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LONGEST CIRCUITS IN TRIANGULAR AND QUADRANGULAR 3-POLYTOPES WITH TWO TYPES OF EDGES

STANISLAV JENDROĽ—ROMAN KEKEŇÁK

ABSTRACT. The paper deals with the longest circuits in triangular and quadrangular 3-polytopes with two types of edges. Hamiltonicity and shortness invariants for several families of the mentioned 3-polytopes are determined. Three relationships among some subfamilies of triangular and quadrangular 3-polytopes are given.

1. Introduction

There are many papers studying circuits in varied families of planar 3-connected graphs (or, equivalently, 3-polytopal graphs), see e.g. Ewald and others [3], Grünbaum [4, 5], Grünbaum and Malkevitch [6], Grünbaum and Walther [7], Harant and Walther [8], Jackson [9], Jucovič [14], Owens [16, 17, 18, 19], Zaks [22] and others. In [7], Grünbaum and Walther introduced several numbers that measure, in a certain sense, the size of the longest circuits in graphs belonging to this family of graphs. Let us mention two of these measures. For a graph $G$ let $v(G)$ denote the number of vertices of $G$ and $h(G)$ the maximum length of simple circuits in $G$. For an infinite family of graphs $\mathcal{F}$, the shortness exponent, $\sigma(\mathcal{F})$ or $\sigma$ and the shortness coefficients, $\varrho(\mathcal{F})$ or $\varrho$, are defined by

$$\sigma(\mathcal{F}) = \liminf_{G \in \mathcal{F}} \frac{\log h(G)}{\log v(G)},$$

and

$$\varrho(\mathcal{F}) = \liminf_{G \in \mathcal{F}} \frac{\log h(G)}{\log v(G)},$$

respectively.

Both $\sigma$ and $\varrho$ lie between 0 and 1 inclusive and $\varrho = 0$ when $\sigma < 1$.

AMS Subject Classification (1985): Primary 05C40, 05C45
Key words: Graph, Connectivity, Circuit
We recall that $G$ is called \textit{hamiltonian} if $v(G) = h(G)$. The family of graphs $\mathcal{F}$ is called \textit{hamiltonian} provided that all its members are hamiltonian and $\mathcal{F}$ is called \textit{nonhamiltonian} if it contains no hamiltonian graph.

For an infinite nonhamiltonian family of graphs $\mathcal{F}$ it is \textit{important} to consider the \textit{length coefficient}, $\tau(\mathcal{F})$ or $\tau$, defined by

$$\tau(\mathcal{F}) = \lim \sup_{G \in \mathcal{F}} \frac{h(G)}{v(G)}.$$ 

Jendröl and Tkáč [13] define an edge of the type $(a, b; p, q)$ in a planar graph to be an edge incident with vertices of valency $a$ and valency $b$ and faces with $p$ and $q$ edges. The present paper is devoted to a study of the longest circuits in 3-polytopal graphs $G$ having $k$-gonal faces only, $k = 3, 4$, and exactly two types of edges. Notice that the vertices of such graphs $G$ can have at most three different valencies because of the connectedness of $G$. So let us denote by $\mathcal{R}(a, b, c)$ the family of all 3-polytopal graphs all edges of which are either of the type $(a, b; k, k)$ or of the type $(b, c; k, k)$. In the sequel let $\mathcal{I}(a, b, c) = \mathcal{R}(a, b, c)$ and $\mathcal{S}(a, b, c) = \mathcal{R}(a, b, c)$.

The present paper is organized as follows. In Section 2 we shall study the longest circuits in simplicial graphs from the families $\mathcal{I}(a, b, c)$. Section 3 contains our results showing some relationships between some subfamilies of triangular and quadrangular 3-polytopal graphs. Section 4 is devoted to the study of the numbers $\sigma, \varrho$ and $\tau$ for some subfamilies of quadrangular 3-polytopal graphs with exactly two types of edges. In Section 5 we shall discuss some open problems.

2. Hamiltonicity of the family $\mathcal{I}(a, b, c)$

In [13], the first step in the study of the combinatorial structure of graphs to $\mathcal{I}(a, b, c)$ has been made. For all triples $(a, b, c)$ of positive integers it has been decided whether the family $\mathcal{I}(a, b, c)$ is finite or not and for each finite family $\mathcal{I}(a, b, c)$, all polytopes belonging to $\mathcal{I}(a, b, c)$ have been constructed. This result is employed in the sequel. We note that the longest circuits in graphs of the families $\mathcal{I}^*(a, b, c)$ dual to those of $\mathcal{I}(a, b, c)$, have been studied in Owens [18, 19] and Jendröl and Mihók [12].

