mu^*(B)$, hence by additivity of μ on $\mathcal{R}$ we have:

$$\mu^*(A\cup B)=\mu^*(A\triangle B)=\lim_{n\to\infty}\mu(A_n\triangle B_n)=\lim_{n\to\infty}\mu(A_n-B_n)+$$

$$\lim_{n\to\infty}\mu(B_n-A_n)=2\mu^*(A\cup B)-\mu^*(A)-\mu^*(B),$$

hence $\mu^*(A\cup B)=\mu^*(A)+\mu^*(B)$.

Concerning subadditively continuous extensions we have

Theorem 3. *The following conditions are equivalent:*

a) *$\hat\mu\colon \bar{\mathcal{R}}\to[0,+\infty)$ is subadditively continuous,*

b) *If $A_n\in\mathcal{R}$, $n=1,2,\dots$ and $\mu(A_n\triangle A_m)\to 0$ as $n, m\to\infty$, then for each $\varepsilon>0$ there is a $\delta>0$ such that $B\in\mathcal{R}$ and $\mu(B)<\delta$ implies $\mu(A_n)-\varepsilon\leqq\mu(A_n-B)\leqq\mu(A_n)\leqq\mu(A_n\cup B)\leqq\mu(A_n)+\varepsilon$ for each $n=1,2,\dots$, and*

c) *If $A_n\in\mathcal{R}$, $n=1,2,\dots$ and $\mu(A_n\triangle A_m)\to 0$ as $n, m\to\infty$, then $\mu(A_n)-\mu(A_m)\to 0$ as $n, m\to\infty$.*

Proof. a)$\Rightarrow$b). Let $A_n\in\mathcal{R}$, $n=1,2,\dots$ be such that $\mu(A_n\triangle A_m)\to 0$ as n, $m\to\infty$. By Corollary 2 of Theorem 1 there is an $A\in\bar{\mathcal{R}}$ such that $\hat\mu(A_n\triangle A)\to 0$. Let $\varepsilon>0$. By the subadditive continuity of $\hat\mu$ on $\bar{\mathcal{R}}$ there is a $\delta_A>0$ such that $B\in\bar{\mathcal{R}}$ and $\hat\mu(B)<\delta_A$ implies $\hat\mu(A)-2^{-1}\cdot\varepsilon\leqq\hat\mu(A-B)\leqq\hat\mu(A)\leqq\hat\mu(A\cup B)\leqq\hat\mu(A)+2^{-1}\cdot\varepsilon$. Further, by the (p.g.p.) of $\hat\mu$ there is a $\delta_0<\delta_A$ such that $B, B_1\in\bar{\mathcal{R}}$ and $\hat\mu(B)$, $\hat\mu(B_1)<\delta_0$ implies $\hat\mu(B\cup B_1)<\delta_A$. Take n_0 so that $\hat\mu(A\triangle A_n)<\delta_0$ for $n\geqq n_0$. Then for $n\geqq n_0$ and for $B\in\bar{\mathcal{R}}$ with $\hat\mu(B)<\delta_0$ we have the inequalities $\hat\mu(A)-2^{-1}\cdot\varepsilon\leqq\hat\mu(A-(B\cup(A-A_n)))\leqq\hat\mu(A_n-B)\leqq\hat\mu(A_n)\leqq\hat\mu(A_n\cup B)\leqq\hat\mu(A\cup(A_n-A)\cup B)\leqq\hat\mu(A)+2^{-1}\cdot\varepsilon$. Hence for such n and B we have the inequalities $\hat\mu(A_n)-\varepsilon\leqq\hat\mu(A_n-B)\leqq\hat\mu(A_n)\leqq\hat\mu(A_n\cup B)\leqq\hat\mu(A_n)+\varepsilon$. Finally, by the subadditive continuity of $\hat\mu$ we take $\delta_1,\dots,\delta_{n_0}$ corresponding to ε and $A_1,\dots,A_{n_0}$ respectively, and we put $\delta=\min\{\delta_0,\delta_1,\dots,\delta_{n_0}\}$.