The main result of this Section is contained in

\textbf{Theorem 2.1.}

(i) The family $\mathcal{I}(a, b, c)$ is hamiltonian for every triple

$(a, b, c) \in \{(4, 4, c), 3 \leq c \neq 4; (5, 5, c), 3 \leq c \neq 5; (6, 6, c), 3 \leq c \leq 5; (7, 7, 3); (7, 7, 4)\}.$
(ii) There is an infinite hamiltonian subfamily of the family \( \mathcal{S}(8, 8, 3) \) and 
\[ q(\mathcal{S}(8, 8, 3)) \leq \frac{14}{15}. \]

(iii) The families \( \mathcal{S}(9, 9, 3) \) and \( \mathcal{S}(10, 10, 3) \) are nonhamiltonian
\[ q(\mathcal{S}(9, 9, 3)) \leq \frac{25}{28}, \quad q(\mathcal{S}(10, 10, 3)) \leq \frac{25}{32} \quad \text{and} \quad \tau(\mathcal{S}(10, 10, 3)) \leq \frac{6}{7}. \]

(iv) Let \( a, b, c \) be integers such that \( a \geq 3, b \geq 3, c \geq 3 \) and at most two of them are equal to each other. If \( (a, b, c) \notin \{(4, 4, c), 3 \leq c \leq 4; (5, 5, c), 3 \leq c \neq 5; (6, 6, c), 3 \leq c \leq 5; (a, a, 3), 7 \leq a \leq 10; (7, 7, 4)\}, \) then \( \mathcal{S}(a, b, c) \) is empty.

The next two theorems will be useful in the proof of Theorem 2.1.

**Theorem 2.2** (Pareek [20], a weaker result in Ewald [1]). Let \( G \) be a triangular planar nonhamiltonian graph. Then \( \Delta(G) \geq 8 \), where \( \Delta(G) \) is a maximum degree of \( G \).

**Theorem 2.3.** Every graph \( G \) belonging to \( \mathcal{S}(5, 5, c), 3 \leq c \neq 5 \), is 4-connected.

**Proof.** Suppose that there is a graph \( G \) in \( \mathcal{S}(5, 5, c) \) which is not 4-connected. It can be easily verified that in \( G \) every minimal separating set consists of three vertices which form a separating triangle \( C \) (i.e., there are vertices of \( G \) both inside and outside of \( C \)). We denote by \( H_1 \) the subgraph consisting of \( C \) and the edges of \( G \) lying in its interior, by \( H_2 \) the subgraph consisting of \( C \) and the edges in its exterior. We may assume that \( v(H_1) \leq v(H_2) \) and that \( H_1 \) is minimal, that is that no separating triangle \( C \) of \( G \) has \( H_1^* \) with \( v(H_1^*) < v(H_1) \). Let \( x_1, x_2 \) and \( x_3 \) be vertices of \( C \). At least two of them, e.g. \( x_1 \) and \( x_2 \) are 5-valent in \( G \). It is easy to see that \( 3 \leq \deg_{H_1}(x_i) \leq 4 \) for any \( i = 1, 2 \) and \( j = 1, 2 \). The assumption \( \deg_{H_1}x_i = \deg_{H_2}x_i = 3 \) for \( i = 1 \) or \( 2 \) and \( j = 1, 2, 3, j \neq k \), leads to a contradiction with the 3-connectedness or the planarity of \( G \), respectively. It is sufficient to consider the case \( \deg_{H_1}x_i \neq \deg_{H_2}x_i \). Evidently \( \deg_{H_1}x_i \geq 4 \).

Since \( H_1 \) is triangular too, there are vertices \( y_1 \) and \( y_2 \) in \( H_1 \) such that the vertices \( x_1, x_2, y_1 \) and \( x_2, x_3, y_2 \), respectively form a face. \( H_1 \) contains only one of the edges \( x_1y_2 \) and \( x_2y_1 \), therefore \( G \) has an edge \( y_1y_2 \) too. Because \( \deg_{H_1}y_i = \deg_{G}y_i \geq 5 \), \( i = 1, 2 \), the vertices \( y_1, y_2 \) and \( x_3 \) create a separating triangle \( C_1 \) in \( G \). For the subgraph \( H_3 \) consisting of \( C_1 \) and edges of \( G \) lying in its interior we have \( v(H_3) < v(H_1) \), which is a contradiction with the minimality of \( H_1 \). □

The proof of the Theorem 2.1 in the case (i) for the triple \( (a, b, c) \in \{(6, 6, c), 3 \leq c \leq 5; (7, 7, c), 3 \leq c \leq 4\} \) follows immediately from Theorem 2.2. By the well-known Tutte theorem (see, e.g., Ore [15]) every 4-connected planar graph is hamiltonian, therefore the family \( \mathcal{S}(5, 5, c) \) for any \( c \geq 3 \), \( c \neq 5 \) is hamiltonian too. The family \( \mathcal{S}(4, 4, c), c \geq 3, c \neq 4 \) consists of exactly one graph-c-sided bipyramid — which is hamiltonian.

The propositions of the case (iv) follow from [13]. □
The Proof of the Theorem 2.1 in the cases (ii) and (iii). Let \( v_k(G) \) denote the number of \( k \)-valent vertices of \( G \). The well-known Euler formula applied to triangular graphs leads to the following equality

\[
\sum_{k \geq 3} (6 - k) v_k(G) = 12.
\]  

This equality and \( v(G) = v_3(G) + v_e(G) \) for \( G \in \mathcal{G}(c, c, 3) \), \( 8 \leq c \leq 10 \), give

\[
v(G) = 4 + \frac{c - 3}{3} v_e(G).
\]  

Since, in \( G \), edges connecting 3-valent vertices are not allowed, we have

\[
h(G) \leq 2v_e(G).
\]  

From (2.1) and (2.2) it is easy to see that the families \( \mathcal{G}(9, 9, 3) \) and \( \mathcal{G}(10, 10, 3) \) are nonhamiltonian and that

\[
\tau(\mathcal{G}(10, 10, 3)) = \limsup_{G \in \mathcal{G}(10, 10, 3)} \frac{h(G)}{v(G)} \leq \lim_{v_{10}(G) \to \infty} \frac{2v_{10}(G)}{4 + \frac{7}{3} v_{10}(G)} = \frac{6}{7}.
\]  

To prove the remaining part of the cases (ii) and (iii) we shall present methods based on inductive constructions of the sequences \( \{G_n\}, n = 0, 1, 2, \ldots \) of graphs with the desired properties. In every case, the graph \( G_n, n = 1, 2, \ldots \) is obtained by replacing certain parts of \( G_{n-1} \) by new graphs of a suitable type.

The construction of a sequence of hamiltonian graphs from \( \mathcal{G}(8, 8, 3) \) starts with a graph \( G_0 \) obtained from graph \( H \) in Fig. 2.1 by adding an edge \( v_1v_{28} \) (numerals in this and further figures denote indices of vertices). To obtain \( G_n \) from \( G_{n-1}, n = 1, 2, \ldots \), we delete from \( G_{n-1} \) the edge \( x_1x_{13} \) and place into a quadrangle thus vacated a copy of graph \( H \) shown in Fig. 2.1; in this we identify the vertices \( x_1, x_2, x_{28}, x_{29} \) of \( H \) with the vertices \( x_{14}, x_{11}, x_{12}, x_{13} \) of \( G_{n-1} \), respectively and the corresponding edges. The labels of all the vertices of \( G_n \) except the labels of the vertices of the "last" subgraph \( H \) of \( G_n \) are deleted.

Now we show that \( G_n \) is hamiltonian if \( G_{n-1} \) is hamiltonian. A hamiltonian circuit in \( G_{n-1} \) pases through the edges \( x_{10}x_{11}, x_{11}x_{12}, x_{12}x_{13}, x_{13}x_{14}, x_{14}x_{15}, \ldots \) of \( G_n \). In \( H \) (and in \( G_0 \)) a hamiltonian circuit passes through the edges \( x_i, x_{i+1} \), \( i = 1, 2, \ldots, 29 \) and \( x_1x_{29} \). A hamiltonian circuit in \( G_n \) consists of the part of the hamiltonian circuit of \( G_{n-1} \) between \( x_{14} \) and \( x_{11} \) and a hamiltonian path from \( x_2 \) to \( x_1 \) in \( H \).

The proof of the bound of the shortness coefficient for the family \( \mathcal{G}(8, 8, 3) \) is based on a construction of an infinite sequence of nonhamiltonian graphs of this class. The construction starts with a graph \( G_0 \) obtained from the graph \( H \) in Fig. 2.2 by adding an edge a b.
To obtain \( G_n \), \( n = 1, 2, \ldots \), from \( G_{n-1} \) each of the quadrangular parts of \( G_{n-1} \) marked dark is replaced by the graph \( H \) of Fig. 2.2 in such a way that the vertices \( a \) and \( b \) are identified with the trivalent vertices of the boundary of the marked part, the vertices \( c \) and \( d \) with 8-valent ones respectively, and the corresponding boundary edges are identified, too.

In \( G_{n-1} \), \( n = 1, 2, \ldots \) there are \( 3^n \) dark marked quadrangular parts, this means that there are at least \( 3^n \) subgraphs in \( G_n \) isomorphic to \( H \). Any two different such subgraphs have at most one vertex in common.

It is easy to verify that \( G_n \) belongs to \( \mathcal{S}(8, 8, 3) \) and that for the number of vertices \( v(G_n) \) of \( G_n \), \( n = 0, 1, \ldots \)

\[
v(G_n) = 4 + 10 \sum_{k=0}^{n} 3^k = 5 \cdot 3^{n+1} - 1.
\]

On the other hand every longest circuit of \( G_n \) contains at most five trivalent vertices from the interior of each copy of \( H \). Therefore

\[
h(G_n) \leq v(G_n) - 3^n = -1 + 5 \cdot 3^{n+1} - 3^n = 14 \cdot 3^n - 1.
\]

The above considerations yield

\[
q(\mathcal{S}(8, 8, 3)) = \liminf_{n \to \infty} \frac{h(G_n)}{v(G_n)} \leq \lim_{n \to \infty} \frac{-1 + 14 \cdot 3^n}{-1 + 15 \cdot 3^n} = \frac{14}{15}.
\]

To establish an upper bound of shortness coefficient for the family \( \mathcal{S}(9, 9, 3) \) (or \( \mathcal{S}(10, 10, 3) \)) we proceed similarly as above. The graph \( G_0 \) is obtained from the graph \( H \) in Fig. 2.3 (or Fig. 2.4) by adding an edge connecting the vertices \( a \) and \( b \).

The graph \( G_n \), \( n = 1, 2, \ldots \), results from \( G_{n-1} \) by replacing each of the dark marked quadrangles of \( G_{n-1} \) by a copy of \( H \) in Fig. 2.3 (or Fig. 2.4) identifying the boundaries of the dark marked quadrangle and of \( H \), respectively. Every longest circuit of \( G_n \) omits at least three vertices (seven vertices, respectively) of each copy of \( H \) of \( G_n \). Since \( G_{n-1} \) contains \( 7^n \) (8\(^n\), respectively) dark marked quadrangles, an easy computation shows that

\[
v(G_n) = 4 + 24 \sum_{k=0}^{n} 7^n = 4 \cdot 7^{n+1} \quad \text{and} \quad h(G_n) \leq v(G_n) - 3 \cdot 7^n = 25 \cdot 7^n
\]

for \( G_n \in \mathcal{S}(9, 9, 3) \) and

\[
v(G_n) = 4 + 28 \sum_{k=0}^{n} 8^n = 4 \cdot 8^{n+1}, \quad h(G_n) \leq 25.8^n \quad \text{for} \quad G_n \in \mathcal{S}(10, 10, 3),
\]

respectively.