Clearly b)$\Rightarrow$c).

c) $\Rightarrow$ a). For $A \in \bar{\mathscr{R}}$ put $\mu(A) = \lim_{n\to\infty} \mu(A_n)$, where $A_n \in \mathscr{R}$, $n = 1, 2, \ldots$ and $\mu(A_n \triangle A) \to 0$. By c) μ is clearly unambiguously defined. First we show that μ: $\bar{\mathscr{R}} \to [0, +\infty)$ is subadditively continuous, and then that $\mu(A) = \hat{\mu}(A)$ for each $A \in \bar{\mathscr{R}}$.

Suppose $\hat{\mu}$: $\bar{\mathscr{R}} \to [0, +\infty)$ is not subadditively continuous. Then there is an $\varepsilon > 0$ and A, $A_n \in \bar{\mathscr{R}}$, $n = 1, 2, \ldots$. such that $\mu(A_n \triangle A) \to 0$ and $|\mu(A_n) - \mu(A)| > \varepsilon$ for each $n = 1, 2, \ldots$. Take $A_{0,k}$, $A_{n,k} \in \mathscr{R}$, k, $n = 1, 2, \ldots$ so that $\hat{\mu}(A_{0,k} \triangle A) \to 0$ and $\hat{\mu}(A_{n,k} \triangle A_n) \to 0$ as $k \to \infty$, for each $n = 1, 2, \ldots$. Then $\mu(A) = \lim_{k\to\infty} \mu(A_{0,k})$, $\mu(A_n) = \lim_{k\to\infty} \mu(A_{n,k})$ for each $n = 1, 2, \ldots$ and $\lim_{n\to\infty} \mu(A \triangle A_n)$ $= \lim_{n\to\infty} \lim_{k\to\infty} \mu(A_{0,k} \triangle A_{n,k}) = 0$.

Take a sequence $\{\delta_i\}_1^\infty$ according to the property c) of μ^*. By the last equality for each $i = 1, 2, \ldots$ there is an n_i such that $\lim_{k\to\infty} \mu(A_{0,k} \triangle A_{n_i,k}) < \delta_i$. But then for each i there is a k_i such that $\mu(A_{0,k_i} \triangle A_{n_i,k_i}) < \delta_i$ and $|\mu(A_{n_i,k_i}) - \mu(A_{n_i})| < i^{-1}$. By the properties of the sequence $\{\delta_i\}_1^\infty$ the first inequality implies that the sequence $\{A_{0,k_1}, A_{n_1,k_1}, \ldots, A_{0,k_i}, A_{n_i,k_i}, \ldots\}$ is ϱ-Cauchy, where $\varrho(E, F) = \mu(E \triangle F)$, hence by c) and the second inequality we have the contradiction

$$\mu(A) = \lim_{i\to\infty} \mu(A_{n_i,k_i}) = \lim_{i\to\infty} \mu(A_{n_i}).$$

There remains to be shown that $\mu(E) = \hat{\mu}(E)$ for each $E \in \bar{\mathscr{R}}$. Let $E \in \bar{\mathscr{R}}$. Take a sequence $E_n \in \mathscr{R}$, $n = 1, 2, \ldots$ so that $\mu(E \triangle E_n) \to 0$, and let have the notations of Theorem 1. Then $\hat{\mu}(E) = \inf\{\mu(B)\colon E \subset B,\ B \in \hat{\mathscr{R}}_\sigma\} \leqq \inf_k \mu(F_k) = \lim_{k\to\infty} \mu(F_k) = \lim_{k\to\infty} \mu(E_{n_k}) = \mu(E)$, since $\mu(F_k \triangle E_{n_k}) \to 0$ as $k \to \infty$.