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Fig. 2.3

Fig. 2.4
So
\[ q(S(9, 9, 3)) \leq \frac{22}{28} \quad \text{and} \quad q(S(10, 10, 3)) \leq \frac{25}{32}. \]

### 3. Relationship among some families of triangular and quadrangular 3-polytopal graphs

Almost all considerations in the sequel use the notion of the radial graph \( r(G) \) of a given planar graph \( G \) (see Jucovič [14], Ore [15]). Given a planar graph \( G \) we associate with \( G \) (with vertex-set \( V(G) \), edge-set \( E(G) \) and face-set \( F(G) \)) a graph \( r(G) \) so that \( V(r(G)) = V(G) \cup F(G) \); \( e = xy \in E(r(G)) \) if and only if \( x \in V(G), y \in F(G) \) and \( x \) is a vertex of the face \( y \) or \( x \in F(G), y \in V(G) \) and \( y \) is a vertex of the face \( x \). As every edge \( e \in E(G) \) is incident with two vertices and with two faces of \( G \), \( e \) determines a quadrangular face of \( r(G) \). So for every graph \( G \), \( r(G) \) is a quadrangular graph whose vertex-set \( V(r(G)) \) is partitioned into two disjoint sets. The valencies of vertices in one set are those of the vertices of \( V(G) \), the valencies of the other second are equal to those of the faces from \( F(G) \).

**Theorem 3.1** (Jendrol, Jucovič and Trenkler [11]). If \( H \in \mathcal{S}(3, 3, c) \), then \( H \) is the radial graph of a \( c \)-gonal pyramid or of a triangular 3-polytopal graph \( G \) belonging to \( \mathcal{S}(c, c, 3) \).

It is easy to see that for every triangular 3-polytopal graph \( G \) the radial graph \( H = r(G) \) of the graph \( G \) is a quadrangular one with the property that at least one of the end-vertices of any edge \( e \) of \( H \) is trivalent. If \( G \) does not contain trivalent vertices, then every edge of \( r(G) \) has exactly one endvertex trivalent.

**Theorem 3.2.** If \( H \) is a quadrangular 3-polytopal graph in which every edge has exactly one trivalent vertex, then there is a triangular 3-polytopal graph \( G \) without trivalent vertices such that

\[ H = r(G) \quad \text{and} \quad v_k(H) = v_k(G) \quad \text{for every} \quad k, \quad k \neq 3. \]

**Proof.** For given \( H \) we shall construct a triangular 3-polytopal graph \( G \). The vertex-set \( V(G) \) of \( G \) consists of the vertices of \( H \) having valencies \( > 3 \) in \( H \). Two vertices \( x \) and \( y \) of \( G \) are connected by an edge provided that there is in \( H \) a face \( \alpha \) incident to the vertices \( x \) and \( y \). Let \( y \) be a \( k \)-valent vertex of \( H \), \( k \geq 3 \). Let \( x_0, x_2, \ldots, x_{k-1} \) be trivalent vertices of \( H \) adjacent to \( y \) such that the vertices \( x_i, y, x_{i+1} \) are incident to the same face \( \beta_i, i = 0, 1, \ldots, k - 1 \). Let \( y_i \) be the fourth vertex of the face \( \beta_i \). (Indices are taken modulo \( k \).) By the assumption of the theorem \( \deg_H y_i > 3 \) and the vertices \( y_i, x_{i+1}, y_{i+1} \) are incident to a face \( \beta_i \). Therefore \( G \) also contains the edges \( y_i y_i, y_i y_{i+1} \) and \( y_i y_{i+1} \). These edges form a triangular face in \( G \). This means that every face of \( G \) is a triangle and there
is an unambiguous correspondence between the vertices of \( V(G) \) and the non-trivalent vertices of \( H \) and between the faces of \( F(G) \) and the trivalent vertices of \( H \), respectively. Obviously \( H = r(G) \). \( G \) is clearly a 3-polytopal triangulation. \( \square \)

**Corollary 3.2.** To every graph \( H \in \mathcal{A}(a, 3, b) \), \( a \neq 3 \neq b \), there is a triangular 3-polytopal graph \( G \) with vertices of valencies \( a \) and \( b \) only and such that \( H = r(G) \). \( \square \)

**Theorem 3.3.** For every triangular graph \( G \)

\[
h(r(G)) = 2h(G).
\]

**Proof.** For the purpose of the proof let \( a_i \) denote a face of \( F(G) \) and a vertex of \( V(r(G)) \) associated to \( a_i \) in \( r(G) \). The indices below are taken modulo \( k \).

First we show that \( h(r(G)) \leq 2h(G) \). Obviously \( r(G) \) is the bipartite graph with a vertex-set \( V(r(G)) = V(G) \cup F(G) \). Let \( x_i \) and \( a_i \) denote the member of \( V(G) \) and \( F(G) \), respectively. Let \( C = x_0, a_0, x_1, a_1, x_2, \ldots, x_k, a_k, x_0 \) be a longest circuit in \( r(G) \). Since \( G \) is triangular one, the vertices \( x_i \) and \( x_{i+1} \) are incident to the face \( a_i \) in \( G \). Therefore the vertices \( x_i \) and \( x_{i+1} \) are adjacent in \( G \), this means that \( C' = x_0, x_1, \ldots, x_k = x_0 \) is a circuit of the length \( k \) in \( G \).

Let \( C' = x_0, e_0, x_1, e_1, x_2, \ldots, x_{h-1}, e_{h-1}, x_h = x_0 \) be the longest circuit in \( G \) with \( h = h(G) \) and \( e_i = x_i x_{i+1} \). Let \( E(C') \) be a set of edges of \( C' \), \( E(a) \) be a set of edges incident to the face \( a \) and \( F(e) \) be a couple of faces incident to the edge \( e \) in \( G \), respectively. If a vertex \( x \) and a face \( a \) are incident in \( G \), then the corresponding vertices \( x \) and \( a \) of \( r(G) \) are adjacent. Let \( \varphi \) be a mapping which maps every edge \( e \) to a face belonging to \( F(e) \). If the mapping \( \varphi \) from \( E(C') \) to \( F(G) \) is an injection, then the sequence \( x_0, \varphi(e_0), x_1, \varphi(e_1), x_2 \ldots x_{h-1}, \varphi(e_{h-1}) \), \( , x_h \) forms a circuit of the length \( 2h \) in \( r(G) \). To finish the proof it is sufficient to show that the mapping can always be chosen in such a way that \( \varphi \) is an injection. The following two facts are evident

\[
F(e_i) \cap F(e_j) = \emptyset \quad \text{for} \quad j \neq \{i - 1, i, i + 1\}, \quad (3.1)
\]

\[
|F(e_i) \cap F(e_{i+1})| \leq 1 \quad \text{for every} \quad i = 0, 1, \ldots, h - 1. \quad (3.2)
\]

If for every \( i = 0, 1, \ldots, h - 1 \) \( F(e_i) \cap F(e_{i+1}) = \emptyset \), then the required mapping \( \varphi \) can be easily chosen. If this is not true, it is sufficient to suppose \( F(e_0) \cap F(e_1) \neq \emptyset \). In this case \( \varphi \) is defined as follows

\[
\varphi(e_0) = F(e_0) \cap F(e_1).
\]

Let \( F_i = \{ \varphi(e_t) \mid t = 0, 1, \ldots, i - 1 \} \), then we put \( \varphi(e_i) = a \in F(e_i) - F_i \), \( a \) arbitrary. (We can do it because \( F(e_i) - F_i \) is always nonempty.) In the opposite case there is a minimum \( i_0 \) such that \( F(e_{i_0}) - F_{i_0} = \emptyset \). Let \( F(e_{i_0}) = \{ a_1, a_2 \} \), then there
must be indices \( j, l < i_0 \) such that \( F(e_j) \cap F(e_l) = \{ \alpha_1 \} \) and \( F(e_j) \cap F(e_l) = \{ \alpha_2 \} \); however, (3.1) and (3.2) imply \( j = l = i_0 - 1 \), which is a contradiction.

4. The longest circuits in the families \( \mathcal{Z}(a, b, c) \)

Basic combinatorial properties of graphs of the families \( \mathcal{Z}(a, b, c) \) have been investigated in Jendroľ and Jucovič [10]. In the sequel we shall consider only triples \( (a, b, c) \) for which the families \( \mathcal{Z}(a, b, c) \) are nonempty.

Theorem 4.1. (i) In the family \( \mathcal{Z}(3, 3, c) \), \( c \geq 4 \), there is a unique hamiltonian graph — a radial graph of a \( c \)-sided pyramid \( M(c) \).

(ii) the family \( \mathcal{Z}(3, 3, c) - \{ M(c) \} \), \( 4 \leq c \leq 4 \), contains a unique nonhamiltonian graph.

(iii) For every graph \( H \in \mathcal{Z}(3, 3, c) - \{ M(c) \} \), \( 6 \leq c \leq 7 \), there is

\[
h(H) = 8 + 2 \left( \frac{c}{3} - 1 \right) v_e(H)
\]

and

\[
\varrho(\mathcal{Z}(3, 3, c)) = \tau(\mathcal{Z}(3, 3, c)) = \frac{2}{3}, \quad \sigma(\mathcal{Z}(3, 3, c)) = 1.
\]

(iv) \( \varrho(\mathcal{Z}(3, 3, 8)) \leq \frac{28}{45} \) and \( \tau(\mathcal{Z}(3, 3, 8)) = \frac{2}{3} \).

(v) \( \varrho(\mathcal{Z}(3, 3, 9)) \leq \frac{25}{42} \) and \( \tau(\mathcal{Z}(3, 3, 9)) \leq \frac{2}{3} \).

(vi) \( \varrho(\mathcal{Z}(3, 3, 10)) \leq \frac{25}{48} \) and \( \tau(\mathcal{Z}(3, 3, 10)) \leq \frac{4}{7} \).

(vii) For every \( c > 10 \) the family \( \mathcal{Z}(3, 3, c) - \{ M(c) \} \) is empty.