On the other hand, for each $\varepsilon > 0$ there is a $B \in \hat{\mathscr{R}}_\sigma$ such that $B \supset F$ and $\hat{\mu}(F) + \varepsilon \geqq \mu(B) \geqq \mu(B \cap F_k) \geqq \mu(F)$ for each k, hence $\hat{\mu}(F) \geqq \mu(F) = \mu(E)$. There remains to be shown that $\hat{\mu}(F) = \hat{\mu}(E)$. Since μ: $\hat{\mathscr{R}} \to [0, +\infty)$ is subadditively continuous, and since $\hat{\mu} = \mu$ on $\hat{\mathscr{R}}_\sigma$, by the definition of $\hat{\mu}$, $\hat{\mu}$: $\hat{\mathscr{R}} \to [0, +\infty)$ is subadditively continuous from the right, i.e., for each $A \in \hat{\mathscr{R}}$ and $\varepsilon > 0$ there is a $\delta > 0$ such that $B \in \hat{\mathscr{R}}$, $\hat{\mu}(B) < \delta$ implies $\hat{\mu}(A \cup B) \leqq \hat{\mu}(B) + \varepsilon$. From this, since $\hat{\mu}(E \triangle F) = 0$ we immediately have the required equality $\hat{\mu}(F) = \hat{\mu}(E)$. The theorem is proved.

From Theorems 2 and 3, and Theorem 3-b) in [1] we immediately have (the uniqueness follows easily from Corollary 3 of Theorem 15 in [1]) our extension theorem for submeasures, compare with Theorem 18 in [1].

Theorem 4. (Extension Theorem for Submeasures.) *A submeasure* $\mu: \mathcal{R} \to [0, +\infty)$ *has a unique extension to* $\sigma(\mathcal{R})$*- the* σ*-ring generated by* $\mathcal{R}$*, if and only if it is exhaustive on* $\mathcal{R}$*,* $A_n \in \mathcal{R}$*,* $n = 1, 2, \dots$ *and* $\mu(A_n \triangle A_m) \to 0$ *as* $n, m \to \infty$ *implies* $\mu(A_n) - \mu(A_m) \to 0$ *as* $n, m \to \infty$*, and for each* $\varepsilon > 0$ *there is a* $\vartheta > 0$ *such that* A*,* $B \in \mathcal{R}$ *and* $\mu(A)$*,* $\mu(B) < \delta$ *implies* $\mu(A \cup B) < \varepsilon$*.*

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О РАСШИРЕНИИ СУБМЕР

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Резюме

Пусть $\mathcal{R}$ кольцо подмножеств непустого множества T. Согласно с [1] функция множеств $\mu: \mathcal{R} \to \langle 0, \infty)$ называется субмерой, если она монотонна, непрерывна ($A_n \searrow \emptyset \Rightarrow \mu(A_n) \to 0$), и полуаддитивно непрерывна ($\forall A \in \mathcal{R}$ и $\forall \varepsilon > 0 \exists \delta > 0$, $B \in \mathcal{R}$, $\mu(B) < \delta \Rightarrow \mu(A) - \varepsilon \leqq \mu(A - B) \leqq \mu(A) \leqq \mu(A \cup B) \leqq \mu(A) + \varepsilon$). Последнее условие можно заменить следующим: A, $A_n \in \mathcal{R}$, $n = 1, 2, \dots$ и $\mu(A_n \triangle A) \to 0 \Rightarrow \mu(A_n) \to \mu(A)$. Необходимые и достаточные условия для расширения субмеры из кольца $\mathcal{R}$ на порожденное им сигма кольцо были установлены Теоремой 18 в [1]. Условия II и III этой теоремы слишком громоздкие. В настоящей работе показывается, что их можно заменить более простыми условиями. А, именно, справедлива следующая

Теорема о расширении субмеры. Субмера $\mu: \mathcal{R} \to \langle 0, +\infty)$ однозначно расширается до субмеры на сигма кольце, порожденном $\mathcal{R}$ тогда и только тогда, когда она не имеет ускользающей нагрузки на $\mathcal{R}$, $A_n \in \mathcal{R}$, $n = 1, 2, \dots$ и $\mu(A_n \triangle A_m) \to 0$ для $n, m \to \infty \Rightarrow \mu(A_n) - \mu(A_m) \to 0$ для $n, m \to \infty$, и для каждого $\varepsilon > 0$ существует $\delta > 0$ так, что $A, B \in \mathcal{R}$ и $\mu(A), \mu(B) < \delta \Rightarrow \mu(A \cup B) < \varepsilon$.