Proof. It is easy to see that the graph \( M(c) \) — a radial graph of a \( c \)-sided pyramid — is hamiltonian. By Theorem 3.1 and Corollary 3.2 there is for every graph \( H \in \mathcal{Z}(3, 3, c) - \{ M(c) \} \) a graph \( G \in \mathcal{F}(3, 3, c) \) such that \( H = r(G) \). Let \( f(M) \) denote the number of faces of a planar graph \( M \). Since \( G \) is triangular we have

\[
v(G) = v_3(G) + v_e(G) = 4 + \left( \frac{c}{3} - 1 \right) v_e(G)
\]

(4.1)

and

\[
f(G) = 4 + 2 \left( \frac{c}{3} - 1 \right) v_e(G).
\]

(4.2)
By Theorems 3.2 and 3.3, (4.1) and (4.2) there is
\[ h(H) = 2h(G) \leq 2v(G) = 8 + 2\left(\frac{c}{3} - 1\right)v_c(G) = 8 + 2\left(\frac{c}{3} - 1\right)v_c(H) \quad (4.3) \]
and
\[ v(H) = v(G) + f(G) = 8 + (c - 3)v_c(G) = 8 + 3(c - 3)v_c(H). \]

From (4.3) and (4.4) it follows that all graphs belonging to the family \( \mathcal{A}(3, 3, c) - \{M(c)\} \) are nonhamiltonian and that \( \tau(\mathcal{A}(3, 3, c)) \leq \frac{2}{3} \). This finishes the proof in the case (i).

By Theorem 2.1 the families \( \mathcal{S}(6, 6, 3) \) and \( \mathcal{S}(7, 7, 3) \) are hamiltonian. For every graph \( H \in \mathcal{A}(3, 3, c), \ 6 \leq c \leq 7 \), the Theorems 3.1 and 3.3 imply
\[ h(H) = 2h(G) = 2v(G) = 8 + 2\left(\frac{c}{3} - 1\right)v_c(G) = 8 + 2\left(\frac{c}{3} - 1\right)v_c(H). \]

The inequalities for \( g(\mathcal{A}(3, 3, c)), \ 8 \leq c \leq 10 \), are obtained by using Theorems 3.1 and 3.3, the relations (4.1), (4.2), (4.3) and (4.4) and sequences of triangular nonhamiltonian graphs belonging to \( \mathcal{S}(c, c, 3) \) which were used in the proof of the Theorem 2.1 (ii) and (iii). For the cases (ii) and (vii) see [10]. □

**Lemma 4.1.** For \( a \neq b \neq c \neq a \) the family \( \mathcal{A}(a, b, c) \) is nonhamiltonian.

**Proof.** Every graph \( H \) belonging to \( \mathcal{A}(a, b, c) \) is bipartite. Its one coloured class of vertices consists of all vertices of the valencies \( a \) and \( c \), while its other class contains all \( b \)-valent vertices of \( H \). This implies
\[ av_a(H) + cv_c(H) = bv_b(H). \quad (4.5) \]
The Euler polyhedral formula for the quadrangular graph \( P \) gives
\[ \sum_{k \geq 3} (4 - k)v_k(P) = 8. \quad (4.6) \]
For \( H \) belonging to the family \( \mathcal{A}(a, b, c) \) the equality (4.6) provides
\[ (4 - a)v_a(H) + (4 - b)v_b(H) + (4 - c)v_c(H) = 8. \quad (4.7) \]
An assumption of hamiltonicity of \( H \) implies
\[ v_a(H) + v_c(H) = v_b(H). \quad (4.8) \]
From (4.5), (4.7) and (4.8) it is easy to obtain a contradiction. □

**Theorem 4.2.** (i) The family \( \mathcal{A}(4, 3, c), c \geq 5, \) is nonhamiltonian.

(ii) The family \( \mathcal{A}(4, 3, 5) \) contains exactly four graphs.

(iii) For every graph \( H \in \mathcal{A}(4, 3, c) \), \( 6 \leq c \leq 7 \) there is
\[ h(H) = 12 + (c - 4)v_c(H) \quad \text{and} \]
(iv) For every \( c \geq 8 \) there is \( \tau(\mathcal{A}(4, 3, e)) = -\).

Proof. By Corollary 3.2 for every graph \( H \in \mathcal{A}(4, 3, c) \) there exists a triangular graph \( G \) with vertices of valencies 4 and \( c \) only and such that \( H = r(G) \). For \( G \) from (2.0) we can easily obtain

\[
f(G) = 8 + (c - 4) v_e(G) \quad \text{and} \quad v(G) = 6 + \frac{1}{2} (c - 4) v_e(G).
\]

Since

\[
h(H) = 2 h(G) \leq 2 v(G) = 12 + (c - 4) v_e(H) \quad \text{(4.9)}
\]

and

\[
v(H) = v(G) + f(G) = 12 + \frac{3}{2} (c - 4) v_e(H)
\]

we can easily obtain \( \tau(\mathcal{A}(4, 3, c)) \leq -\).

By Theorem 2.2 all triangular planar graphs with maximum degree \( \leq 7 \) are hamiltonian. therefore for \( 6 \leq c \leq 7 \) there is an equality in (4.9). The equality for \( \varrho, \sigma \) and \( \tau \) can now be easily obtained. The case (iii) is exhausted.

To prove the equation in (iv) it is sufficient to construct an infinite sequence of triangular hamiltonian graphs \( G_n \) with 4-valent and \( e \)-valent vertices only. A construction of such sequence begins with a graph \( G_0 \) of a \( c \)-sided bipyramid. Let \( G_{n-1}, n = 1, 2, \ldots \) be a triangular hamiltonian graph with the required property. Choose three 4-valent vertices \( x, y, z \) in such a way that the distance between \( x \) and \( z \) is two and \( y \) is a vertex adjacent to both of them. Let \( w \) be a vertex of \( G_{n-1} \) adjacent to \( y, w \neq x, z \). We add \( c - 4 \) new vertices \( z_1, z_2, \ldots, z_{c-4} \) in the edge \( yw \) and join them with the vertices \( x \) and \( z \). A graph \( G_n \) thus obtained has two \( c \)-valent vertices and \( c - 2 \) 4-valent vertices more than the graph \( G_{n-1} \). It can be verified that \( G_n \) is hamiltonian provided that \( G_{n-1} \) is. The cases (i) and (ii) follow from Lemma 4.1 and [10], respectively. \( \square \)

Theorem 4.3. (i) The family \( \mathcal{A}(5, 3, c), c \geq 6 \), is nonhamiltonian.
(ii) For every graph \( H \in \mathcal{A}(5, 3, c), 6 \leq c \leq 7 \),

\[
h(H) = 24 + 2(c - 5) v_e(H) \quad \text{and}
\]

\[
\varrho(\mathcal{A}(5, 3, c)) = \tau(\mathcal{A}(5, 3, c)) = \frac{2}{3}, \quad \sigma(\mathcal{A}(5, 3, c)) = 1
\]

(iii) For every \( c \geq 12 \), \( \tau(\mathcal{A}(5, 3, c)) = \frac{2}{3} \).
Proof. The proof in the cases (i) and (ii) is similar to the proof of the parts (i) and (iii) of the previous Theorem 4.2. We omit it. The equality 
\( \tau(2(5, 3, c)) = \frac{2}{3} \) can be obtained by using (2.0), Theorems 3.2 and 3.3, Corollary 3.2 and the fact that the family \( \mathcal{P}(5, 5, c), c \geq 12 \), is hamiltonian. \( \square \)

Theorem 4.4. (i) In the family \( \mathcal{P}(3, 4, 4) \) there is an infinite hamiltonian subfamily and an infinite nonhamiltonian subfamily.

(ii) \( \sigma(2(3, 4, 4)) = 1 \)

(iii) The family \( \mathcal{P}(3, 4, c), c \geq 5 \), is nonhamiltonian and 
\( \tau(2(3, 4, c)) = 1 \).

Proof. A construction of an infinite sequence of hamiltonian graphs starts with a graph \( G_0 \) in Fig. 4.1. A circuit \( C_0 = u_1, u_2, \ldots, u_7, u_{0,1}, u_{0,2}, u_0, u_{0,3}, \ldots, u_{0,8}, u_8, u_9, u_1 \) is a hamiltonian circuit in \( G_0 \).

To obtain a graph \( G_n \) from the graph \( G_{n-1} \) we delete from \( G_{n-1} \) the vertex \( u_{n-1,0} \) (and edges incident with it) and fill an 8-gon \( u_{n-1,1}, u_{n-1,3}, u_{n-1,4}, \ldots, u_{n-1,8}, u_{n-1,2} \), thus vacated in the manner as shown in Fig. 4.2.

A hamiltonian circuit \( C_n \) of \( G_n \) is obtained from the hamiltonian circuit \( C_{n-1} \) of \( G_{n-1} \) by replacing its part \( u_{n-1,2}, u_{n-1,0}, u_{n-1,3} \) by the path \( u_{n-1,2}, u_{n,1}, u_{n,2}, u_{n,0}, u_{n,3}, \ldots, u_{n,8}, u_{n-1,3} \).
To prove the existence of an infinite nonhamiltonian subfamily of the family \( \tau(3, 4, 4) \) it is sufficient to consider the family of all 4-regular 3-polytopal graphs with triangular and quadrangular faces only. For every graph \( G \) from this family there is \( v(G) \neq f(G) \), therefore the graph \( r(G) \in \tau(3, 4, 4) \) and it is nonhamiltonian (see, e.g., Jucovič [14]).

![Fig. 4.2](image_url)

The proof that \( \sigma(\tau(3, 4, 4)) = 1 \) may be found in Ewald [2]. The family \( \tau(3, 4, c), c \geq 5 \) is nonhamiltonian by Lemma 4.1. In order to prove the second part of (iii) it is sufficient to construct an infinite sequence \( \{G_n\} \) of 4-regular 3-polytopal graphs with triangular and \( c \)-gonal faces only. It can be verified that \( r(G_n) \in \tau(3, 4, c) \) and \( f(G_n) = v(G_n) + 2 \).

Since the graph \( r(G_n) \) is bipartite, the vertices of the one colour class of \( r(G_n) \) correspond to the vertices of \( G_n \) and the vertices of the second correspond to the faces of \( G_n \). Therefore it is sufficient to show that in \( G_n \) there exists an alternating sequence \( C_n \) of vertices and faces of \( G_n \), \( x_0, a_0, x_1, a_1, x_2, \ldots, x_m, a_m, x_0 \) such that 

\[ m = v(G_n), \ a_j \neq a_i, \ x_i \neq x_j \text{ if } i \neq j \text{ and } a_i \text{ is incident to } x_i \text{ and } x_{i+1}, \ a_m \text{ is incident to } x_{m} \text{ and } x_0 \text{ for every } i = 0, 1, \ldots, m. \]

The sequence \( C_n \) specifies a circuit \( C_n \) in \( r(G_n) \) of the length \( h(r(G_n)) = 2v(G_n) \). Since \( v(r(G_n)) = v(G_n) + f(G_n) = 2v(G_n) + 2 \) we have \( \tau(\tau(3, 4, c)) = 1 \) for every \( c \geq 5 \). The construction of a required sequence \( G_n \) begins with a graph of \( c \)-sided antiprisma in Fig. 4.3 taken as \( G_0 \). To obtain the graph \( G_n \) we delete the edge \( x_{n-1,1}y_{n-1,2} \) of \( G_{n-1} \) and add \( c - 3 \) new vertices \( x_{n,1}, \ldots, x_{n,c-3} \) into the edge \( x_{n-1,1}y_{n-1,1} \) and \( c - 3 \) new vertices \( y_{n,1}, \ldots, y_{n,c-3} \) into the edge \( x_{n-1,2}y_{n-1,2} \) and connect by an edge the couples of vertices \( x_{n-1,1} \) and \( y_{n,1} \), \( x_{n,c-3} \) and \( y_{n-1,2} \), for every \( i = 1, 2, \ldots, c - 3 \) the couples \( x_{n,i} \) and \( y_{n,i} \), for every \( i = 1, 2, \ldots, c - 4 \) the couples \( x_{n,i} \) and \( y_{n,i+1} \), respectively. See Fig. 4.4.
Fig. 4.4

To find a required sequence $C_n$ in $G_n$ is easy and is left to the reader. \( \square \)

**Theorem 4.5.** (i) The family $\mathcal{C}(3, 5, c)$, $c \geq 5$ is nonhamiltonian.

(ii) \[ \varphi(\mathcal{C}(3, 5, 5)) \leq \frac{4}{5} \]

(iii) \[ \tau(\mathcal{C}(3, 5, c)) \leq \frac{4}{5} \text{ for every } c \geq 6. \]

**Proof.** Nonhamiltonicity of the family $\mathcal{C}(3, 5, c)$ for $c \geq 6$ follows from Lemma 4.1. The proof of nonhamiltonicity of the family $\mathcal{C}(3, 5, 5)$ is based on the fact that no graph $H$ from $\mathcal{C}(3, 5, 5)$ contains an edge with both end-vertices trivalent. This and (4.6) implies

\[ h(H) \leq 2v_3(H) \leq v(H) = v_3(H) + v_3(H) = 2v_3(H) + 8. \]

To prove the case (ii) consider 5-regular polyhedral graphs $G$ containing triangles and pentagons only. (For an existence of an infinite family of such graphs see Jucovič [14] or Trenkler [21]). Clearly $r(G) \in \mathcal{C}(3, 5, 5)$. Denote by $f_k(P)$ the number of $k$-gonal faces of a 3-polytopal graph $P$. Using the Euler polyhedral formula we have
\[ f(G) = 20 + 6f_5(G) \quad \text{and} \quad v(G) = 12 + 4f_5(G). \]

Since \( v(r(G)) = v(G) + f(G) = 32 + 10f_5(G) \) and
\[ h(r(G)) \leq 2v(G) = 24 + 8f_5(G), \]
we can easily obtain the proposition of (ii).

For every graph \( H \in \mathcal{A}(3, 5, c), c \geq 6, \) there is
\[ 3v_3(H) + cv_c(H) = 5v_3(H) \quad \text{and} \quad h(H) \leq 2 \min \{v_3(H) + v_c(H), v_3(H)\} \]
because of the biparticity of \( H. \) The relation (4.6) implies
\[ v_3(H) - v_3(H) + (4 - c)v_c(H) = 8. \]

These three relations lead to
\[ h(H) \leq 24 + 4(c - 3)v_c(H) \]
and
\[ v(H) \leq 32 + 5(c - 3)v_c(H), \]
from which we easily obtain \( \tau(\mathcal{A}(3, 5, c)) \leq \frac{4}{5}. \)

5. Remarks

The results presented leave many open questions, in particular for families of quadrangular 3-polytopal graphs with exactly two types of edges. Some of them concern the cases of the families of triangular graphs \( \mathcal{S}(a, a, 3), 8 \leq a \leq 10, \) too. We believe (in agreement with the conjecture of Grünbaum and Walther [7]) that in all these cases the shortness exponent is equal to 1; more precisely we state

**Conjecture 1.** \( \sigma(\mathcal{S}(a, a, 3)) = \sigma(\mathcal{A}(3, 3, c)) = 1 \) for every \( c, 8 \leq c \leq 10. \)

The following question would be interesting: What is the minimum number \( c_0 \) such that \( \sigma(\mathcal{A}(a, 3, c_0)) < 1, 4 \leq a \leq 5? \)

Theorem 4.2 (and Theorem 4.3 if \( a = 5 \)) implies \( c_0 \geq 8. \) A similar question can be posed for the families \( \mathcal{A}(3, b, c) 4 \leq b \leq 5. \)

**Conjecture 2.** \( \sigma(\mathcal{A}(3, b, c)) = 1 \) for any \( c \leq 7 \) and \( 4 \leq b \leq 5. \)

We should like to remind the reader that many problems concerning shortness parameters for various families of 3-polytopal graphs formulated by Grünbaum and Walther [7] are still open.
REFERENCES


Received June 27, 1989